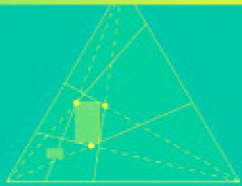
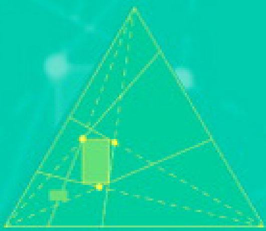


JULIUS B. BARBANEL

The GEOMETRY of EFFICIENT FAIR DIVISION



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The Geometry of Efficient Fair Division

What is the best way to divide a "cake" and allocate the pieces among some finite collection of players? In this book, the cake is a measure space, and each player uses a countably additive, non-atomic probability measure to evaluate the size of the pieces of cake, with different players generally using different measures. The author investigates efficiency properties (such as Pareto maximality: is there another partition that would make everyone at least as happy, and would make at least one player happier, than the present partition?) and fairness properties (such as envy-freeness: do all players think that their piece is at least as large as every other player's piece?). He focuses exclusively on abstract existence results rather than algorithms, and on the geometric objects that arise naturally in this context. By examining the shape of these objects and the relationship between them, he demonstrates results concerning the existence of efficient and fair partitions.

This is a work of mathematics that will be of interest to both mathematicians and economists.

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The Geometry of Efficient Fair Division

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Dedicated to
my mother, Evelyn Barbanel,
my wife, Nancy Niefield, and
my (step) children, Daniel Somerfield and Beth Somerfield
and
In memory of my father, Joseph Barbanel, whose memory continues to be a source of inspiration.

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Contents

Introduction

by Alan D. Taylor

Notions of obvious breadth and importance should, when possible, be examined under a number of different disciplinary lenses. This is the spirit in which the present offering by Julius Barbanel (a mathematician) joins recent books by Hervé Moulin (an economist) [32] and Nicholas Rescher (a philosopher) [35]. But fairness – or, more explicitly, fair division – comes in a number of different flavors, and we should begin by setting forth something of a general framework in which we can place the present book.

One of the more important dichotomies in the treatment of fairness is the extent to which the treatment is normative. Is the author trying to argue that certain methods of allocation are superior to others? The treatment of fair division by economists, philosophers, and political scientists tends to lie in the normative camp. Mathematicians, on the other hand, focus on what is possible and what is not, and often leave subjective judgments to others, as Barbanel does here.

Yet there is a normative aspect of the present work that sets it apart from the great majority of mathematical treatments, and it is revealed in Barbanel's choice of title. The work is not called "The Geometry of Fair Division" but "The Geometry of Efficient Fair Division." Efficiency – also called Pareto optimality, after the nineteenth-century Italian scholar Vilfredo Pareto – is, according to Hervé Moulin, "the single most important tool of normative economics" [32, pg. 8].

Economists also tend to focus (although not exclusively) on issues of fairness in the context of a finite collection of divisible homogeneous goods. Mathematicians, however, far more often work with a single divisible heterogeneous good and typically phrase the discussion in terms of the cake-cutting metaphor that dates back to the seventeen-century English political theorist James Harrington [26].

Discussions of cake cutting almost always begin with the procedure known as divide-and-choose. Historically, this two-person scheme traces its origins back 5000 years to the Bible's account of land division between Abram (later to be called Abraham) and Lot, and it resurfaces more explicitly two-and-a-half millennia ago as Hesiod, in his *Theogony*, describes the division of meat into two piles by Prometheus, with Zeus then choosing the pile that he preferred.

But even narrowed to the context of a mathematician's non-normative treatment of cake cutting, there is an important second dichotomy that sets the present work apart from earlier efforts such as those by Steven Brams and myself [16] and by Jack Robertson and William Webb [36]. This dichotomy is, in a sense, one of process versus product. Is one searching for a constructive procedure – a process – that will, in a step-by-step fashion, lead to desirable allocations, or is one trying to establish, by any mathematical means possible, the very existence of the desired allocation itself – the product?

The Brams–Taylor book and the Roberston–Webb book both focus on constructive procedures. The present work, on the other hand, is the first one on fair division to sit squarely in the existence camp. Yet economists will find it remarkably accessible – and an absolute gem in terms of illustrating how much insight the hands of an expert can wring from a couple of abstract results.

This distinction between constructive procedures and existence results is also reflected in the assumptions made in formalizing the preferences of the various participants in a fair-division situation. But in order to illustrate these differences, we need a few procedures on which to hang such a discussion. A quick historical tour will provide what is required.

Mathematical investigations of fair division date from the early 1940s. The constructive vein was first opened by the Polish mathematician Hugo Steinhaus (see [40]) and his colleagues Stefan Banach and Bronislaw Knaster. Steinhaus appears to have been the first to ask if there is an obvious extension of divide-and-choose to the case wherein there are three participants instead of two, and he derived the scheme referred to in a number of mathematical texts for non-majors (see [18] and [42]) as "the lone-divider method." But extending this procedure to four or more participants is somewhat complicated, and was not actually achieved until Harold Kuhn [30] did so in 1967. Banach and Knaster, however, took an entirely different tack and devised a fair-division scheme for any number of participants that is known today as the "last-diminisher method."

Each of these schemes generalizes divide-and-choose in the sense of providing a finite constructive procedure by which a group of people can allocate a "cake" among themselves in such a way that each has a strategy that ensures his or her own "satisfaction" even in the face of a conspiracy by all of the others. The word "protocol" is often used to capture both the algorithmic and the strategic aspects of such procedures, and this game-theoretic view results in the use of "player" in place of "participant."

Yet it turns out that the devil is in the details. “Satisfied” in what sense? For the procedures of Steinhaus, Banach, and Knaster, the answer is something called “proportionality” – each of n players is assured of receiving a piece that he or she thinks is at least $\frac{1}{n}$th of the total in size or value. Divide-and-choose is obviously proportional: if the divider makes it a 50–50 division, he or she will get exactly one-half; the chooser can’t go wrong. Proportionality, however, is only the easy answer.

In 1959, the physicist George Gamow and the mathematician Marvin Stern published a book [24] in which they pointed out that with divide-and-choose, each of the two players is assured of getting a piece that he or she thinks is at least tied for largest (or tied for most valuable). They asked if there were procedures that would do the same for three or more players. The name attached to such allocations today is “envy-free” or “no-envy,” a notion that economists trace back to Duncan Foley [22] in 1967. Envy-freeness is harder to come by than proportionality, although the existence results we turn to momentarily show that much more is, in some sense, possible.

Within a year of the Gamow–Stern question, John Conway of Princeton and John Selfridge of Northern Iowa University independently constructed an elegant envy-free protocol for three parties (see [16]), although the general question for four or more parties remained open until it was settled in the affirmative in 1992 [15]. There is, however, an important issue that still awaits attention: The three-person scheme never requires more than five cuts, whereas the general procedure, even if there are only four players, has no upper bound on the number of cuts needed that is independent of the preferences of the people involved.

But how do we formalize these “preferences” of the players, and what kind of an object do we take this “cake” to be? If we begin with the most general context that suggests itself, the “cake” C would be an arbitrary set and each player’s preferences over (certain) subsets of C would be given by a binary relation R that is reflexive, transitive, and complete (with XRY intuitively meaning that this player finds the subset X to be at least as desirable as the subset Y). And, as first pointed out by David Gale [23], discrete cake-cutting protocols implicitly assume only three additional postulates: (i) a partitioning postulate, asserting that a player can divide a piece of cake into any number of smaller pieces that he or she considers equivalent to each other, (ii) a trimming postulate asserting that if a player prefers one piece of cake to another, then there is a subset of the former that he or she considers equivalent to the latter, and (iii) a weak-additivity postulate asserting that if a player prefers piece 1 to piece 2, and piece 3 to piece 4, and pieces 1 and 3 are disjoint, then that player will prefer the union of pieces 1 and 3 to the union of pieces 2 and 4.

The easiest way to obtain such a relation is to let Player i's preferences be given by a finitely additive, non-atomic probability measure over some algebra of subsets of the arbitrary set C. That is, one starts with a collection of subsets of C that is closed under complementation, finite unions, and finite intersections – this is what an algebra is – and a function μ that assigns a real number in the interval [0, 1] to each set in the algebra so that if $A_1, \ldots, A_n$ is a finite collection of pairwise disjoint sets in the algebra, then $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ – this is finite additivity – and such that, if $\mu(A) > 0$, then there is some $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$ – this is what it means to be non-atomic.

In point of fact, there is only one difference between working in the general context of a preference relation satisfying Gale's three postulates and working with a finitely additive, non-atomic probability measure: the latter ensures that the players' preferences satisfy an Archimedean property asserting that, if a subset of C is strictly preferred to the empty set, then the entire cake C can be partitioned into finitely many pieces, all of which are less desirable than the given piece. This is the only difference in the sense that one can prove [10] that any Archimedean preference relation satisfying Gale's three postulates is induced by a finitely additive, non-atomic probability measure.

Protocols – or, more generally, all cake-division schemes with a legitimate claim to being finite and constructive – work in the context of finitely additive, non-atomic probability measures. Existence results, on the other hand, both assume more and deliver more.

Historically, the first existence result to explicitly address fair division may have been Jerzy Neyman's 1946 result [34] asserting that a cake can be divided among n players in such a way that every player thinks every piece is $\frac{1}{n}$th of the total. This theorem assumes, as do virtually all of what are called "existence results" in this context, that the players' preferences are given by non-atomic probability measures that are not only finitely additive, but countably additive: If $A_1, A_2, \ldots$ is a collection of pairwise disjoint sets in the algebra indexed by the set of natural numbers, then $\mu(A_1 \cup A_2 \cup \cdots) = \mu(A_1) + \mu(A_2) + \cdots$.

There are stepping stones between protocols and existence results that deserve mention. These are the so-called "moving-knife schemes" that date back to the observation of Lester Dubins and E. H. Spanier [20] that the Banach–Knaster scheme can be envisioned as one in which a knife is slowly moved across the cake, with each player having the option to call "cut" at any time and to exit the game with the resulting piece. A moving-knife alternative to the three-player envy-free Selfridge–Conway procedure was found by Walter Stromquist [41] in 1980, and, in 1982, A. K. Austin [3] introduced a moving-knife version of the $n = 2$ case of Neyman's theorem. A number of questions in the context of moving-knife schemes remain open (see [17] and [9]). The reader seeking an additional challenge can try to extend to the moving-knife arena the myriad of results set forth by Barbanel in what follows.

So now we have the context: Barbanel is giving a non-normative, mathematical treatment of existence results that deal with efficiency as well as fairness, in the context of a single heterogeneous good with the preferences of players given by countably additive, non-atomic, probability measures. All that remains is to address the question of how geometry enters the picture.

Geometry is the study of size and shape. Thus, one might expect Barbanel to study the size and shape of, well, the cake (or at least pieces thereof). But that's not at all what he does. His study of the geometry of fair division is much more in the spirit of Donald Saari's study of the geometry of voting [39]. For Saari, a ballot in an election corresponds to a point in n-space. For Barbanel, an allocation of the cake corresponds to a point of n-space in one of the two main geometric objects considered. In the other, each point of the cake corresponds to a point in n-space, but in a non-obvious manner. Either way, once he has a set of points in n-space, he is geometrically off and running.

The book is laid out as follows. After introducing some basic notation, terminology, and background in Chapter 1, Barbanel defines the first geometric object on which he focuses: the Individual Pieces Set (IPS). He introduces the IPS for two players in Chapter 2 and then exploits it in the context of fairness and efficiency in Chapter 3.

In Chapter 4, Barbanel moves on to the general case of n players, where he generalizes the IPS to the FIPS, the Full Individual Pieces Set, and he proves an important result concerning the possible shapes of the FIPS. In Chapter 5, he considers what the IPS and FIPS reveal about fairness and efficiency in the general n-player context.

Barbanel next focuses exclusively on efficiency, and he presents three quite different characterizations of Pareto optimality. After some introductory notions in Chapter 6, he characterizes Pareto optimality using the optimization of convex combinations of measures (Chapter 7) and partition ratios (Chapter 8). In Chapter 9, Barbanel introduces the second of his two main geometric objects: the Radon–Nikodym Set (RNS), and he uses it, together with an idea of Dietrich Weller, to present a third characterization of Pareto optimality in Chapter 10.

In Chapter 11, Barbanel considers the possible shapes of the IPS, and he provides a complete characterization in the case of two players and a partial result in the general n-player context. In Chapters 12 and 13, he studies the relationship between the IPS and the RNS, and he provides a new presentation of the fundamental result that ensures the existence of a partition that is both Pareto optimal and envy-free.

In Chapter 14, Barbanel introduces a strengthening of Pareto optimality that he calls "strong Pareto optimality," and he presents both characterization theorems and existence results. He also discusses the relationships between the number of strongly Pareto optimal partitions and the number of Pareto optimal partitions that are not strongly Pareto optimal.

Barbanel's characterizations of Pareto optimality in Chapters 7 and 10 involve what is essentially an iterative procedure. In Chapter 15, he shows that these ideas can be greatly simplified by the use of hyperreal numbers and non-standard analysis.

Finally, in Chapter 16, Barbanel shows that the IPS can be viewed as a piece of a larger structure that he calls the Multicake Individual Pieces Set (MIPS). Earlier chapters reveal certain peculiarly non-symmetric possibilities for the IPS; symmetry reasserts itself in the MIPS.

1

Notation and Preliminaries

Our "cake" C is some set. We wish to partition C among n players, whom we shall refer to as Player 1, Player 2, ..., Player n. For each $i = 1, 2, \ldots, n$, Player i uses a measure m_i to evaluate the size of pieces of cake (i.e., subsets of C). Unless otherwise noted, we shall always assume that C is non-empty.

Definition 1.1 A σ*-algebra* on C is a collection of subsets W of C satisfying that

a. $C \in W$,
b. if $A \in W$ then $C \backslash A \in W$, and
c. if $A_i \in W$ for every $i \in \mathbf{N}$, then $(\bigcup_{i \in \mathbf{N}} A_i) \in W$ (where $\mathbf{N}$ denotes the set of natural numbers).

Definition 1.2 Assume that some σ-algebra W has been defined on C. A *countably additive measure* on W is a function $\mu : W \to \mathbf{R}$ (where $\mathbf{R}$ denotes the set of real numbers) satisfying that

a. $\mu(A) \geq 0$ for every $A \in W$,
b. $\mu(\emptyset) = 0$, and
c. if $A_1, A_2, \ldots$ is a countable collection of elements of W and this collection is pairwise disjoint, then $\mu(\bigcup_{i \in \mathbf{N}} A_i) = \sum_{i \in \mathbf{N}} \mu(A_i)$.

In addition, μ is

d. *non-atomic* if and only if, for any $A \in W$, if $\mu(A) > 0$ then for some $B \subseteq A$, $B \in W$ and $0 < \mu(B) < \mu(A)$ and
e. a *probability measure* if and only if $\mu(C) = 1$.

Unless otherwise noted, all measures that we shall consider will be countably additive, non-atomic probability measures, and we shall simply use the term "measure" to refer to them. Notice that for any measure μ and any $a \in C$, the non-atomic nature of μ implies that $\mu(a) = 0$.

Also, unless otherwise specified, C shall denote an arbitrary cake. We assume that there are n players, Player 1, Player 2, ..., Player n, with corresponding measures $m_1, m_2, \ldots, m_n$, respectively. At times, we shall work with specific examples and shall give specific definitions of C and $m_1, m_2, \ldots, m_n$.

Whenever a subset of C is mentioned, we assume it is a member of some common σ-algebra on which all of the measures are defined. We shall never explicitly define a specific σ-algebra.

We will be concerned with partitions of the cake C among the players. When we consider an ordered partition $\langle P_1, P_2, \ldots, P_n \rangle$ of C, our intention is that P_1 goes to Player 1, P_2 goes to Player 2, etc. The term "partition" always means "ordered partition." *Part* denotes the set of all partitions of the appropriated size, which will always be clear by context.

Consider the set $\{(m_1(A), m_2(A), \ldots, m_n(A)) : A \subseteq C\}$, which is a subset of $\mathbf{R}^n$. This set will be important for us. A central tool concerning this set is Lyapounov's theorem.

Theorem 1.3 (Lyapounov's Theorem [31]) $\{(m_1(A), m_2(A), \ldots, m_n(A)) : A \subseteq C\}$ *is a closed and convex subset of* $\mathbf{R}^n$.

Another important set is $\{[m_i(P_j)]_{i,j \le n} : \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of $C\}$. This is a subset of the set of all $n \times n$ matrices and can be viewed as a subset of $\mathbf{R}^{(\mathbf{n}^2)}$. An element of this set gives each player's evaluation of the size of each piece of cake in a given partition. A central tool concerning this set is Dvoretsky, Wald, and Wolfovitz's theorem.

Theorem 1.4 (Dvoretsky, Wald, and Wolfovitz's Theorem [21]) $\{[m_i(P_j)]_{i,j \le n} : \langle P_1, P_2, \ldots, P_n \rangle$ *is a partition of* $C\}$ *is a closed and convex subset of the set of all* $n \times n$ *matrices.*

(Dvoretsky, Wald, and Wolfovitz's theorem actually is more general than the preceding statement. The number of players need not equal the number of pieces of the partition, and thus the set under consideration is $\{[m_i(P_j)]_{i \le m; j \le n} : \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of $C\}$. The theorem says that this set is a closed and compact subset of the set of all m $\times$ n matrices. We shall always have the number of players equal to the number of pieces of partitions, and so we have stated the theorem in this more restricted form.)

Notice that $\{(m_1(A), m_2(A), \ldots, m_n(A)) : A \subseteq C\}$ is the set of all first (or second, or third, etc.) columns of $\{[m_i(P_j)]_{i,j \le n} : \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of $C\}$. This tells us that Lyapounov's theorem follows immediately from Dvoretsky, Wald, and Wolfovitz's theorem.

We shall frequently need to find subsets of C having certain sizes on which all players agree. The following corollary to Lyapounov's theorem will often provide exactly what we need.

Corollary 1.5 *Fix non-negative real numbers* $p_1, p_2, \ldots, p_n$ *such that* $p_1 + p_2 + \cdots + p_n = 1$. *There is a partition* $P = \langle P_1, P_2, \ldots, P_n \rangle$ *of* C *such that for all* $i, j = 1, 2, \ldots, n$, $m_i(P_j) = p_j$.

Proof: Fix $p_1, p_2, \ldots, p_n$ as in the statement of the corollary and let $G = \{[m_i(P_j)]_{i,j \le n} : \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of $C\}$. For each $i = 1, 2, \ldots, n$, let M_i be the matrix with all ones in column i and zeros everywhere else. Then, by considering the partitions $\langle C, \emptyset, \emptyset, \ldots, \emptyset, \emptyset \rangle$, $\langle \emptyset, C, \emptyset, \ldots, \emptyset, \emptyset \rangle, \ldots, \langle \emptyset, \emptyset, \emptyset, \ldots, \emptyset, C \rangle$, we see that each M_i is in G. By Dvoretsky, Wald, and Wolfovitz's theorem, G is convex and hence $p_1 M_1 + p_2 M_2 + \cdots + p_n M_n \in G$. But $p_1 M_1 + p_2 M_2 + \cdots + p_n M_n$ is the matrix with every entry in the first column equal to p_1, every entry in the second column equal to p_2, etc. This implies that there is a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of C such that for all $i, j = 1, 2, \ldots, n$, $m_i(P_j) = p_j$, as desired. □

Corollary 1.5 has many simple applications. Two are given by the following two corollaries.

Corollary 1.6 *For any* $A \subseteq C$ *and non-negative real numbers* $q_1, q_2, \ldots, q_n$ *with* $q_1 + q_2 + \cdots + q_n = 1$, *there is a partition* $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ *of* A *such that for all* $i, j = 1, 2, \ldots, n$, $m_i(Q_j) = q_j m_i(A)$.

Proof: Fix A and $q_1, q_2, \ldots, q_n$ as in the statement of the corollary and let $\delta = \{i \le n : m_i(A) > 0\}$. For each $i \in \delta$, we define m'_i on A as follows:

$$\text{for each } B \subseteq A, m'_i(B) = \frac{m_i(B)}{m_i(A)}$$

Each such m'_i is a measure on A. For each $i \notin \delta$, let m'_i be any measure on A.

It follows from Corollary 1.5, with A playing the role of C, that there is a partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ of A satisfying that $m'_i(Q_j) = q_j$ for all $i, j = 1, 2, \ldots, n$. We claim that for all $i, j = 1, 2, \ldots, n$, $m_i(Q_j) = q_j m_i(A)$. Fix such an i and j. We consider two cases.

Case 1: $i \in \delta$. Then $m_i(Q_j) = m'_i(Q_j) m_i(A) = q_j m_i(A)$.
Case 2: $i \notin \delta$. Then $m_i(A) = 0$ and hence, since $Q_j \subseteq A$, $m_i(Q_j) = 0$.

Therefore, $m_i(Q_j) = 0 = (q_j)(0) = q_j m_i(A)$.

This establishes that for all $i, j = 1, 2, \ldots, n$, $m_i(Q_j) = q_j m_i(A)$, as desired. □

Corollary 1.7 *Fix some $A \subseteq C$ and $k = 1, 2, \ldots, n$. If $m_k(A) > 0$, then for any r with $0 \leq r \leq m_k(A)$, there is a $B \subseteq A$ with $m_k(B) = r$.*

Proof: Let A and k be as in the statement of the corollary, assume that $m_k(A) > 0$, and fix some r with $0 \leq r \leq m_k(A)$. Set $q_k = \frac{r}{m_k(A)}$. Then $0 \leq q_k \leq 1$. For each $i = 1, 2, \ldots, n$ with $i \neq k$, let q_i be an arbitrary non-negative real number, subject to the condition that $q_1 + q_2 + \cdots + q_n = 1$. By Corollary 1.6, there is a partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ of A such that for all $i, j = 1, 2, \ldots, n$, $m_i(Q_j) = q_j m_i(A)$. Set $B = Q_k$. Then $m_k(B) = m_k(Q_k) = q_k m_k(A) = r$, as desired. □

We will be interested in what it means for a partition of C to be a "good" partition. Various notions of what good means in this context have been considered. These notions are of two types. One is concerned with fairness and the other with efficiency. Before we can define fairness and efficiency properties, we must first consider a more basic question: Do players want as much cake as possible or do they want as little cake as possible? For example, if the cake represents money to be distributed among the players, then it is reasonable to assume that each player wants as much of the cake as possible. On the other hand, if the cake represents some task that all players view as unpleasant, then each player wants as little of the cake as possible. We shall refer to the first setting, in which "bigger is better," as the *standard setting*, and shall refer to the latter setting, in which "smaller is better," as the *chores setting* (since, in this case, pieces of cake may be viewed as "chores"). Unless otherwise noted, we shall assume that we are working in the standard setting. Our approach for most sections is to first concentrate on the standard setting and then on the chores setting. (However, there will be some sections where we find it most convenient to consider the standard setting and the chores setting at the same time.) Most of the time, results about the chores setting will simply be symmetric adjustments of results about the standard setting. However, there will be important exceptions.

What does it mean to say that a partition of the cake is fair? We shall say that a partition is fair if and only if every player thinks that it is fair, and so the question becomes: When does a player think that a partition is fair? Consider the following five answers for the standard setting. A player thinks that a partition is fair if and only if that player thinks that his or her piece of cake is

a. at least of average size.
b. of bigger-than-average size.
c. at least as big as every other piece.

d. bigger than every other piece.
e. of bigger-than-average size, and everyone else's piece is of smaller-than-average size.

Notice that for a given partition P, each player's decision as to whether or not P is fair (in each of the five aforementiond senses) entails comparisons involving this player's evaluation of his or her own piece, this player's evaluation of the pieces of the other players, and the number $\frac{1}{n}$ (the average size of a piece of cake, where n is the number of players).

Each of these five answers yields a different notion of fairness for a partition. These are given by the following definition (where parts a, b, c, d, and e of the definition correspond to previously listed answers a, b, c, d, and e, respectively).

Definition 1.8 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of C. P is

a. *proportional* if and only if, for each $i = 1, 2, \ldots, n$, $m_i(P_i) \geq \frac{1}{n}$.
b. *strongly proportional* if and only if, for each $i = 1, 2, \ldots, n$, $m_i(P_i) > \frac{1}{n}$.
c. *envy-free* if and only if, for all $i, j = 1, 2, \ldots, n$, $m_i(P_i) \geq m_i(P_j)$.
d. *strongly envy-free* if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $m_i(P_i) > m_i(P_j)$.
e. *super envy-free* if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $m_i(P_i) > \frac{1}{n}$ and $m_i(P_j) < \frac{1}{n}$.

The reason for the name "envy-free" should be clear: A partition is envy-free if and only if no player envies another player or, in other words, no player would be happier if he or she traded pieces with some other player.

It follows immediately from the definition that, for any partition P,

- if P is super envy-free, then P is strongly envy-free.
- if P is strongly envy-free, then P is envy free and strongly proportional.
- if P is envy-free, then P is proportional.
- if P is strongly proportional, then P is proportional.

As we shall see (in Chapters 4 and 5), the converses of these implications need not hold.

The fairness properties for the chores setting are as follows.

Definition 1.9 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of C. P is

a. *chores proportional* if and only if, for each $i = 1, 2, \ldots, n$, $m_i(P_i) \leq \frac{1}{n}$.
b. *strongly chores proportional* if and only if, for each $i = 1, 2, \ldots, n$, $m_i(P_i) < \frac{1}{n}$.
c. *chores envy-free* if and only if, for all $i, j = 1, 2, \ldots, n$, $m_i(P_i) \leq m_i(P_j)$.

d. *strongly chores envy-free* if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $m_i(P_i) < m_i(P_j)$.
e. *super chores envy-free* if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $m_i(P_i) < \frac{1}{n}$ and $m_i(P_j) < \frac{1}{n}$.

We shall generally abbreviate the terms "chores proportional," "strongly chores proportional," "chores envy-free," "strongly chores envy-free," and "super chores envy-free" by writing "c-proportional," "strongly c-proportional," "c-envy-free," "strongly c-envy-free," and "super c-envy-free," respectively.

Next, we consider efficiency. In contrast with fairness, efficiency involves comparing different partitions. *A* partition P is efficient if and only if no other partition makes every player at least as happy as does P, and makes some player happier. This leads to the following definitions for the standard and chores settings.

Definition 1.10 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of C. P is *Pareto maximal* if and only if for no partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ is it true that, for each $i = 1, 2, \ldots, n$, $m_i(Q_i) \geq m_i(P_i)$, with at least one of these inequalities being strict.

Definition 1.11 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of C. P is *Pareto minimal* if and only if for no partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ is it true that, for each $i = 1, 2, \ldots, n$, $m_i(Q_i) \leq m_i(P_i)$, with at least one of these inequalities being strict.

We shall say that a partition is *Pareto optimal* if and only if it is either Pareto maximal or Pareto minimal. Many references use the term Pareto optimal for what we have called Pareto maximal. Our present terminology seems more natural, since we shall be considering both the standard and the chores setting.

We shall say that partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is *Pareto bigger* than partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ if and only if, for each $i = 1, 2, \ldots, n$, $m_i(P_i) \geq m_i(Q_i)$, with at least one of these inequalities being strict. Similarly, partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is *Pareto smaller* than partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ if and only if, for each $i = 1, 2, \ldots, n$, $m_i(P_i) \leq m_i(Q_i)$, with at least one of these inequalities being strict. Then, a partition P is Pareto maximal if and only if no partition is Pareto bigger than P, and a partition P is Pareto minimal if and only if no partition is Pareto smaller than P.

Sometimes, we shall assume that any piece of cake that has value zero to one player has value zero to all players, whereas at other times we shall not make this assumption. This is the notion of absolute continuity.

Definition 1.12 Measure m_i is *absolutely continuous* with respect to measure m_j if and only if for any $A \subseteq C$, if $m_j(A) = 0$ then $m_i(A) = 0$.

Our approach for most chapters is to first assume that all measures are absolutely continuous with respect to each other and, after completing this study, to consider the situation without this assumption. Notice that if the measures are all absolutely continuous with respect to each other, then we may use terminology such as "almost all $a \in C$" or "A is a set of measure zero" without specifying a measure, since the given statement is true with respect to one measure if and only if it is true with respect to all measures. If the measures are not absolutely continuous with respect to each other, then such statements cannot me made without specifying to which measure they refer. We shall often simply say that "absolute continuity holds" to mean that the measures are all absolutely continuous with respect to each other, and that "absolute continuity fails" to mean that this is not so.

The following lemma tells us that if absolute continuity holds, then we may change the inequalities in Definitions 1.10 and 1.11 to strict inequalities. In other words, if absolute continuity holds and a partition P is not Pareto maximal, then there exists a partition that makes each player happier than he or she was with partition P. A similar statement holds for Pareto minimality.

Lemma 1.13 *Assume that all measures are absolutely continuous with respect to each other.*

a. Partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is Pareto maximal if and only if for no partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ is it true that, for each $i = 1, 2, \ldots, n$, $m_i(Q_i) > m_i(P_i)$.

b. Partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is Pareto minimal if and only if for no partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ is it true that, for each $i = 1, 2, \ldots, n$, $m_i(Q_i) < m_i(P_i)$.

Proof: The forward direction of part a is trivial and does not require absolute continuity. For the reverse direction, assume that P is not Pareto maximal and let $R = \langle R_1, R_2, \ldots, R_n \rangle$ be a partition that is Pareto bigger than P. Then, for some $k = 1, 2, \ldots, n$, $m_k(R_k) > m_k(P_k)$. It follows by repeated use of Corollary 1.7 that there is a partition $S = \langle S_1, S_2, \ldots, S_n \rangle$ of R_k such that $m_k(S_k) > m_k(P_k)$ and, for each $i = 1, 2, \ldots, n$, $m_k(S_i) > 0$. (We first apply the corollary to find $S_k \subseteq R_k$ with $m_k(P_k) < m_k(S_k) < m_k(R_k)$, and then we find the S_i for $i \neq k$.)

Define a new partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ of C as follows: for each $i = 1, 2, \ldots, n$,

$$Q_i = \begin{cases} R_i \cup S_i & \text{if } i \neq k \\ S_k & \text{if } i = k \end{cases}$$

Since $m_k(S_i) > 0$ for each $i = 1, 2, \ldots, n$, it follows by absolute continuity that, for each such $i, m_i(S_i) > 0$. Hence, for $i \neq k$, we have $m_i(Q_i) = m_i(R_i \cup S_i) = m_i(R_i) + m_i(S_i) > m_i(R_i) \geq m_i(P_i)$. Since $m_k(Q_k) = m_k(S_k) > m_k(P_k)$, we have shown that, for every $i = 1, 2, \ldots, n$, $m_i(Q_i) > m_i(P_i)$.

The proof for part b is analogous and we omit it. □

The idea behind the lemma is quite simple. Every player is at least as happy with partition Q as with partition P, and at least one player is happier. A player that is happier can give away a piece of cake of positive measure to each player and still be happier than he or she was with partition P. By absolute continuity, all of the other players will now be happier than they were with partition Q and, hence, happier than they were with partition P.

Many of our geometric constructions will involve the simplex. When there are n players, the relevant simplex is $(n-1)$-*simplex*, which is the set $\{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \geq 0 \text{ and } x_1 + x_2 + \cdots + x_n = 1\}$. Thus, the number of players determines which is the relevant simplex. We shall just refer to the "simplex" rather than the "$(n-1)$-simplex" when the "$n-1$" is clear by context. We will generally let S denote the simplex and S^+ its interior. Thus, $S^+ = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n > 0 \text{ and } x_1 + x_2 + \cdots + x_n = 1\}$. (We use "+" because, for $\omega \in S$, $\omega \in S^+$ if and only if each coordinate of ω is positive.) The $(n-1)$-simplex is an $(n-1)$-dimensional subset of $\mathbf{R}^n$.

The one-simplex, two-simplex, and three-simplex are one-dimensional, two-dimensional, and three-dimensional objects, respectively, and hence can easily be pictured. These are shown in Figure 1.1. The one-simplex two-simplex, and three-simplex are shown in Figures 1.1a, 1.1b, and 1.1c, respectively. In these figures, we have shown only the simplex. Often, we shall want to view

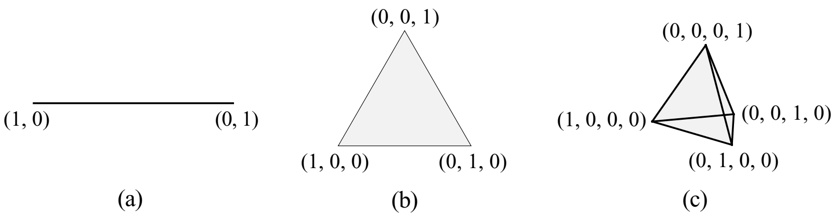


Figure 1.1

the one-simplex and the two-simplex in the context of the xy- and the xyz-coordinate system, respectively.

Another geometric object that will be important is the *unit hypercube*, which we always assume includes its interior. In $\mathbf{R}^n$, this is the set $\{(x_1, x_2, \ldots, x_n) : \text{for each } i = 1, 2, \ldots, n, 0 \leq x_i \leq 1\}$. When $n = 2$, this is the square with vertices $(0, 0), (0, 1), (1, 1)$, and $(1, 0)$, together with its interior and, when $n = 3$, this is the cube with vertices $(0, 0, 0)$, $(0, 0, 1), (0, 1, 1), (0, 1, 0), (1, 0, 0), (1, 0, 1), (1, 1, 1)$, and $(1, 1, 0)$, together with its interior.

For any $G \subseteq \mathbf{R}^n$, a point $p \in \mathbf{R}^n$ is a *convex combination* of the elements of G if and only if $p = \alpha_1 p^1 + \alpha_2 p^2 + \cdots + \alpha_m p^m$ for some $p^1, p^2, \ldots, p^m \in G$ and non-negative $\alpha_1, \alpha_2, \ldots, \alpha_m$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$ (or, equivalently, if and only if $p = \alpha_1 p^1 + \alpha_2 p^2 + \cdots + \alpha_m p^m$ for some $p^1, p^2, \ldots, p^m \in G$ and $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in S$). The *convex hull* of G, which we denote by $\mathrm{CH}(G)$, is the set of all $p \in \mathbf{R}^n$ that are convex combinations of the elements of G.

For any set G, $|G|$ denotes the cardinality of G.

Some of the material in this book previously appeared in other work by the author (see [4–8, 11]).

2

Geometric Object #1a

The Individual Pieces Set (IPS) for Two Players

In this chapter, we introduce the first of two geometric objects that we associate with cake division. We also introduce various notions and questions that will be important in later chapters. We call this geometric object the Individual Pieces Set, or IPS. Our present focus is the two-player context. In Chapter 4, we consider the general case of n players, where we shall also introduce a generalized version of the IPS, called the Full Individual Pieces Set. Throughout this chapter, the measures m_1 and m_2 may or may not be absolutely continuous with respect to each other.

Definition 2.1 For any partition $P = \langle P_1, P_2 \rangle$ of C, let $m(P) = (m_1(P_1), m_2(P_2))$. The *Individual Pieces Set*, or *IPS*, is the set $\{m(P) : P \in \text{Part}\}$.

Notice that IPS $\subseteq \mathbf{R}^2$.

Of course, the IPS depends upon C, m_1, and m_2, and thus we shall always need to be sure that when we write "the IPS" the corresponding cake and measures are clear by context.

We wish to understand the general shape and geometric properties of the IPS. What do we know about points in the IPS? We can imagine all of the cake being given to Player 1. The associated partition is $\langle C, \emptyset \rangle$ and the corresponding point in the IPS is $(1, 0)$. Similarly, we see that $(0, 1) \in$ IPS. These facts, together with Dvoretsky, Wald, and Wolfovitz's theorem (Theorem 1.4), imply that the convex hull of these points, which is simply the line segment between them, is in the IPS. Without further assumptions, this is all we can say.

Theorem 2.2
a. The IPS contains the closed line segment between $(1, 0)$ *and* $(0, 1)$.
b. The IPS consists precisely of this line segment if and only if $m_1 = m_2$.

Proof: We have already seen that the IPS includes the closed line segment between $(1, 0)$ and $(0, 1)$.

For the forward direction of part b, we first note that a point (p_1, p_2) is on the closed line segment connecting $(1, 0)$ and $(0, 1)$ if and only if $p_1 \geq 0$, $p_2 \geq 0$, and $p_1 + p_2 = 1$. Suppose that $m_1 \neq m_2$. Then for some $A \subseteq C$, $m_1(A) \neq m_2(A)$. Since $m_2(A) = 1 - m_2(C \backslash A)$, we know that $m_1(A) \neq 1 - m_2(C \backslash A)$, and hence $m_1(A) + m_2(C \backslash A) \neq 1$. Since $\langle A, C \backslash A \rangle \in \text{Part}$, it follows that $(m_1(A), m_2(C \backslash A))$ is a point in the IPS that is not on the closed line segment connecting $(1, 0)$ and $(0, 1)$.

For the reverse direction of part b, we assume that $m_1 = m_2$. Any point of the IPS is of the form $(m_1(P_1), m_2(P_2))$ for some partition $\langle P_1, P_2 \rangle$. Then $m_1(P_1) + m_2(P_2) = m_1(P_1) + m_1(P_2) = m_1(P_1 \cup P_2) = m_1(C) = 1$. Clearly, $m_1(P_1) \geq 0$ and $m_2(P_2) \geq 0$. It follows that $(m_1(P_1), m_2(P_2))$ is on the closed line segment between $(1, 0)$ and $(0, 1)$. □

Continuing our study of the general shape of the IPS, we find that it possesses a nice symmetry property.

Lemma 2.3 *The IPS is symmetric about the point* $(\frac{1}{2}, \frac{1}{2})$.

Proof: Suppose $(p_1, p_2) \in \text{IPS}$. Then, for some partition $\langle P_1, P_2 \rangle$ of C, $(m_1(P_1), m_2(P_2)) = (p_1, p_2)$, and it follows that $(m_1(P_2), m_2(P_1)) = (m_1(C \backslash P_1), m_2(C \backslash P_2)) = (1 - m_1(P_1), 1 - m_2(P_2)) = (1 - p_1, 1 - p_2) \in \text{IPS}$. Since $(1 - p_1, 1 - p_2)$ is the reflection of (p_1, p_2) about the point $(\frac{1}{2}, \frac{1}{2})$, the lemma follows. □

After we have defined the IPS for $n > 2$ in Chapter 4, we will consider the symmetry of the IPS for this more general setting. For now, we simply note that there is no obvious generalization of the proof of Lemma 2.3 for $n > 2$.

Theorem 2.2 and Lemma 2.3 begin to give us a picture of what the IPS looks like. What else can we say about the shape of the IPS? Since the measures take on values in the closed interval $[0, 1]$, we know that the IPS is a subset of $[0, 1]^2$, the unit square together with its interior. Also, we recall that Dvoretsky, Wald, and Wolfovitz's theorem implies that the IPS is closed and convex.

Let us assemble the facts that we presently know about the IPS.

Theorem 2.4 *The IPS*

a. *is a subset of* $[0, 1]^2$,
b. *contains the points* $(1, 0)$ *and* $(0, 1)$,
c. *is closed,*
d. *is convex, and*
e. *is symmetric about the point* $(\frac{1}{2}, \frac{1}{2})$.

Are there other facts that must hold of any IPS? In Chapter 11, we will show that the answer to this question is "no." For the case of two players, these five properties characterize the possible shapes of the IPS. In other words, given any $A \subseteq \mathbf{R}^2$ satisfying these five conditions, we can find a cake C and measures m_1 and m_2 on C so that A is the IPS corresponding to C, m_1, and m_2. In Chapter 11 we shall also see that the situation is very different when there are more than two players.

Six sets satisfying the five previously listed properties are shown in Figure 2.1. Once we have established the result mentioned in the previous paragraph, we will know that each of these regions is the IPS for some cake C and corresponding measures m_1 and m_2. Notice that

> there is a point p in the IPS that is in the interior of the line segment between (0, 1) and (1, 1) or that is in the interior of the line segment between (1, 0) and (1, 1),
>
> if and only if
>
> there is a piece of cake A such that $m_1(A) > 0$ and $m_2(A) = 0$ or such that $m_2(A) > 0$ and $m_1(A) = 0$, respectively,
>
> if and only if
>
> m_1 is not absolutely continuous with respect m_2 or m_2 is not absolutely continuous with respect m_1, respectively.

Thus, the IPSs in Figures 2.1a, 2.1b, 2.1c, and 2.1d correspond to measures that are absolutely continuous with respect to each other, the IPS in Figure 2.1e corresponds to a situation in which neither measure is absolutely continuous with respect to the other, and the IPS in Figure 2.1f corresponds to a situation in which m_1 is not absolutely continuous with respect m_2, but m_2 is absolutely continuous with respect m_1. We also observe that parts of the boundaries of these IPSs are straight and parts are curved. We shall study the significance of this distinction in Chapter 12.

The function m given in Definition 2.1 maps Part onto the IPS. Is this mapping one-to-one? The answer is: certainly not. Given any partition $P = \langle P_1, P_2 \rangle$, let $Q = \langle Q_1, Q_2 \rangle$ be a partition obtained by transferring a single point of cake from one player to the other. Since the measures are non-atomic, any single point has measure zero, and it follows that $m(P) = m(Q)$. (It is easy to see that there are infinitely many such partitions $Q = \langle Q_1, Q_2 \rangle$ such that $P \neq Q$ but $m(P) = m(Q)$.) However, we want to consider P and Q to be essentially the same partition in this case, because they differ only on a set of measure zero. This leads us to the following definition.

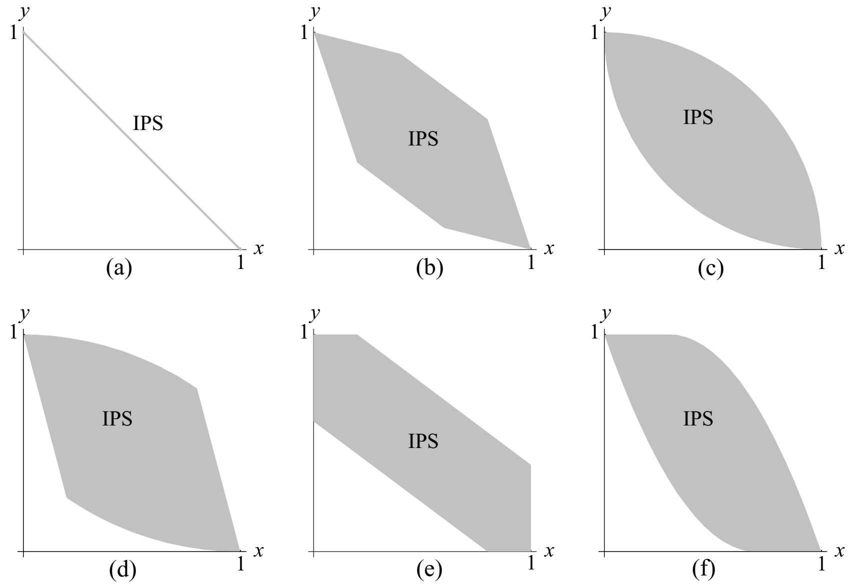


Figure 2.1

Definition 2.5 Two partitions $P = \langle P_1, P_2\rangle$ and $Q = \langle Q_1, Q_2\rangle$ are *s-equivalent* ("s" for "set") if and only if $m_1(P_1 \Delta Q_1) = 0$ and $m_2(P_2 \Delta Q_2) = 0$, where Δ denotes the symmetric difference.

Given two partitions P and Q, we may view each of these partitions as arising from the other by two transfers of cake, one from Player 1 to Player 2 and one from Player 2 to Player 1. Then, P and Q are s-equivalent if and only if each player views each of these transfers as consisting of a piece of cake of measure zero.

Clearly, s-equivalence is an equivalence relation. We shall refer to the associated equivalence classes as *s-classes*.

It is obvious that the function m from Part to the IPS respects s-equivalence. (That is, if partitions P and Q are s-equivalent, then $m(P) = m(Q)$). We are interested in whether or not m is one-to-one for non-s-equivalent partitions. Or, equivalently, we are interested in whether the function induced by m that maps s-classes of Part to the IPS is one-to-one.

Theorem 2.6 *Let p be a point of the IPS. The following are equivalent:*

a. p is the image, under m, of infinitely many mutually non-s-equivalent partitions.

b. p is the image, under m, of at least two non-s-equivalent partitions.
c. p lies in the interior of a line segment contained in the IPS.

Before beginning the proof of the theorem, we establish a lemma.

Lemma 2.7 *For any piece of cake A, there is a collection of subsets of A such that, for each player, each subset in the collection has size half that of A, and any player who believes that A has positive measure also believes that all pairwise symmetric differences from this collection have positive measure. Also, if either player believes that A has positive measure, then this collection is infinite. In other words, for any $A \subseteq C$, there is a collection $\Gamma(A)$ of subsets of A such that*

a. for any $B \in \Gamma(A)$ and $i = 1, 2$, $m_i(B) = \frac{1}{2}m_i(A)$,
b. for $i = 1, 2$, if $m_i(A) > 0$, then for distinct $B_1, B_2 \in \Gamma(A)$, $m_i(B_1 \Delta B_2) > 0$, and
c. if either $m_1(A) > 0$ or $m_2(A) > 0$, then $\Gamma(A)$ is infinite.

Proof: We repeatedly use Corollary 1.6. Fix $A \subseteq C$ and let $\langle D_{1,1}, D_{1,2}, D_{1,3}\rangle$ be a partition of A into sets that both players agree are each one-third of A. Next, let $\langle D_{2,1}, D_{2,2}, D_{2,3}\rangle$ be a partition of $D_{1,3}$ into sets that both players agree are each one-third of $D_{1,3}$. Continuing in this manner, we obtain sets $D_{j,k}$ for each $j = 1, 2, \ldots$ and $k = 1, 2, 3$ so that for each such j and k, $\langle D_{j,1}, D_{j,2}, D_{j,3}\rangle$ is a partition of $D_{j-1,3}$ such that, for each $i = 1, 2$, $m_i(D_{j,k}) = (\frac{1}{3})m_i(D_{j-1,3})$, where we set $D_{0,3} = A$.

Let $\Gamma(A) = \{(\bigcup_{j\in K} D_{j,1}) \cup (\bigcup_{j\notin K} D_{j,2}) : K \subseteq \mathbf{N}\}$. Then $\Gamma(A)$ is a collection of subsets of A. We claim that $\Gamma(A)$ satisfies the conditions of the lemma.

For condition a, fix $B \in \Gamma(A)$. For some $K \subseteq \mathbf{N}$, $B = (\bigcup_{j\in K} D_{j,1}) \cup (\bigcup_{j\notin K} D_{j,2})$. We know that for each $i = 1, 2$, $j = 1, 2, 3, \ldots,$ and $k = 1, 2, 3$, $m_i(D_{j,k}) = (\frac{1}{3})m_i(D_{j-1,3})$. Then, recalling that $D_{0,3} = A$ and noting that the $D_{j,k}$ for $k = 1, 2$ are pairwise disjoint, we have

$$\begin{aligned} m_i(B) &= \left[\frac{1}{3}m_i(D_{0,3})\right] + \left[\frac{1}{3}m_i(D_{1,3})\right] + \left[\frac{1}{3}m_i(D_{2,3})\right] + \cdots \\ &= \left[\frac{1}{3}m_i(A)\right] + \left[\left(\frac{1}{3}\right)^2 m_i(A)\right] + \left[\left(\frac{1}{3}\right)^3 m_i(A)\right] + \cdots \end{aligned}$$

$$= \left[\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots \right] m_i(A)$$

$$= \left(\frac{\frac{1}{3}}{1 - \frac{1}{3}} \right) m_i(A) = \frac{1}{2} m_i(A)$$

where the fourth equality uses the standard formula for summing a geometric series. (The geometric series $a + ar + ar^2 + ar^3 + \cdots$, with $|r| < 1$, sums to $\frac{a}{1-r}$.)

For condition b, fix $i = 1, 2$, suppose that $m_i(A) > 0$, and choose distinct $B_1, B_2 \in \Gamma(A)$. Since $B_1 \neq B_2$, we know that some $D_{j,k}$ is in one of these sets and not in the other and, since $m_i(A) > 0$, it follows that $m_i(D_{j,k}) > 0$. Hence, $m_i(B_1 \Delta B_2) > 0$.

Finally, for condition c, assume that either $m_1(A) > 0$ or $m_2(A) > 0$. Then all of the $D_{j,k}$ are non-empty. This implies that each choice for $K \subseteq \mathbf{N}$ results in a different element of $\Gamma(A)$, and there are infinitely many such choices for K. □

Proof of Theorem 2.6: Fix some $p \in$ IPS. Part a obviously implies part b. We shall show that part b implies part c and that part c implies part a.

To show that part b implies part c, suppose that partitions $P = \langle P_1, P_2 \rangle$ and $Q = \langle Q_1, Q_2 \rangle$ are non-s-equivalent partitions and $m(P) = m(Q) = p$. Let $R_{12} = P_1 \cap Q_2$ and $R_{21} = Q_1 \cap P_2$. We can view R_{12} and R_{21} as the portions of cake that Player 1 must transfer to Player 2, and that Player 2 must transfer to Player 1, respectively, in changing from partition P to partition Q.

The non-s-equivalence of P and Q and the fact that $m(P) = m(Q)$ together imply that, for at least one of the two players, R_{12} and R_{21} are each sets of positive measure. Also, since $m(P) = m(Q)$ and Q is obtained from P by swapping the sets R_{12} and R_{21} between the two players, it must be that $m_1(R_{12}) = m_1(R_{21})$ and $m_2(R_{12}) = m_2(R_{21})$.

Consider the following two partitions: $S = \langle P_1 \cup R_{21}, P_2 \backslash R_{21} \rangle$ and $T = \langle P_1 \backslash R_{12}, P_2 \cup R_{12} \rangle$. We may view S and T as each having been obtained from partition P by completing one of the two transfers of cake discussed in the preceding paragraph. Since, for at least one player, R_{12} and R_{21} are each sets of positive measure, we know that $m(S) \neq m(T)$.

We claim that p is the midpoint of the line segment connecting $m(S)$ and $m(T)$. We establish this as follows:

midpoint of the line segment connecting $m(S)$ and $m(T)$

$=$ midpoint of the line segment connecting $(m_1(P_1 \cup R_{21}), m_2(P_2 \backslash R_{21}))$

$$\begin{aligned}
&\text{and } (m_1(P_1 \backslash R_{12}), m_2(P_2 \cup R_{12}))\\
&= \left(\frac{m_1(P_1 \cup R_{21}) + m_1(P_1 \backslash R_{12})}{2}, \frac{m_2(P_2 \backslash R_{21}) + m_2(P_2 \cup R_{12})}{2} \right)\\
&= \left(\frac{m_1(P_1) + m_1(R_{21}) + m_1(P_1) - m_1(R_{12})}{2}, \right.\\
&\qquad \left. \frac{m_2(P_2) - m_2(R_{21}) + m_2(P_2) + m_2(R_{12})}{2} \right)\\
&\quad = (m_1(P_1), m_2(P_2)) = p
\end{aligned}$$

This establishes that p is the midpoint of the line segment between $m(S)$ and $m(T)$. By convexity, this line segment is contained in the IPS. Hence, p lies in the interior of a line segment contained in the IPS.

Next, we show that part c implies part a. Suppose that p is a point that lies in the interior of some line segment contained in the IPS. Let $P = \langle P_1, P_2 \rangle$ and $Q = \langle Q_1, Q_2 \rangle$ be two partitions such that p is the midpoint of the line segment connecting $m(P)$ and $m(Q)$. We can imagine Q as being obtained from P by two transfers of cake, one from Player 1 to Player 2, and one from Player 2 to Player 1.

Since p is the midpoint of the line segment connecting $m(P)$ and $m(Q)$, $p = m(R)$ where R is any partition obtained by completing "half" of each of these two transfers. In other words, suppose that in switching from partition P to partition Q, Player 1 transfers piece S_{12} to Player 2, and Player 2 transfers piece S_{21} to Player 1. Let $\Gamma(S_{12})$ and $\Gamma(S_{21})$ be as in Lemma 2.7. If instead of completing the full transfer of S_{12} and S_{21}, Player 1 and Player 2 each transfer any piece chosen from $\Gamma(S_{12})$ and $\Gamma(S_{21})$, respectively, then $p = m(R)$, where R is the resulting partition. It remains for us to show that there are infinitely many mutually non-s-equivalent such partitions R with $m(R) = p$.

Since $m(P) \neq m(Q)$, we know that P and Q are not s-equivalent. This implies that at least one of S_{12} and S_{21} has positive measure to at least one player. Assume, without loss of generality, that for some $i = 1, 2$, $m_i(S_{12}) > 0$. Then, by Lemma 2.7, $\Gamma(S_{12})$ is infinite and, for all distinct $B_1, B_2 \in \Gamma(S_{12})$, $m_i(B_1 \Delta B_2) > 0$. This implies that the infinitely many choices that Player i has in transferring a piece of cake from $\Gamma(S_{12})$ to the other player result in infinitely many mutually non-s-equivalent partitions, each of which is sent by the function m to the point p. This completes the proof of the theorem. □

Theorem 2.6 may be restated in terms of the composition of certain natural equivalence classes.

Definition 2.8 Two partitions $P = \langle P_1, P_2\rangle$ and $Q = \langle Q_1, Q_2\rangle$ are *p-equivalent* ("p" for "partition") if and only if $m_1(P_1) = m_1(Q_1)$ and $m_2(P_2) = m_2(Q_2)$. Or, equivalently, P and Q are *p-equivalent* if and only if $m(P) = m(Q)$.

Clearly, p-equivalence is an equivalence relation. We shall refer to the associated equivalence classes as *p-classes*.

As in the case of s-equivalence, it is obvious that the function m from Part to the IPS respects p-equivalence in the sense that if partitions P and Q are p-equivalent, then $m(P) = m(Q)$. It follows that m induces a bijection from the set of p-classes of Part to the IPS. In addition to its original meaning as a function from Part to the IPS, we shall also use "m" to denote the induced function from the set of s-classes of Part to the IPS, and the induced function from the set of p-classes of Part to the IPS.

If partitions P and Q are s-equivalent, then they are certainly p-equivalent. (The converse is sometimes true and sometimes false. This follows from the equivalence class version of Theorem 2.6, to follow) Hence, every p-class is a union of s-classes.

Let $[P]_s$ and $[P]_p$ denote the s-class and the p-class, respectively, of partition P. Then, Theorem 2.6 may be restated as follows.

Theorem 2.6 – Equivalence Class Version *For any partition P, the following are equivalent:*

a. $[P]_p$ is the union of infinitely many s-classes.
b. $[P]_p$ is the union of at least two s-classes.
c. $m(P)$ lies in the interior of a line segment contained in the IPS.

Theorem 2.6 sheds light on the question of whether m is one-to-one with respect to s-equivalence. Since we know that m is one-to-one on p-classes, it follows from the theorem that a point p in the IPS is the image under m of more than one s-class if and only if p lies in the interior of a line segment contained in the IPS. Hence, m is never one-to-one with respect to s-equivalence, since certainly there are line segments (such as the line segment connecting $(1, 0)$ and $(0, 1)$) in the IPS. On the other hand, depending on the shape of the boundary of the IPS, it may be that m is one-to-one with respect to s-equivalence on some subsets of Part. For example, m is one-to-one with respect to s-equivalence on all partitions corresponding to points on the boundary of the IPS of Figure 2.1c, because the boundary of this IPS contains no line segments.

We shall have more to say about points on the boundary of the IPS in the next chapter. We close this section with a result concerning points not on the boundary of the IPS.

Corollary 2.9 *Any point of the IPS that is not on the boundary of the IPS is the image, under m, of infinitely many mutually non-s-equivalent partitions.*

Proof: The corollary follows immediately from the theorem, since any point not on the boundary of the IPS certainly lies in the interior of a line segment that is contained in the IPS. □

Corollary 2.9 – Equivalence Class Version *For any partition P, if $m(P)$ is not on the boundary of the IPS, then $[P]_p$ is the union of infinitely many s-classes.*

3

What the IPS Tells Us About Fairness and Efficiency in the Two-Player Context

In this chapter, we continue to restrict our attention to the two-player context and we consider how the fairness or efficiency of partitions is reflected in the IPS. In other words, if a partition P has some fairness property or some efficiency property, what can be said about the location of $m(P)$ in the IPS? In Section 3A, we consider fairness; in Section 3B, we consider efficiency; and in Section 3C, we consider fairness and efficiency together. In these sections, we assume that measures m_1 and m_2 on some cake C are absolutely continuous with respect to each other. In Section 3D, we consider the situation when absolute continuity fails.

3A. Fairness

We begin by noting that when there are only two players, proportionality and envy-freeness correspond:

$\langle P_1, P_2 \rangle$ is a proportional partition if and only if
$\langle P_1, P_2 \rangle$ is an envy-free partition if and only if
$m_1(P_1) \geq \frac{1}{2}$ and $m_2(P_2) \geq \frac{1}{2}$

Similarly, strong proportionality, strong envy-freeness, and super envy-freeness correspond:

$\langle P_1, P_2 \rangle$ is a strongly proportional partition if and only if
$\langle P_1, P_2 \rangle$ is a strongly envy-free partition if and only if
$\langle P_1, P_2 \rangle$ is a super envy-free partition if and only if
$m_1(P_1) > \frac{1}{2}$ and $m_2(P_2) > \frac{1}{2}$

In Chapter 4, we shall see that these notions are all distinct if there are more than two players. For the remainder of this chapter, in which we study only the two-player context, we shall only use the terms "proportional" and "strongly

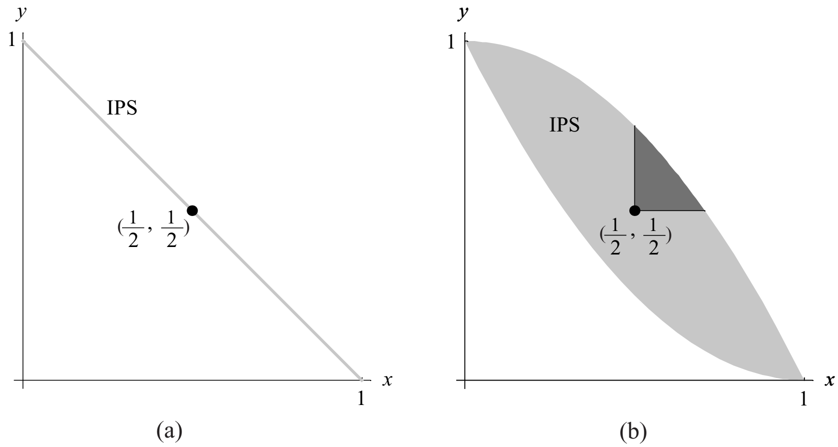


Figure 3.1

proportional." It should be kept in mind that these two notions are equivalent to other fairness notions, as described in the two preceding paragraphs. We begin our study with the notion of proportionality.

Partition $P = \langle P_1, P_2 \rangle$ of C is proportional if and only if $m_1(P_1) \geq \frac{1}{2}$ and $m_2(P_2) \geq \frac{1}{2}$. Thus, P is proportional if and only if the corresponding point in the IPS (i.e., $m(P)$) is to the right of and above $(\frac{1}{2}, \frac{1}{2})$, where neither, one, or both of these relationships may be strict. In the IPS of Figure 3.1a, the only such point is $(\frac{1}{2}, \frac{1}{2})$, whereas in the IPS of Figure 3.1b, there are infinitely many such points. In this figure, the proportional points are the darker points, including the points on the two line segments that bound the darker region.

Partition $P = \langle P_1, P_2 \rangle$ is strongly proportional if and only if $m_1(P_1) > \frac{1}{2}$ and $m_2(P_2) > \frac{1}{2}$. Thus, $\langle P_1, P_2 \rangle$ is strongly proportional if and only if the corresponding point in the IPS is to the right of and above $(\frac{1}{2}, \frac{1}{2})$, where both of these relationships are strict. In the IPS of Figure 3.1a, there are no such points, whereas in the IPS of Figure 3.1b, there are infinitely many such points. In this figure, the strongly proportional points are the darker points, not including the points on the two line segments that bound the darker region.

Because a point's location in the IPS determines whether partitions associated with that point are proportional or strongly proportional, it makes sense to refer to "proportional points" or "strongly proportional points" in the IPS.

Definition 3.1 Suppose $p = (p_1, p_2) \in$ IPS.

a. p is a *proportional point* if and only if $p_1 \geq \frac{1}{2}$ and $p_2 \geq \frac{1}{2}$.
b. p is a *strongly proportional point* if and only if $p_1 > \frac{1}{2}$ and $p_2 > \frac{1}{2}$.

Thus, a point in the IPS is a proportional point if and only if any and all corresponding partitions are proportional, and a point in the IPS is a strongly proportional point if and only if any and all corresponding partitions are strongly proportional. Next we consider the possible numbers of proportional points and the possible numbers of strongly proportional points.

Theorem 3.2

a. *If $m_1 = m_2$, then*
   i. *the IPS has exactly one proportional point and that point is $(\frac{1}{2}, \frac{1}{2})$.*
   ii. *the IPS has no strongly proportional points.*

b. *If $m_1 \neq m_2$, then*
   i. *the IPS has infinitely many proportional points. In particular, for any κ with $0 \leq \kappa \leq \infty$, there are infinitely many points $(p_1, p_2) \in IPS$ that are proportional and are such that $\frac{p_2-\frac{1}{2}}{p_1-\frac{1}{2}} = \kappa$ (where we set $\frac{\lambda}{0} = \infty$ for any real number $\lambda \neq 0$).*
   ii. *the IPS has infinitely many strongly proportional points. In particular, for any κ with $0 < \kappa < \infty$, there are infinitely many points $(p_1, p_2) \in IPS$ that are strongly proportional and are such that $\frac{p_2-\frac{1}{2}}{p_1-\frac{1}{2}} = \kappa$.*

Proof: For part a, we assume that $m_1 = m_2$. By Theorem 2.2, the IPS consists of the closed line segment between $(1, 0)$ and $(0, 1)$. Hence $(\frac{1}{2}, \frac{1}{2}) \in$ IPS and any point of the IPS other than $(\frac{1}{2}, \frac{1}{2})$ has one coordinate less than $\frac{1}{2}$. This implies that $(\frac{1}{2}, \frac{1}{2})$ is the only proportional point of the IPS. Also, there are no points in the IPS with both coordinates greater than $\frac{1}{2}$. Thus the IPS has no strongly proportional points. This establishes parts ai and aii.

For part b, we assume that $m_1 \neq m_2$. Theorem 2.2 tells us that the closed line segment between $(1, 0)$ and $(0, 1)$ is a proper subset of the IPS. By the symmetry and convexity of the IPS (see Theorem 2.4), it follows that $(\frac{1}{2}, \frac{1}{2})$ is an interior point of the IPS. This implies that any line that contains the point $(\frac{1}{2}, \frac{1}{2})$ and has non-negative slope κ (where we count "∞" as a non-negative number) will pass through infinitely many proportional points. Hence, for any κ with $0 \leq \kappa \leq \infty$, there are infinitely many proportional points $(p_1, p_2) \in$ IPS with $\frac{p_2-\frac{1}{2}}{p_1-\frac{1}{2}} = \kappa$. This establishes part bi.

The proof for bii is the same except that here we must not allow $\kappa = 0$ or $\kappa = \infty$, since this would correspond to a point (p_1, p_2) with either $p_1 = \frac{1}{2}$ or $p_2 = \frac{1}{2}$, and such a point is not strongly proportional. □

The theorem is illustrated in Figure 3.2. The line segment in the figure contains the point $(\frac{1}{2}, \frac{1}{2})$ and has slope κ. Every point along the solid part of this line segment, including $(\frac{1}{2}, \frac{1}{2})$, is proportional, and every point along this part of the line segment, not including $(\frac{1}{2}, \frac{1}{2})$, is strongly proportional.

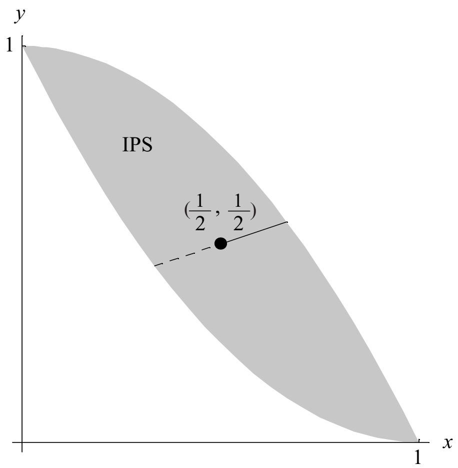


Figure 3.2

Corollary 3.3

a. If $m_1 = m_2$*, then*

- *i. there are infinitely many mutually non-*s*-equivalent proportional partitions.*
- *ii. all proportional partitions are* p*-equivalent.*
- *iii. there are no strongly proportional partitions.*

b. If $m_1 \neq m_2$*, then*

- *i. there are infinitely many mutually non-*s*-equivalent proportional partitions.*
- *ii. there are infinitely many mutually non-*p*-equivalent proportional partitions.*
- *iii. there are infinitely many mutually non-*s*-equivalent strongly proportional partitions.*
- *iv. there are infinitely many mutually non-*p*-equivalent strongly proportional partitions.*

Proof: For part ai, note that $(\frac{1}{2}, \frac{1}{2})$ is a point corresponding to at least one proportional partition P. Then, since $m(P) = (\frac{1}{2}, \frac{1}{2})$ is an interior point of the line segment connecting (1, 0) and (0, 1), and this line segment lies in the IPS, Theorem 2.6 tells us that $(\frac{1}{2}, \frac{1}{2})$ is the image, under m, of infinitely many mutually non-s-equivalent partitions. Thus there are infinitely many mutually non-s-equivalent partitions that are p-equivalent to P. Clearly any partition that is p-equivalent to P is a proportional partition. Hence, there are infinitely many mutually non-s-equivalent proportional partitions.

Part aii follows easily from part ai of the theorem.

Part aiii is just a rewording of part aii of the theorem.

Next, we observe that parts bi, bii, and biii all follow trivially from part biv. For part biv, we recall that distinct points of the IPS are the image, under m, of non-p-equivalent partitions. Hence, the infinitely many strongly proportional points of the IPS given to us by part bii of the theorem correspond to infinitely many mutually non-p-equivalent strongly proportional partitions. □

It is easy to see that proportionality and strong proportionality respect s-equivalence and p-equivalence. In other words, if partitions P and Q are either s-equivalent or p-equivalent, then

- P is proportional if and only if Q is proportional and
- P is strongly proportional if and only if Q is strongly proportional.

(We used this fact for p-equivalence and proportionality in the proof of part ai of the preceding corollary.) Then, it makes sense to refer to "proportional s-classes," "strongly proportional p-classes," etc. We can now restate the previous corollary as follows.

Corollary 3.3 – Equivalence Class Version

a. If $m_1 = m_2$, then

i. there are infinitely many proportional s-classes.

ii. there is exactly one proportional p-class.

iii. there are no strongly proportional s-classes or p-classes.

b. If $m_1 \neq m_2$, then

i. there are infinitely many proportional s-classes.

ii. there are infinitely many proportional p-classes.

iii. there are infinitely many strongly proportional s-classes.

iv. there are infinitely many strongly proportional p-classes.

The definitions and results of this section have analogous chores versions. If $p = (p_1, p_2) \in$ IPS, then p is a *chores proportional point* if and only if $p_1 \leq \frac{1}{2}$ and $p_2 \leq \frac{1}{2}$ and is a *strongly chores proportional point* if and only if $p_1 < \frac{1}{2}$ and $p_2 < \frac{1}{2}$. *Chores proportional s-classes*, *strongly chores proportional p-classes*, etc. are defined in the obvious way. We shall abbreviate this terminology by writing "c-proportional point," "strongly c-proportional point," "c-proportional s-class," "strongly c-proportional p-class," etc. The following are the chores versions of Theorem 3.2 and Corollary 3.3. The proofs are entirely analogous, and we omit them.

Theorem 3.4

a. If $m_1 = m_2$, then

i. the IPS has exactly one c-proportional point and that point is $(\frac{1}{2}, \frac{1}{2})$.

ii. the IPS has no strongly c-proportional points.

b. If $m_1 \neq m_2$, then

i. the IPS has infinitely many c-proportional points. In particular, for any κ with $0 \leq \kappa \leq \infty$, there are infinitely many points $(p_1, p_2) \in$ IPS that are c-proportional and are such that $\frac{p_2 - \frac{1}{2}}{p_1 - \frac{1}{2}} = \kappa$ (where we set $\frac{\lambda}{0} = \infty$ for any real number $\lambda \neq 0$).

ii. the IPS has infinitely many strongly c-proportional points. In particular, for any κ with $0 < \kappa < \infty$, there are infinitely many points $(p_1, p_2) \in$ IPS that are strongly c-proportional and are such that $\frac{p_2 - \frac{1}{2}}{p_1 - \frac{1}{2}} = \kappa$.

In Figure 3.2, we assumed that the line segment has slope κ. Every point along the dashed part of this line segment, including $(\frac{1}{2}, \frac{1}{2})$, is c-proportional, and every point along this part of the line segment, not including $(\frac{1}{2}, \frac{1}{2})$, is strongly c-proportional.

Corollary 3.5

a. If $m_1 = m_2$, then

i. there are infinitely many mutually non-s-equivalent c-proportional partitions.

ii. all c-proportional partitions are p-equivalent.

iii. there are no strongly c-proportional partitions.

b. If $m_1 \neq m_2$, then

i. there are infinitely many mutually non-s-equivalent c-proportional partitions.

ii. there are infinitely many mutually non-p-equivalent c-proportional partitions.

iii. there are infinitely many mutually non-s-equivalent strongly c-proportional partitions.

iv. there are infinitely many mutually non-p-equivalent strongly c-proportional partitions.

Corollary 3.5 – Equivalence Class Version

a. If $m_1 = m_2$, then

i. there are infinitely many c-proportional s-classes.

ii. there is exactly one c-proportional p-class.

iii. there are no strongly c-proportional s-classes or p-classes.

b. If $m_1 \neq m_2$, then

i. there are infinitely many c-proportional s-classes.

ii. there are infinitely many c-proportional p-classes.

iii. *there are infinitely many strongly c-proportional s-classes.*
iv. *there are infinitely many strongly c-proportional p-classes.*

3B. Efficiency

We turn now from fairness to efficiency. Suppose that $\langle P_1, P_2\rangle$ is a partition of C. We recall that $\langle P_1, P_2\rangle$ is Pareto maximal if and only if for no partition $\langle Q_1, Q_2\rangle$ of C do we have $m_1(Q_1) \geq m_1(P_1)$ and $m_2(Q_2) \geq m_2(P_2)$, with at least one of these inequalities being strict, and $\langle P_1, P_2\rangle$ is Pareto minimal if and only if for no partition $\langle Q_1, Q_2\rangle$ of C do we have $m_1(Q_1) \leq m_1(P_1)$ and $m_2(Q_2) \leq m_2(P_2)$, with at least one of these inequalities being strict. Of course, Pareto minimality is the chores version of Pareto maximality.

Let $P = \langle P_1, P_2\rangle$ be a partition of C. As we did for fairness properties in the previous section, we consider what the Pareto maximality or the Pareto minimality of P says about the location of $m(P)$ in the IPS. It is clear that P is Pareto maximal if and only if there is no point in the IPS that is to the right of and above $m(P)$, with at least one of these relationships being strict. Similarly, P is Pareto minimal if and only if there is no point in the IPS that is to the left of and below $m(P)$, with at least one of these relationships being strict.

Just as we did for proportionality and strong proportionality, we note that because a point's location in the IPS determines whether partitions associated with that point are Pareto maximal, Pareto minimal, or neither, it makes sense to refer to "Pareto maximal points" or "Pareto minimal points" in the IPS.

Definition 3.6 Suppose $p = (p_1, p_2) \in$ IPS.

a. p is a *Pareto maximal point* if and only if there is no $q = (q_1, q_2) \in$ IPS such that $q_1 \geq p_1$ and $q_2 \geq p_2$, with at least one of these inequalities being strict.
b. p is a *Pareto minimal point* if and only if there is no $q = (q_1, q_2) \in$ IPS such that $q_1 \leq p_1$ and $q_2 \leq p_2$, with at least one of these inequalities being strict.

Thus, a point in the IPS is a Pareto maximal point if and only if any and all corresponding partitions are Pareto maximal, and a point in the IPS is a Pareto minimal point if and only if any and all corresponding partitions are Pareto minimal.

We wish to name, and geometrically describe, the region of the IPS that consists of Pareto maximal points and the region of the IPS that consists of Pareto minimal points. We shall call these regions the outer Pareto boundary and the inner Pareto boundary, respectively. It will be convenient for us to also define the outer boundary and the inner boundary of the IPS.

First, for any $(x_0, y_0) \in \mathbf{R}^2$, we define $B^+(x_0, y_0) = \{(x, y) \in \mathbf{R}^2 : x \geq x_0$ and $y \geq y_0\}$. Similarly, we define $B^-(x_0, y_0) = \{(x, y) \in \mathbf{R}^2 : x \leq x_0$ and

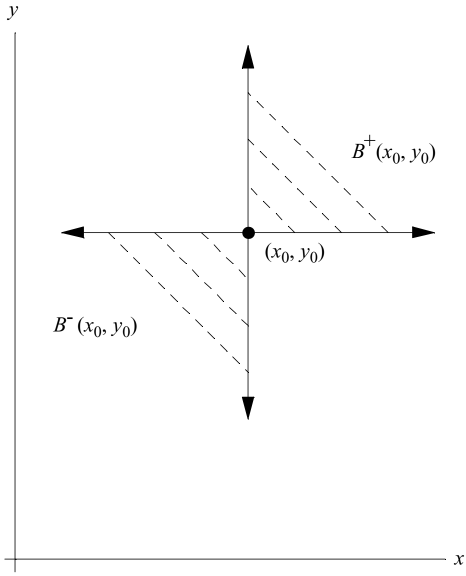


Figure 3.3

$y \leq y_0\}$. Hence, $B^+(x_0, y_0)$ is the set of all points that are to the right of and above (x_0, y_0), where neither, one, or both of these relationships may be strict, and $B^-(x_0, y_0)$ is the set of all points that are to the left of and below (x_0, y_0), where neither, one, or both of these relationships may be strict. These notions are illustrated in Figure 3.3. In the figure, $B^+(x_0, y_0)$ and $B^-(x_0, y_0)$ are the regions with the dashed lines, as indicated. Each of these sets includes the region's boundary, which consists of the point (x_0, y_0) and a horizontal and a vertical half-line.

Definition 3.7

a. The *outer boundary* of the IPS consists of all points (p_1, p_2) on the boundary of the IPS for which $p_1 + p_2 \geq 1$, and the *inner boundary* of the IPS consists of all points (p_1, p_2) on the boundary of the IPS for which $p_1 + p_2 \leq 1$.
b. The *outer Pareto boundary* of the IPS consists of all points $(p_1, p_2) \in$ IPS for which $B^+(p_1, p_2) \cap \text{IPS} = \{(p_1, p_2)\}$, and the *inner Pareto boundary* of the IPS consists of all points $(p_1, p_2) \in$ IPS for which $B^-(p_1, p_2) \cap \text{IPS} = \{(p_1, p_2)\}$. The *Pareto boundary* is the union of the outer Pareto boundary and the inner Pareto boundary.

Obviously, the union of the outer boundary and the inner boundary of the IPS is equal to the boundary of the IPS. If $m_1 = m_2$, then the outer boundary and the inner boundary are each equal to the simplex and, if the two measures are not equal, then the intersection of the outer boundary and the inner boundary is the set consisting of the two points $(1, 0)$ and $(0, 1)$.

Definition 3.7 makes the following theorem trivial.

Theorem 3.8

a. *The outer Pareto boundary of the IPS consists precisely of the set of all Pareto maximal points of the IPS.*
b. *The inner Pareto boundary of the IPS consists precisely of the set of all Pareto minimal points of the IPS.*

It is easy to see that every point on the Pareto boundary of the IPS is on the boundary of the IPS. Are there points on the boundary of the IPS that are not on the Pareto boundary? The answer is "no" in our present context, in which we are assuming that m_1 and m_2 are absolutely continuous with respect to each other.

Theorem 3.9

a. *The outer Pareto boundary of the IPS is equal to the outer boundary of the IPS.*
b. *The inner Pareto boundary of the IPS is equal to the inner boundary of the IPS.*

In order to establish Theorem 3.9, we first prove a lemma.

Lemma 3.10 *The only points of the IPS that lie on the unit square are* $(1, 0)$ *and* $(0, 1)$.

Proof: Suppose, by way of contradiction, that $p = (p_1, p_2) \in \text{IPS}$, $p \neq (1, 0)$, $p \neq (0, 1)$, and p is on the unit square. Since $p \in \text{IPS}$, there is a partition $P = \langle P_1, P_2 \rangle$ of C such that $m(P) = p$.

First, let us assume that $0 \leq p_1 < 1$ and $p_2 = 0$. Then, $m_1(P_1) < 1$ and $m_2(P_2) = 0$. But then $m_1(P_2) = 1 - m_1(P_1) > 0$. This violates absolute continuity.

The argument for $p_1 = 0$ and $0 \leq p_2 < 1$ is similar. The other two cases ($p_1 = 1$ with $0 < p_2 \leq 1$, and $0 < p_1 \leq 1$ with $p_2 = 1$) follow by symmetry (i.e., by Theorem 2.4, part e). □

Proof of Theorem 3.9: We prove part a. The proof for part b is similar.

Let $p = (p_1, p_2)$ be a point on the outer Pareto boundary of the IPS. Then $B^+(p) \cap \text{IPS} = \{p\}$. We have already observed that such a point is on the boundary of the IPS. Suppose, by way of contradiction, that p is not on the outer boundary. It follows that p is on the inner boundary, and thus $p_1 + p_2 < 1$. This implies that $B^+(p)$ contains points of the simplex. (In particular, if we set

$\varepsilon = \frac{1-p_1-p_2}{2}$ then, since $p_1 + p_2 < 1$, it follows that $\varepsilon > 0$. This implies that $(p_1 + \varepsilon, p_2 + \varepsilon) \in B^+(p)$. It is easy to check that $p_1 + \varepsilon + p_2 + \varepsilon = 1$; hence, $(p_1 + \varepsilon, p_2 + \varepsilon)$ is a point in the simplex.) Since any point of the simplex is in the IPS, this contradicts the fact that $B^+(p) \cap \text{IPS} = \{p\}$.

Next, we assume that $p = (p_1, p_2)$ is a point on the outer boundary of the IPS. We consider three cases.

Case 1: The measures are equal. In this case, the outer boundary and the outer Pareto boundary are equal (and are both equal to the simplex). Hence p is on the outer Pareto boundary.

Case 2: The measures are not equal and either $p = (1, 0)$ or $p = (0, 1)$. Since the IPS is a subset of $[0, 1]^2$, Lemma 3.10 implies that $B^+(1, 0) \cap \text{IPS} = \{(1, 0)\}$ and $B^+(0, 1) \cap \text{IPS} = \{(0, 1)\}$. Hence, p is on the outer Pareto boundary.

Case 3: The measures are not equal, $p \neq (1, 0)$, and $p \neq (0, 1)$. Since the measures are not equal, the outer boundary intersects the simplex only at the points $(1, 0)$ and $(0, 1)$. Since p is equal to neither of these points, p is not on the simplex. Then, since p is on the outer boundary, we must have $p_1 + p_2 > 1$. Assume, by way of contradiction, that p is not on the outer Pareto boundary. Then there is a point $q \in B^+(p) \cap \text{IPS}$ with $q \neq p$. Consider the set $G = \{q, (1, 0), (0, 1)\}$ and recall that $\text{CH}(G)$ denotes the convex hull of G. Since each point in G is in the IPS and the IPS is convex, we know that $\text{CH}(G) \subseteq \text{IPS}$. But it is easy to see that p is an interior point of $\text{CH}(G)$ and thus p is an interior point of the IPS. This contradicts the fact the p is on the outer boundary of the IPS. □

The theorem tells us that the Pareto boundary of the IPS is the same as the boundary of the IPS. As we shall see in Section 3D (see Theorem 3.22), this correspondence does not hold if absolute continuity fails and, as we shall see in Chapter 5 (see Theorem 5.13), this correspondence also does not hold whenever there are more than two players and the measures are not all equal (regardless of whether absolute continuity holds).

Our geometric perspective makes it easy to see that there are many Pareto maximal partitions and many Pareto minimal partitions. This fact is a corollary to the following theorem.

Theorem 3.11 *The IPS has infinitely many Pareto maximal points and infinitely many Pareto minimal points. In particular, for any κ with $0 \leq \kappa \leq \infty$, there is a point $(p_1, p_2) \in$ IPS and a point $(q_1, q_2) \in$ IPS that are Pareto maximal and Pareto minimal, respectively, and are such that $\frac{p_2}{p_1} = \frac{q_2}{q_1} = \kappa$ (where we set $\frac{\lambda}{0} = \infty$ for any real number $\lambda \neq 0$).*

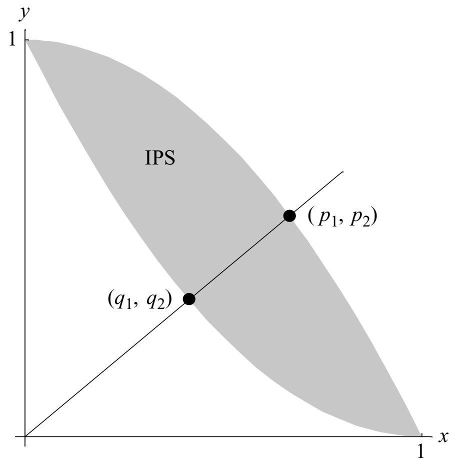


Figure 3.4

Proof: The first statement clearly follows from the second. For the second statement, we first note that the points (1, 0) and (0, 1) are on the outer and inner boundaries of the IPS and the outer and inner boundaries are curves that connect these two points. Every line that goes through the origin and has non-negative slope (where we count "∞" as a non-negative number) intersects each of these curves exactly once. Letting κ denote the slope of any such line, the theorem follows easily from Theorems 3.8 and 3.9. □

The theorem is illustrated in Figure 3.4. The line segment in the figure passes through the origin and has slope κ. The point (p_1, p_2) is the point of intersection of this line segment with the outer boundary of the IPS and is therefore a Pareto maximal point. Similarly, the point (q_1, q_2) is the point of intersection of this line segment with the inner boundary of the IPS and is therefore a Pareto minimal point.

We note that the line through the origin having slope $\kappa = 0$ (i.e., the horizontal line through the origin) intersects the outer and inner boundaries at the same point, namely (1, 0). Similarly, the line through the origin having slope $\kappa = \infty$ (i.e., the vertical line through the origin) intersects the outer and inner boundaries at the same point, namely (0, 1). The points (1, 0) and (0, 1) are each Pareto maximal and Pareto minimal.

The reader may wonder why we stated Theorem 3.11 in terms of the slope κ of a line containing the origin rather than a line containing the point $(\frac{1}{2}, \frac{1}{2})$, as in Theorems 3.2 and 3.4. It is clear that these theorems, which involved proportionality, strong proportionality, and the corresponding chores notions, need to use the point $(\frac{1}{2}, \frac{1}{2})$ rather than the origin. On the other hand, we could have presented Theorem 3.11 using either point (with appropriate differences in the allowed values of the slope κ). We chose to use the origin because this version will generalize in a natural way to the context of more than two players, whereas the version involving the point $(\frac{1}{2}, \frac{1}{2})$ will not. (See Theorem 5.18 and the discussion following the statement of the theorem.)

Corollary 3.12

a. *There are infinitely many mutually non-s-equivalent Pareto maximal partitions and infinitely many mutually non-s-equivalent Pareto minimal partitions.*
b. *There are infinitely many mutually non-p-equivalent Pareto maximal partitions and infinitely many mutually non-p-equivalent Pareto minimal partitions.*

Proof: Part a follows immediately from part b. For part b, we recall that distinct points of the IPS are the images, under m, of non-p-equivalent partitions. Hence, each of the two infinite collections of points given to us by the theorem yields an infinite collection of mutually non-p-equivalent partitions. □

Corollary 3.12 – Equivalence Class Version

a. *There are infinitely many Pareto maximal s-classes and infinitely many Pareto minimal s-classes.*
b. *There are infinitely many Pareto maximal p-classes and infinitely many Pareto minimal p-classes.*

We close this section with an easy result on partitions that are neither Pareto maximal nor Pareto minimal.

Theorem 3.13 *If partition P is neither Pareto maximal nor Pareto minimal, then P is p-equivalent to infinitely many mutually non-s-equivalent partitions.*

Proof: Suppose that P is neither Pareto maximal nor Pareto minimal. Then $m(P)$ is on neither the outer nor the inner Pareto boundary of the IPS. It follows from Theorem 3.9 that $m(P)$ is on neither the outer nor the inner boundary of the IPS and, hence, is not on the boundary of the IPS. Corollary 2.9 tells us that $m(P)$ is the image, under m, of infinitely many mutually non-s-equivalent partitions. Each of these partitions is p-equivalent to P. □

Theorem 3.13 – Equivalence Class Version *If partition P is neither Pareto maximal nor Pareto minimal, then $[P]_p$ is the union of infinitely many s-classes.*

3C. Fairness and Efficiency Together: Part 1a

In this section, we first consider the existence of partitions that are both Pareto maximal and either proportional or strongly proportional, and then we consider the corresponding chores properties.

We have learned that a point in the IPS is

- proportional if and only if it is to the right of and above $(\frac{1}{2}, \frac{1}{2})$, where neither, one, or both of these relationships may be strict;
- strongly proportional if and only if it is to the right of and above $(\frac{1}{2}, \frac{1}{2})$, where both of these relationships are strict; and
- Pareto maximal if and only if it is on the outer Pareto boundary of the IPS.

The following theorem and its corollary generalize Theorems 3.2 and 3.11 and their corollaries.

Theorem 3.14

a. If $m_1 = m_2$, then

i. the IPS has exactly one point that is both proportional and Pareto maximal, and that point is $(\frac{1}{2}, \frac{1}{2})$.

ii. the IPS has no points that are both strongly proportional and Pareto maximal.

b. If $m_1 \neq m_2$, then

i. the IPS has infinitely many points that are both proportional and Pareto maximal. In particular, for any κ with $0 \leq \kappa \leq \infty$, there is a point $(p_1, p_2) \in$ IPS that is both proportional and Pareto maximal and is such that $\frac{p_2 - \frac{1}{2}}{p_1 - \frac{1}{2}} = \kappa$ (where we set $\frac{\lambda}{0} = \infty$ for any real number $\lambda \neq 0$).

ii. the IPS has infinitely many points that are both strongly proportional and Pareto maximal. In particular, for any κ with $0 < \kappa < \infty$, there is a point $(p_1, p_2) \in$ IPS that is both strongly proportional and Pareto maximal and is such that $\frac{p_2 - \frac{1}{2}}{p_1 - \frac{1}{2}} = \kappa$.

Proof: For part a, we assume that $m_1 = m_2$. Then, by Theorem 3.2, the IPS has exactly one point that is proportional and that point is $(\frac{1}{2}, \frac{1}{2})$. Theorem 2.2 implies that $(\frac{1}{2}, \frac{1}{2})$ is on the outer boundary of the IPS and therefore, by Theorem 3.9, we know that $(\frac{1}{2}, \frac{1}{2})$ is on the outer Pareto boundary and thus is Pareto maximal. This establishes part ai.

Part aii follows trivially from part aii of Theorem 3.2.

For part b, we assume that $m_1 \neq m_2$. For part bi, we first note that the first statement clearly follows from the second. For the second statement, we simply note that for each κ with $0 \leq \kappa \leq \infty$, the infinite collection of proportional points given to us by Theorem 3.2 contains one point that is also Pareto maximal. This is the point obtained by moving in the direction given by κ until we reach the outer boundary of the IPS (which, by Theorem 3.9, is also the outer Pareto boundary of the IPS).

The proof for part bii is similar. □

Part b of the theorem is illustrated in Figure 3.5. In the figure, the proportional and strongly proportional points are the points in the darker region of the IPS. (The set of proportional points includes the points on the horizontal and the vertical line segments that bound this region; the set of strongly proportional points includes none of these points.) The outer Pareto boundary is darkened. The set of points that are proportional or strongly proportional, and also Pareto maximal, is the intersection of these two sets and is the thicker curve in the figure. (Each of the two endpoints of this thicker curve is proportional, but neither is strongly proportional.)

Corollary 3.15

a. If $m_1 = m_2$, then

i. there are infinitely many mutually non-s-equivalent partitions that are both proportional and Pareto maximal.

ii. all partitions that are both proportional and Pareto maximal are p-equivalent.

iii. there are no partitions that are both strongly proportional and Pareto maximal.

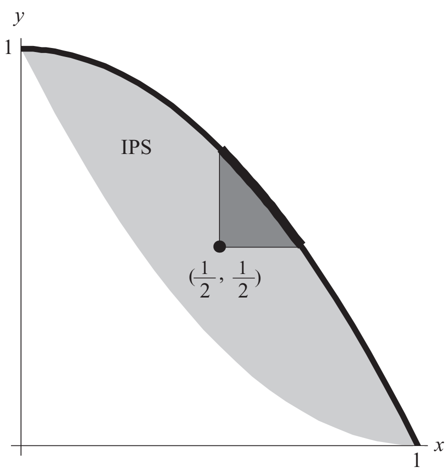


Figure 3.5

b. If $m_1 \neq m_2$, then

i. there are infinitely many mutually non-s-equivalent partitions that are both proportional and Pareto maximal.

ii. there are infinitely many mutually non-p-equivalent partitions that are both proportional and Pareto maximal.

iii. there are infinitely many mutually non-s-equivalent partitions that are both strongly proportional and Pareto maximal.

iv. there are infinitely many mutually non-p-equivalent partitions that are both strongly proportional and Pareto maximal.

Proof: For part a, we assume that $m_1 = m_2$. For part ai, part ai of the theorem implies that $(\frac{1}{2}, \frac{1}{2})$ is a point of the IPS that is both proportional and Pareto maximal. Since $(\frac{1}{2}, \frac{1}{2})$ is an interior point of the line segment connecting $(1, 0)$ and $(0, 1)$, it follows from Theorem 2.6 that there are infinitely many mutually non-s-equivalent partitions that are mapped by m to $(\frac{1}{2}, \frac{1}{2})$. Each of these partitions is proportional and Pareto maximal.

For part aii, part ai of the theorem tells us that $(\frac{1}{2}, \frac{1}{2})$ is the only point in the IPS that is both proportional and Pareto maximal. The result then follows from the fact that m maps non-p-equivalent partitions to different points in the IPS.

Part aiii is just a rewording of part aii of the theorem.

For part b, we assume that $\mathrm{m}_1 \neq m_2$. Parts bi, bii and biii are implied by part biv. For part biv, we note that since distinct points of the IPS are the image, under m, of non-p-equivalent partitions, it follows from part bii of the theorem that there are infinitely many mutually non-p-equivalent partitions that are both strongly proportional and Pareto maximal. □

Corollary 3.15 – Equivalence Class Version

a. If $m_1 = m_2$, then

i. there are infinitely many s-classes that are both proportional and Pareto maximal.

ii. there is exactly one p-class that is both proportional and Pareto maximal.

iii. there are no s-classes or p-classes that are both strongly proportional and Pareto maximal.

b. If $m_1 \neq m_2$, then

i. there are infinitely many s-classes that are both proportional and Pareto maximal.

ii. there are infinitely many p-classes that are both proportional and Pareto maximal.

iii. there are infinitely many s-classes that are both strongly proportional and Pareto maximal.

iv. there are infinitely many p-classes that are both strongly proportional and Pareto maximal.

The chores versions of Theorem 3.14 and Corollary 3.15 are as follows. The proofs are similar and we omit them.

Theorem 3.16

a. If $m_1 = m_2$, then

i. the IPS has exactly one point that is both c-proportional and Pareto minimal, and that point is $(\frac{1}{2}, \frac{1}{2})$.

ii. the IPS has no points that are both strongly c-proportional and Pareto minimal.

b. If $m_1 \neq m_2$, then

i. the IPS has infinitely many points that are both c-proportional and Pareto minimal. In particular, for any κ with $0 \leq \kappa \leq \infty$, there is a point $(p_1, p_2) \in IPS$ that is both c-proportional and Pareto minimal and is such that $\frac{p_2 - \frac{1}{2}}{p_1 - \frac{1}{2}} = \kappa$ (where we set $\frac{\lambda}{0} = \infty$ for any real number $\lambda \neq 0$).

ii. the IPS has infinitely many points that are both strongly c-proportional and Pareto minimal. In particular, for any κ with $0 < \kappa < \infty$, there is a point $(p_1, p_2) \in IPS$ that is both strongly c-proportional and Pareto minimal and is such that $\frac{p_2 - \frac{1}{2}}{p_1 - \frac{1}{2}} = \kappa$.

Corollary 3.17

a. If $m_1 = m_2$, then

i. there are infinitely many mutually non-s-equivalent partitions that are both c-proportional and Pareto minimal.

ii. all partitions that are both c-proportional and Pareto minimal are p-equivalent.

iii. there are no partitions that are both strongly c-proportional and Pareto minimal.

b. If $m_1 \neq m_2$, then

i. there are infinitely many mutually non-s-equivalent partitions that are both c-proportional and Pareto minimal.

ii. there are infinitely many mutually non-p-equivalent partitions that are both c-proportional and Pareto minimal.

iii. there are infinitely many mutually non-s-equivalent partitions that are both strongly c-proportional and Pareto minimal.

iv. there are infinitely many mutually non-p-equivalent partitions that are both strongly c-proportional and Pareto minimal.

Corollary 3.17 – Equivalence Class Version

a. If $m_1 = m_2$, then

i. there are infinitely many s-classes that are both c-proportional and Pareto minimal.

ii. there is exactly one p-class that is both c-proportional and Pareto minimal.

iii. there are no s-classes or p-classes that are both strongly c-proportional and Pareto minimal.

b. If $m_1 \neq m_2$, then

i. there are infinitely many s-classes that are both c-proportional and Pareto minimal.

ii. there are infinitely many p-classes that are both c-proportional and Pareto minimal.

iii. there are infinitely many s-classes that are both strongly c-proportional and Pareto minimal.

iv. there are infinitely many p-classes that are both strongly c-proportional and Pareto minimal.

3D. The Situation Without Absolute Continuity

In this section, we reconsider the results of previous sections, dropping our assumption of absolute continuity. For this section, we explicitly assume that at least one of the measures is not absolutely continuous with respect to the other. Hence, either there is a piece of cake that has positive value to Player 1 but has no value to Player 2 or there is a piece of cake that has positive value to Player 2 but has no value to Player 1, or both. We recall that in the first case we say that m_1 is not absolutely continuous with respect to m_2, and in the second case we say that m_2 is not absolutely continuous with respect to m_1.

How does the assumption that the measures are not absolutely continuous with respect to each other affect the results of the previous sections? We begin by considering fairness issues. The idea here is quite easy, since none of the results in Section 3A rely on absolute continuity and, hence, all hold in our present context. We observe that the failure of absolute continuity certainly implies that $m_1 \neq m_2$. Thus, the appropriate adjustments of Theorem 3.2, Corollary 3.3, Theorem 3.4, and Corollary 3.5 are the following.

Theorem 3.18

a. The IPS has infinitely many proportional points. In particular, for any κ with $0 \leq \kappa \leq \infty$, there are infinitely many points $(p_1, p_2) \in$ IPS that are *proportional and are such that* $\frac{p_2-\frac{1}{2}}{p_1-\frac{1}{2}} = \kappa$ *(where we set* $\frac{\lambda}{0} = \infty$ *for any real number* $\lambda \neq 0$*).*

b. *The IPS has infinitely many strongly proportional points. In particular, for any* κ *with* $0 < \kappa < \infty$*, there are infinitely many points* $(p_1, p_2) \in IPS$ *that are strongly proportional and are such that* $\frac{p_2-\frac{1}{2}}{p_1-\frac{1}{2}} = \kappa$*.*

Corollary 3.19

a. *There are infinitely many mutually non-s-equivalent proportional partitions.*
b. *There are infinitely many mutually non-p-equivalent proportional partitions.*
c. *There are infinitely many mutually non-s-equivalent strongly proportional partitions.*
d. *There are infinitely many mutually non-p-equivalent strongly proportional partitions.*

Corollary 3.19 – Equivalence Class Version

a. *There are infinitely many proportional s-classes.*
b. *There are infinitely many proportional p-classes.*
c. *There are infinitely many strongly proportional s-classes.*
d. *There are infinitely many strongly proportional p-classes.*

Theorem 3.20

a. *The IPS has infinitely many c-proportional points. In particular, for any* κ *with* $0 \leq \kappa \leq \infty$*, there are infinitely many points* $(p_1, p_2) \in IPS$ *that are c-proportional and are such that* $\frac{p_2-\frac{1}{2}}{p_1-\frac{1}{2}} = \kappa$ *(where we set* $\frac{\lambda}{0} = \infty$ *for any real number* $\lambda \neq 0$*).*
b. *The IPS has infinitely many strongly c-proportional points. In particular, for any* κ *with* $0 < \kappa < \infty$*, there are infinitely many points* $(p_1, p_2) \in IPS$ *that are strongly c-proportional and are such that* $\frac{p_2-\frac{1}{2}}{p_1-\frac{1}{2}} = \kappa$*.*

Corollary 3.21

a. *There are infinitely many mutually non-s-equivalent c-proportional partitions.*
b. *There are infinitely many mutually non-p-equivalent c-proportional partitions.*
c. *There are infinitely many mutually non-s-equivalent strongly c-proportional partitions.*
d. *There are infinitely many mutually non-p-equivalent strongly c-proportional partitions.*

Corollary 3.21 – Equivalence Class Version

a. *There are infinitely many c-proportional s-classes.*
b. *There are infinitely many c-proportional p-classes.*

c. *There are infinitely many strongly c-proportional s-classes.*
d. *There are infinitely many strongly c-proportional p-classes.*

Next, we reconsider our results on efficiency from Section 3B. It follows trivially from the corresponding definitions that the union of the outer and inner boundaries of the IPS is the boundary of the IPS and that Theorem 3.8 holds regardless of any assumptions about absolute continuity. However, this is not the case for Theorem 3.9, which told us that the outer Pareto boundary is equal to the outer boundary and the inner Pareto boundary is equal to the inner boundary. In particular, the first use of absolute continuity occurs in the proof of Lemma 3.10, and this lemma is used in the proof of Theorem 3.9.

In contrast with Lemma 3.10, our present assumption, together with the fact that the IPS is symmetric about $(\frac{1}{2}, \frac{1}{2})$, implies that the IPS includes points of the form $(1, a)$ and $(0, 1-a)$, or else points of the form $(a, 1)$ and $(1-a, 0)$, for some a with $0 < a < 1$. Examples of possible IPSs are shown in Figure 3.6. Figure 3.6a shows an IPS corresponding to m_1 not absolutely continuous with respect to m_2, but m_2 absolutely continuous with respect to m_1. Figure 3.6b shows an IPS corresponding to the reverse situation. Figure 3.6c shows an IPS corresponding to neither m_1 nor m_2 being absolutely continuous with respect to the other. It shall follow from our work in Chapter 11 that these are correct examples; i.e., for each of these figures, there exists a cake C and corresponding measures m_1 and m_2 such that the given figure is the IPS corresponding to C, m_1, and m_2. Notice that in Figure 3.6a the point $(1, 0)$ is Pareto maximal but not Pareto minimal, and the point $(0, 1)$ is Pareto minimal but not Pareto maximal. In Figure 3.6b, the situation is exactly the opposite. In Figure 3.6c, neither the point $(1, 0)$ nor the point $(0, 1)$ is Pareto maximal or Pareto minimal.

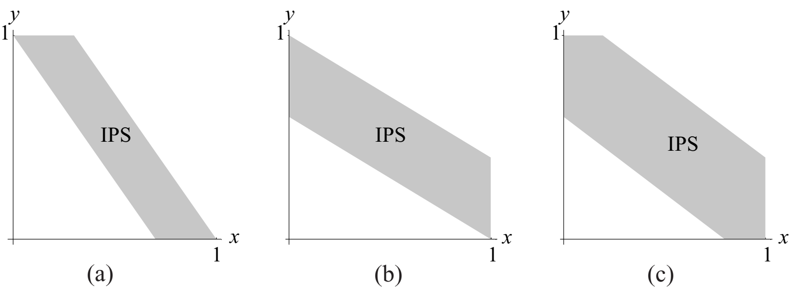


Figure 3.6

Theorem 3.22

a. *The outer Pareto boundary of the IPS is a proper subset of the outer boundary of the IPS.*

b. *The inner Pareto boundary of the IPS is a proper subset of the inner boundary of the IPS.*

Proof: For part a, the proof that the outer Pareto boundary is a subset of the outer boundary is precisely as in Theorem 3.9, because that part of the proof did not use absolute continuity. We will find a point that is on the outer boundary but is not on the outer Pareto boundary.

As noted previously, the failure of absolute continuity implies that the IPS either contains points of the form $(1, a)$ and $(0, 1 - a)$ or else points of the form $(a, 1)$ and $(1 - a, 0)$ for some a with $0 < a < 1$. Let us first assume that it contains points of the form $(1, a)$ and $(0, 1 - a)$ for some a with $0 < a < 1$. In particular, we shall focus on the point $(1, a)$.

Pick any b with $0 < b < a$. Since $(1, 0) \in$ IPS and the IPS is convex, it follows that $(1, b) \in$ IPS and, since this point is on the unit square, it is clearly on the boundary of the IPS. Also, since $1 + b > 1$, it is on the outer boundary.

We claim that $(1, b)$ is not on the outer Pareto boundary. Note that $(1, b)$ is an interior point of the line segment between $(1, 0)$ and $(1, a)$. Then (using the notation developed just before Definition 3.7) it is clear that $B^+(1, b)$ contains many points from this line segment. Finally, since this line segment lies completely in the IPS, it follows that $B^+(1, b) \cap \text{IPS} \neq \{(1, b)\}$. Thus, $(1, b)$ is not on the outer Pareto boundary.

If the IPS does not contain a point of the form $(1, a)$ for some a with $0 < a < 1$, then it must contain a point of the form $(1 - a, 1)$ for some a with $0 < a < 1$. The argument in this case is similar and we omit it.

This establishes part a. Part b follows from part a and the fact that the IPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$. □

This theorem, together with Theorem 3.8, immediately yields the following corollary.

Corollary 3.23 *There are points on the boundary of the IPS that do not correspond to Pareto maximal or to Pareto minimal partitions.*

In particular, the point $(1, b)$ in the proof of the theorem (or the analogous point $(b, 1)$, if we are working with $(1 - a, 1)$ instead of $(1, a)$) is an example of a boundary point that is neither Pareto maximal nor Pareto minimal. It turns out that points of this form are the only such examples. This fact will be needed in Chapter 11 and is made precise in the following theorem and corollary.

Theorem 3.24 *Suppose that $p = (p_1, p_2) \in IPS$, $0 < p_1 < 1$, and $0 < p_2 < 1$.*

a. *p is on the outer boundary of the IPS if and only if it is on the outer Pareto boundary of the IPS.*

b. p is on the inner boundary of the IPS if and only if it is on the inner Pareto boundary of the IPS.

Proof: Assume that $p = (p_1, p_2) \in$ IPS, $0 < p_1 < 1$, and $0 < p_2 < 1$. We prove part a. The proof for part b is similar.

For the forward direction, we first note that because the measures are not absolutely continuous with respect to each other, they are not equal, and hence the outer boundary of the IPS is not equal to the simplex. In particular, the outer boundary intersects the simplex only at the points $(1, 0)$ and $(0, 1)$. Assume that $p = (p_1, p_2)$ is on the outer boundary of the IPS. Our assumptions tell us that $p \neq (1, 0)$ and $p \neq (0, 1)$. It follows that $p_1 + p_2 > 1$.

We proceed as we did in part of the proof of Theorem 3.9. Suppose, by way of contradiction, that p is not on the outer Pareto boundary of the IPS. Then there is a point $q \in B^+(p) \cap$ IPS with $q \neq p$, and it is not hard to see that p is an interior point of CH(G), the convex hull of G, where $G = \{q, (1, 0), (0, 1)\}$. Since each point in G is in the IPS, the convexity of the IPS implies that CH(G) $\subseteq$ IPS. Thus p is an interior point of the IPS. This contradicts the fact the p is on the outer boundary of the IPS.

The reverse direction follows from part a of Theorem 3.22. □

Corollary 3.25 *If p is on the boundary but not on the Pareto boundary of the IPS, then p lies on the unit square.*

In Figure 2.1, we showed six IPSs. In Figure 3.7, we repeat these IPSs, darkening the outer boundary and outer Pareto boundary. The outer Pareto boundary is shown with the thicker curve. In Figures 3.7a, 3.7b, 3.7c, and 3.7d, the associated measures are absolutely continuous with respect to each other, so the outer boundary and the outer Pareto boundary are the same. In Figures 3.7e and 3.7f, this is not the case and, therefore, the outer Pareto boundary is a proper subset of the outer boundary.

Next we consider the question of how many points of the IPS are Pareto maximal or Pareto minimal. In other words, we consider how the first statement in Theorem 3.11 looks when absolute continuity fails. It turns out that this statement is still true, with one exception. That exception involves the most extreme case of the failure of absolute continuity.

Definition 3.26 Measures m_1 and m_2 *concentrate on disjoint sets* if and only if there is a partition $\langle P_1, P_2 \rangle$ of C with $m_1(P_1) = 1$ and $m_2(P_2) = 1$.

If P_1 and P_2 are as above, then we shall say that measures m_1 and m_2 concentrate on the disjoint sets P_1 and P_2, respectively. It is not really necessary for us to insist that $\langle P_1, P_2 \rangle$ be a partition of C. It is only necessary that P_1 and P_2

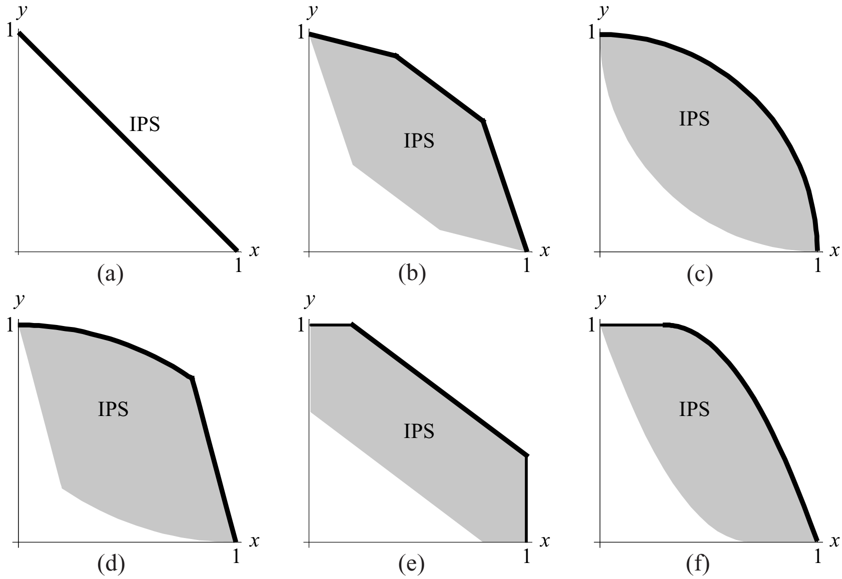


Figure 3.7

be disjoint and that $m_1(P_1) = 1$ and $m_2(P_2) = 1$. If $P_1 \cup P_2 \neq C$, then we can simply replace P_1 by $P_1 \cup (C \backslash (P_1 \cup P_2))$, or P_2 by $P_2 \cup (C \backslash (P_1 \cup P_2))$, since $C \backslash (P_1 \cup P_2)$ has measure zero to each player. The resulting two sets would then form a partition of C that is the same as the partition $\langle P_1, P_2 \rangle$, except for a set that has measure zero to each player. Insisting that $\langle P_1, P_2 \rangle$ be a partition of C is simply a convenience.

Theorem 3.27

a. *If m_1 and m_2 concentrate on disjoint sets, then the IPS has exactly one Pareto maximal point and exactly one Pareto minimal point, and these points are* $(1, 1)$ *and* $(0, 0)$*, respectively.*

b. *If m_1 and m_2 do not concentrate on disjoint sets, then the IPS has infinitely many Pareto maximal points and infinitely many Pareto minimal points.*

Proof: We will need the following observation for both parts: m_1 and m_2 concentrate on disjoint sets if and only if $(1, 1) \in$ IPS.

For part a, we assume that m_1 and m_2 concentrate on disjoint sets. By Theorem 3.8, it suffices for us to show that the outer Pareto boundary consists of the one point $(1, 1)$ and the inner Pareto boundary consists of the one point $(0, 0)$.

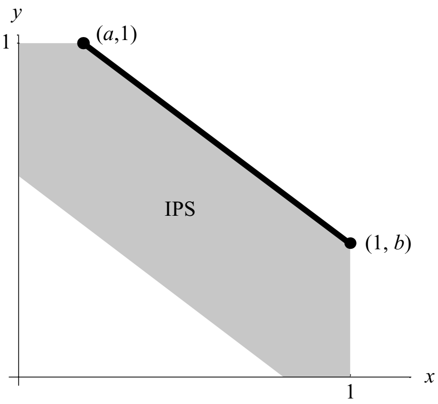


Figure 3.8

Since m_1 and m_2 concentrate on disjoint sets, we know that $(1, 1) \in$ IPS and, by symmetry, this implies that $(0, 0) \in$ IPS. Since the IPS is a subset of the unit square including its interior, it follows easily that $(1, 1)$ is Pareto maximal and no other point is Pareto maximal. (In this particular case, since we know that $(0, 0), (0, 1), (1, 0), (1, 1) \in$ IPS, convexity implies that the IPS is equal to the unit square, together with its interior.) Similarly, $(0, 0)$ is the only Pareto minimal point. Therefore, the outer Pareto boundary consists of the one point $(1, 1)$, and the inner Pareto boundary consists of the one point $(0, 0)$.

For part b, we assume that m_1 and m_2 do not concentrate on disjoint sets. Let $a = \sup\{x : (x, 1) \in \text{IPS}\}$ and let $b = \sup\{y : (1, y) \in \text{IPS}\}$. We know that these suprema are taken over non-empty sets, since $(0, 1) \in$ IPS and $(1, 0) \in$ IPS. Also, since the IPS is closed, $(a, 1) \in$IPS and $(1, b) \in$ IPS. And, since we are assuming that m_1 and m_2 do not concentrate on disjoint sets, we know that $(1, 1) \notin$ IPS and, hence, $(a, 1) \neq (1, 1)$ and $(1, b) \neq (1, 1)$. This implies that $(a, 1) \neq (1, b)$. It is then geometrically clear that the outer Pareto boundary is precisely the (shorter) portion of the boundary of the IPS between $(a, 1)$ and $(1, b)$, including the points $(a, 1)$ and $(1, b)$. This is illustrated in Figure 3.8, where we have darkened the outer Pareto boundary. It follows that the IPS containsinfinitely many Pareto maximal points. The result for the inner Pareto boundary and Pareto minimal points follows from symmetry considerations. □

Notice that in the preceding proof it might be that $(a, 1) = (0, 1)$ or that $(1, b) = (1, 0)$, but these two equalities cannot both hold, since we are assuming that one of the measures fails to be absolutely continuous with respect to the other. In the situation illustrated in Figure 3.8, neither of these equalities holds since neither measure is absolutely continuous with respect to the other.

Figure 3.8 also illustrates the fact that, in contrast with the absolute continuity context of Theorem 3.11, we cannot move from the origin along a straight line in every possible direction into the first quadrant and expect to hit a Pareto maximal point. In the figure, a line through the origin that either passes through the point $(c, 1)$ for some c with $0 < c < a$, or passes through the point $(1, d)$ for some d with $0 < d < b$, fails to hit any such point.

Corollary 3.28

a. If m_1 and m_2 concentrate on the disjoint sets P_1 and P_2, respectively, then

i. all Pareto maximal partitions are s-equivalent (and are s-equivalent to $\langle P_1, P_2\rangle$) and all Pareto minimal partitions are s-equivalent (and are s-equivalent to $\langle P_2, P_1\rangle$).

ii. all Pareto maximal partitions are p-equivalent (and are p-equivalent to $\langle P_1, P_2\rangle$) and all Pareto minimal partitions are p-equivalent (and are p-equivalent to $\langle P_2, P_1\rangle$).

b. If m_1 and m_2 do not concentrate on disjoint sets then

i. there are infinitely many mutually non-s-equivalent Pareto maximal partitions and infinitely many mutually non-s-equivalent Pareto minimal partitions.

ii. there are infinitely many mutually non-p-equivalent Pareto maximal partitions and infinitely many mutually non-p-equivalent Pareto minimal partitions.

Proof: For part a, we assume that m_1 and m_2 concentrate on the disjoint sets P_1 and P_2, respectively. Set $P = \langle P_1, P_2\rangle$. Then P is Pareto maximal and $m(P) = (1, 1)$. Suppose that Q is any Pareto maximal partition. By part a of the theorem, $(1, 1)$ is the only Pareto maximal point and, hence, $m(P) = m(Q)$. Since no points of the IPS lie outside of the unit square, we know that the point $(1, 1)$ is not an interior point of any line segment in the IPS. Hence, it follows from Theorem 2.6 that P and Q are s-equivalent. This establishes that all Pareto maximal partitions are s-equivalent to $P = \langle P_1, P_2\rangle$. The proof that all Pareto minimal partitions are s-equivalent to $\langle P_2, P_1\rangle$ is similar. (Notice that $m(\langle P_2, P_1\rangle) = (0, 0)$.) This establishes part ai.

Part aii follows from part ai.

Part bi follows from part bii. Part bii follows from part b of the theorem and the fact that distinct points of the IPS are the image, under m, of non-p-equivalent partitions. □

Corollary 3.28 – Equivalence Class Version

a. If m_1 and m_2 concentrate on the disjoint sets P_1 and P_2, respectively, then

i. there is exactly one Pareto maximal s-class (and that class is $[\langle P_1, P_2\rangle]_s$) and exactly one Pareto minimal s-class (and that class is $[\langle P_2, P_1\rangle]_s$).

ii. there is exactly one Pareto maximal p-class (and that class is $[\langle P_1, P_2\rangle]_p$) and exactly one Pareto minimal p-class (and that class is $[\langle P_2, P_1\rangle]_p$).

b. If m_1 and m_2 do not concentrate on disjoint sets then

i. there are infinitely many Pareto maximal s-classes and infinitely many Pareto minimal s-classes.

ii. there are infinitely many Pareto maximal p-classes and infinitely many Pareto minimal p-classes.

We can see the truth of part a of the corollary by more direct reasoning. It is clear that if m_1 and m_2 concentrate on disjoint sets, then any Pareto maximal partition must give each player a piece of cake on which that player's measure concentrates. Any remaining cake has measure zero to each player. Hence, all possible choices of how to distribute the remaining cake result in partitions that are s-equivalent and p-equivalent. And, all Pareto maximal partitions are of this form. This idea is similar for Pareto minimality, with each player taking the piece of cake on which the other player's measure concentrates.

Next, we consider Theorem 3.13 in light of the failure of absolute continuity. Whether or not this theorem is true in our current setting depends on whether just one of the measures fails to be absolutely continuous with respect to the other, or whether both fail to be absolutely continuous with respect each other. Recall that Figures 3.6a and 3.6b illustrate the first of these situations and Figure 3.6c illustrates the second. Our main result on this topic is Theorem 3.30. First we prove a lemma from which the theorem will follow easily.

Lemma 3.29

a. Suppose that one of the measures is absolutely continuous with respect to the other and p is a point in the IPS that is neither Pareto maximal nor Pareto minimal. Then p is an interior point of a line segment contained in the IPS.

b. Suppose that neither measure is absolutely continuous with respect to the other.

i. The points $(1, 0)$ and $(0, 1)$ are each neither Pareto maximal nor Pareto minimal.

ii. *Suppose that p is a point in the IPS that is neither Pareto maximal nor Pareto minimal. Then p is an interior point of a line segment contained in the IPS if and only if $p \neq (1, 0)$ and $p \neq (0, 1)$.*

Proof: For part a, we assume, without loss of generality, that m_2 is not absolutely continuous with respect to m_2, but m_2 is absolutely continuous with respect to m_1. This is the situation illustrated in Figure 3.6a. Assume $p \in$ IPS is neither Pareto maximal nor Pareto minimal.

If p is not on the boundary of the IPS, then clearly it is an interior point of a line segment contained in the IPS.

Assume then that p is on the boundary of the IPS. Since p is neither Pareto maximal nor Pareto minimal, p is on neither the outer nor the inner Pareto boundary of the IPS. Our assumptions about m_1 and m_2, together with Theorem 3.24, tell us that for some a with $0 < a < 1$, the parts of the boundary that are on neither the outer nor the inner Pareto boundary consist of two open line segments, one of the form $\{(x, 1) : 0 < x < a\}$ and one of the form $\{(x, 0) : 1 - a < x < 1\}$. (We note that each of the endpoints of each of these line segments is either a Pareto maximal point or a Pareto minimal point and hence each is part of the outer or the inner Pareto boundary.) Thus, p must be in one of these open line segments and therefore is an interior point of a line segment contained in the IPS.

For part b, we assume that neither measure is absolutely continuous with respect to the other. This is the situation illustrated in Figure 3.6c. For part bi, we observe that neither of the points $(1, 0)$ or $(0, 1)$ is on the outer Pareto boundary or the inner Pareto boundary of the IPS. Hence (as we pointed out earlier in this section in our discussion of Figure 3.6c), neither is Pareto maximal or Pareto minimal.

For part bii, we assume that p is a point in the IPS that is neither Pareto maximal nor Pareto minimal. For the forward direction, assume that either $p = (1, 0)$ or $p = (0, 1)$. Since the IPS is contained in $[0, 1]^2$, it is clear that p is not an interior point of a line segment contained in the IPS.

For the reverse direction of part bii, we assume that $p \neq (1, 0)$ and $p \neq (0, 1)$. The proof is similar to the proof of part a. If p is in the interior of the IPS, then it certainly lies on a line segment contained in the IPS. Suppose then that p is on the boundary of the IPS. Since p is neither Pareto maximal nor Pareto minimal, we know that p is not on the outer or the inner Pareto boundary. Since $p \neq (1, 0)$ and $p \neq (0, 1)$, it follows, as in part a, that p lies on an open line segment on the boundary of the IPS. Therefore, p is an interior point of a line segment contained in the IPS. □

Theorem 3.30 *Suppose that P is a partition that is neither Pareto maximal nor Pareto minimal.*

a. Assume that one of the measures is absolutely continuous with respect to the other. Then P is p-equivalent to infinitely many mutually non-s-equivalent partitions.
b. Assume that neither measure is absolutely continuous with respect to the other.
  i. If P is not p-equivalent to either of the partitions $\langle C, \emptyset\rangle$ or $\langle \emptyset, C\rangle$, then P is p-equivalent to infinitely many mutually non-s-equivalent partitions.
  ii. If P is p-equivalent to either $\langle C, \emptyset\rangle$ or $\langle \emptyset, C\rangle$, then any partition that is p-equivalent to P is also s-equivalent to P.

Proof: The proof follows easily from Lemma 3.29 and Theorem 2.6. □

Theorem 3.30 – Equivalence Class Version *Suppose that P is a partition that is neither Pareto maximal nor Pareto minimal.*

a. Assume that one of the measures is absolutely continuous with respect to the other. Then $[P]_p$ is the union of infinitely many s-classes.
b. Assume that neither measure is absolutely continuous with respect to the other.
  i. If $[P]_p \neq [\langle C, \emptyset\rangle]_p$ and $[P]_p \neq [\langle \emptyset, C\rangle]_p$, then $[P]_p$ is the union of infinitely many s-classes.
  ii. If $[P]_p = [\langle C, \emptyset\rangle]_p$ or $[P]_p = [\langle \emptyset, C\rangle]_p$, then $[P]_p$ consists of a single s-class. (Or, equivalently, $[\langle C, \emptyset\rangle]_p = [\langle C, \emptyset\rangle]_s$ and $[\langle \emptyset, C\rangle]_p = [\langle \emptyset, C\rangle]_s$).

We conclude this section by considering how the results of Section 3C, where we combined fairness and efficiency notions, adjust to the present setting in which absolute continuity fails. This failure of absolute continuity implies that $m_1 \neq m_2$ and, hence, in considering Theorem 3.14, we need only investigate how to adjust part b of the theorem. The appropriate adjustment is the following.

Theorem 3.31
a. If m_1 and m_2 concentrate on disjoint sets, then
  i. the IPS has exactly one point that is both proportional and Pareto maximal, and that point is $(1, 1)$.
  ii. the IPS has exactly one point that is both strongly proportional and Pareto maximal, and that point is $(1, 1)$.

b. If m_1 and m_2 do not concentrate on disjoint sets, then

i. the IPS has infinitely many points that are both proportional and Pareto maximal.

ii. the IPS has infinitely many points that are both strongly proportional and Pareto maximal.

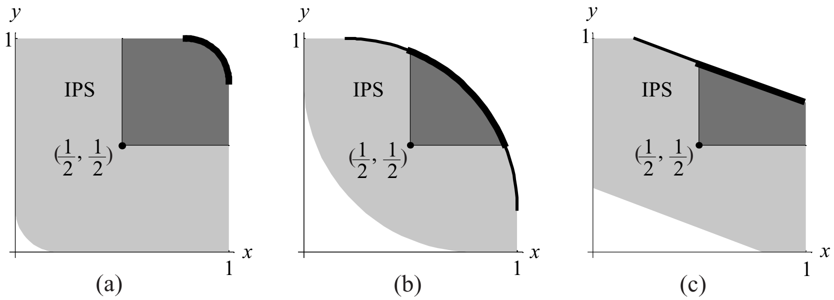


Figure 3.9

The proof follows easily from Theorem 3.27 and the geometric perspectives discussed previously, and we omit the details. Part b of the theorem is illustrated in Figure 3.9. As was the case in Figure 3.5, which we used to illustrate Theorem 3.14, the proportional and strongly proportional points are the points in the darker regions of the IPSs (where the sets of proportional points include the points on the horizontal and the vertical line segments that bound these regions, and the sets of strongly proportional points include none of these points) and the outer Pareto boundaries are darkened. The sets of points that are proportional or strongly proportional, and are also Pareto maximal, are the intersections of these sets and are the thicker curves in the figures. In Figure 3.9a, this intersection is the entire outer Pareto boundary, whereas in Figures 3.9b and 3.9c, part of the outer Pareto boundary extends beyond this region.

Corollary 3.32

a. If m_1 and m_2 concentrate on the disjoint sets P_1 and P_2, respectively, then

i. all partitions that are both proportional and Pareto maximal are s-equivalent and are p-equivalent (and are s-equivalent and p-equivalent to $\langle P_1, P_2 \rangle$).

ii. all partitions that are both strongly proportional and Pareto maximal are s-equivalent and are p-equivalent (and are s-equivalent and p-equivalent to $\langle P_1, P_2 \rangle$).

b. If m_1 and m_2 do not concentrate on disjoint sets, then

i. there are infinitely many mutually non-s-equivalent partitions that are both proportional and Pareto maximal.

ii. there are infinitely many mutually non-p-equivalent partitions that are both proportional and Pareto maximal.
iii. there are infinitely many mutually non-s-equivalent partitions that are both strongly proportional and Pareto maximal.
iv. there are infinitely many mutually non-p-equivalent partitions that are both strongly proportional and Pareto maximal.

Proof: Part a follows from part a of the theorem, Theorem 2.6, and the fact that $(1, 1)$ is not an interior point of a line segment contained in the IPS. Part b follows immediately from part b of the theorem. □

Corollary 3.32 – Equivalence Class Version

a. If m_1 and m_2 concentrate on the disjoint sets P_1 and P_2, respectively, then
i. there is exactly one s-class that is both proportional and Pareto maximal (and that class is $[\langle P_1, P_2\rangle]_s$), and there is exactly one p-class that is both proportional and Pareto maximal (and that class is $[\langle P_1, P_2\rangle]_p$).
ii. there is exactly one s-class that is both strongly proportional and Pareto maximal (and that class is $[\langle P_1, P_2\rangle]_s$), and there is exactly one p-class that is both strongly proportional and Pareto maximal (and that class is $[\langle P_1, P_2\rangle]_p$).

b. If m_1 and m_2 do not concentrate on disjoint sets, then
i. there are infinitely many s-classes that are both proportional and Pareto maximal.
ii. there are infinitely many p-classes that are both proportional and Pareto maximal.
iii. there are infinitely many s-classes that are both strongly proportional and Pareto maximal.
iv. there are infinitely many p-classes that are both strongly proportional and Pareto maximal.

Of course, the s-classes and the p-classes in parts ai and aii are all the same set.

We conclude by stating the chores versions of Theorem 3.31 and Corollary 3.32. The proofs are similar and we omit them. (Theorem 3.33 can also be proved by using Theorem 3.31 and the symmetry of the IPS.)

Theorem 3.33

a. If m_1 and m_2 concentrate on disjoint sets, then
i. the IPS has exactly one point that is both c-proportional and Pareto minimal, and that point is $(0, 0)$.

ii. *the IPS has exactly one point that is both strongly c-proportional and Pareto minimal, and that point is* (0, 0).

b. *If* m_1 *and* m_2 *do not concentrate on disjoint sets, then*

i. *the IPS has infinitely many points that are both c-proportional and Pareto minimal.*

ii. *the IPS has infinitely many points that are both strongly c-proportional and Pareto minimal.*

Corollary 3.34

a. *If* m_1 *and* m_2 *concentrate on the disjoint sets* P_1 *and* P_2*, respectively, then*

i. *all partitions that are both c-proportional and Pareto minimal are s-equivalent and are p-equivalent* (*and are s-equivalent and p-equivalent to* $\langle P_2, P_1 \rangle$).

ii. *all partitions that are both strongly c-proportional and Pareto minimal are s-equivalent and are p-equivalent* (*and are s-equivalent and p-equivalent to* $\langle P_2, P_1 \rangle$).

b. *If* m_1 *and* m_2 *do not concentrate on disjoint sets, then*

i. *there are infinitely many mutually non-s-equivalent partitions that are both c-proportional and Pareto minimal.*

ii. *there are infinitely many mutually non-p-equivalent partitions that are both c-proportional and Pareto minimal.*

iii. *there are infinitely many mutually non-s-equivalent partitions that are both strongly c-proportional and Pareto minimal.*

iv. *there are infinitely many mutually non-p-equivalent partitions that are both strongly c-proportional and Pareto minimal.*

Corollary 3.34 – Equivalence Class Version

a. *If* m_1 *and* m_2 *concentrate on disjoint sets, then*

i. *there is exactly one s-class that is both c-proportional and Pareto minimal* (*and that class is* $[\langle P_2, P_1 \rangle]_s$), *and there is exactly one p-class that is both c-proportional and Pareto minimal* (*and that class is* $[\langle P_2, P_1 \rangle]_p$).

ii. *there is exactly one s-class that is both strongly c-proportional and Pareto minimal* (*and that class is* $[\langle P_2, P_1 \rangle]_s$), *and there is exactly one p-class that is both strongly c-proportional and Pareto minimal* (*and that class is* $[\langle P_2, P_1 \rangle]_p$).

b. *If* m_1 *and* m_2 *do not concentrate on disjoint sets, then*

i. *there are infinitely many s-classes that are both c-proportional and Pareto minimal.*

ii. *there are infinitely many p-classes that are both c-proportional and Pareto minimal.*

iii. there are infinitely many s-classes that are both strongly c-proportional and Pareto minimal.

iv. there are infinitely many p-classes that are both strongly c-proportional and Pareto minimal.

As was the case in Corollary 3.32, the s-classes and the p-classes in parts ai and aii are all the same set.

4

The Individual Pieces Set (IPS) and the Full Individual Pieces Set (FIPS) for the General n-Player Context

In this chapter, we consider the general case of n players. In Section 4A we consider the IPS, and then in Section 4B we shall see that the IPS is not a sufficient structure for studying all fairness properties when there are more than two players. In Section 4C, we generalize the IPS to the FIPS, the Full Individual Pieces Set. In Section 4D, we prove a general result about the possibilities for the FIPS. This result will be a central tool in our work in Chapter 5. We make no general assumptions about absolute continuity in this chapter.

4A. Geometric Object #1b: The IPS for *n* Players

Before considering fairness and efficiency issues, we consider more general geometric issues, as we did in the two-player context. More specifically, we examine Theorems 2.2, 2.4, and 2.6 and consider appropriate generalizations to the n-player context. The definition of the IPS, of s-equivalence, and of p-equivalence are the obvious generalizations of the corresponding definitions for two players.

Definition 4.1

a. For any partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of C, let $m(P) = (m_1(P_1), m_2(P_2), \ldots, m_n(P_n))$. The *Individual Pieces Set*, or *IPS*, is the set $\{m(P) : P \in \text{Part}\}$.

b. Two partitions $P = \langle P_1, P_2, \ldots, P_n \rangle$ and $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ are *s-equivalent* if and only if, for all $i = 1, 2, \ldots, n$, $m_i(P_i \Delta Q_i) = 0$.

c. Two partitions $P = \langle P_1, P_2, \ldots, P_n \rangle$ and $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ are *p-equivalent* if and only if, for every $i = 1, 2, \ldots, n$, $m_i(P_i) = m_i(Q_i)$. Or, equivalently, P and Q are *p-equivalent* if and only if $m(P) = m(Q)$.

We note that IPS $\subseteq \mathbf{R}^n$. Also, as in the two-player context of Chapter 2, s-equivalence and p-equivalence are equivalence relations, and we shall refer to the associated equivalence classes as *s-classes* and *p-classes*, respectively, and shall let $[P]_s$ and $[P]_p$ denote the s-class and the p-class, respectively, of partition P. The function m from Part to the IPS respects s-equivalence and p-equivalence, and the induced function from the set of p-classes to the IPS is a bijection.

Next, we consider Theorem 2.2 and the obvious generalization of this result. Recall that the idea behind part a of this theorem is that, by giving all of the cake to Player 1 or by giving all of the cake to Player 2, we see that $(1, 0)$ and $(0, 1)$ are in the IPS. Dvoretsky, Wald, and Wolfovitz's theorem (Theorem 1.4) tells us that the IPS is convex and so the line segment connecting $(1, 0)$ and $(0, 1)$ is in the IPS. This idea generalizes in a natural way to the n-player context. Arguing precisely as in the two-player case, we see that $(1, 0, 0, \ldots, 0, 0), (0, 1, 0, \ldots, 0, 0), \ldots, (0, 0, 0, \ldots, 0, 1)$ are all in the associated IPS and hence, by convexity, the $(n - 1)$-simplex is a subset of the IPS. This establishes part a of the following theorem.

Theorem 4.2
a. The IPS contains the simplex.
b. The IPS consists precisely of the simplex if and only if $m_1 = m_2 = \cdots = m_n$.

The proof for part b is analogous to the proof of part b of Theorem 2.2, and we omit it. We turn next to Theorem 2.4.

The natural generalizations of parts a, b, c, and d of Theorem 2.4 are easily seen to be true in the n-player context. In other words, we have the following result.

Theorem 4.3 *The IPS*
a. is a subset of $[0, 1]^n$.
b. contains the simplex.
c. is closed.
d. is convex.

Part a is obvious, part b is a restatement of part a of Theorem 4.2, and parts c and d follow from Dvoretsky, Wald, and Wolfovitz's theorem. We have not included a generalization of part e of Theorem 2.4 in Theorem 4.3. We shall discuss this issue shortly.

Theorem 2.6 is true as stated for n players. We repeat in here.

Theorem 4.4 *Let p be a point of the IPS. The following are equivalent:*

a. p is the image, under m, of infinitely many mutually non-s-equivalent partitions.

b. p is the image, under m, of at least two non-s-equivalent partitions.
c. p lies in the interior of a line segment contained in the IPS.

The proof of Theorem 4.4 is a generalization of the proof of Theorem 2.6. The proof of Theorem 2.6 required Lemma 2.7. We shall require the n-player version of this lemma.

Lemma 4.5 *For any piece of cake A, there is a collection of subsets of A such that, for each player, each subset in the collection has size half that of A, and any player who believes that A has positive measure also believes that all pairwise symmetric differences from this collection have positive measure. Also, if any player believes that A has positive measure, then this collection is infinite. In other words, for any $A \subseteq C$, there is a collection $\Gamma(A)$ of subsets of A such that,*

a. for any $B \in \Gamma(A)$ and $i = 1, 2, \ldots, n$, $m_i(B) = \frac{1}{2} m_i(A)$;
b. for $i = 1, 2, \ldots, n$, if $m_i(A) > 0$, then for distinct $B_1, B_2 \in \Gamma(A)$, $m_i(B_1 \Delta B_2) > 0$; and
c. if $m_i(A) > 0$ for some $i = 1, 2, \ldots, n$, then $\Gamma(A)$ is infinite.

The proof is similar to the proof of Lemma 2.7, and we omit it.

Proof of Theorem 4.4: Fix some $p \in$ IPS. It is obvious that part a implies part b. We will show that part b implies part c and that part c implies part a.

To show that part b implies part c, assume that $P = \langle P_1, P_2, \ldots, P_n \rangle$ and $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ are two non-s-equivalent partitions and $m(P) = m(Q) = p$. We must show that p lies in the interior of a line segment contained in the IPS.

For distinct $i, j = 1, 2, \ldots, n$, define $R_{ij} = P_i \cap Q_j$. Then R_{ij} is the piece of cake that Player i must transfer to Player j in going from partition P to partition Q.

For each $i = 1, 2, \ldots, n$, we define a partition S^i as follows:

$$\begin{aligned} S^i = \langle & P_1 \cup R_{i1}, P_2 \cup R_{i2}, \ldots, P_{i-1} \cup R_{i,i-1}, \\ & P_i \backslash (R_{i1} \cup R_{i2} \cup \cdots \cup R_{i,i-1} \cup R_{i,i+1} \cup \cdots \cup R_{in}), \\ & P_{i+1} \cup R_{i,i+1}, \ldots, P_n \cup R_{in} \rangle \end{aligned}$$

We may view each S^i as arising by starting with partition P and then having Player i complete all of the transfers to other players that would have to be completed to go to partition Q (and having all other players give up nothing). Since $m_i(P_i) = m_i(Q_i)$ and Player i obtains Q_i from P_i by giving up the sets $R_{i1}, R_{i2}, \ldots, R_{i,i-1}, R_{i,i+1}, \ldots, R_{in}$ and receiving the sets $R_{1i}, R_{2i}, \ldots, R_{i-1,i}, R_{i+1,i}, \ldots, R_{ni}$, it must be the case that $m_i(R_{i1} \cup R_{i2} \cup \cdots \cup R_{i,i-1} \cup R_{i,i+1} \cup \cdots \cup R_{in}) = m_i(R_{1i} \cup R_{2i} \cup \cdots \cup R_{i-1,i} \cup R_{i+1,i} \cup \cdots \cup R_{ni})$. Also, the non-$s$-equivalence of P and Q implies that, in going from partition P to partition Q, at least one player receives a piece of cake and gives up a piece of cake that are each of positive measure to that player. In other words, for some i, j, and k, $m_i(R_{ji}) > 0$ and $m_i(R_{ik}) > 0$, and it follows that there are at least two non-p-equivalent S^i.

Obviously, for each $i = 1, 2, \ldots, n$, $m(S^i) \in$ IPS. Consider the following convex combination of elements of the IPS:

$$\left(\frac{1}{n}\right) m(S^1) + \left(\frac{1}{n}\right) m(S^2) + \cdots + \left(\frac{1}{n}\right) m(S^n)$$

We claim that this convex combination is equal to p. We establish this as follows:

$$\begin{aligned}
&\left(\frac{1}{n}\right) m(S^1) + \left(\frac{1}{n}\right) m(S^2) + \cdots + \left(\frac{1}{n}\right) m(S^n) \\
&= \left(\frac{1}{n}\right) (m(S^1) + m(S^2) + \cdots + m(S^n)) \\
&= \left(\frac{1}{n}\right) \begin{pmatrix} (nm_1(P_1) + m_1(R_{21} \cup R_{31} \cup \cdots \cup R_{n1}) \\ \quad - m_1(R_{12} \cup R_{13} \cup \cdots \cup R_{1n})), \\ (nm_2(P_2) + m_2(R_{12} \cup R_{32} \cup \cdots \cup R_{n2}) \\ \quad - m_2(R_{21} \cup R_{23} \cup \cdots \cup R_{2n})), \ldots, \\ (nm_n(P_n) + m_n(R_{1n} \cup R_{2n} \cup \cdots \cup R_{n-1,n}) \\ \quad - m_n(R_{n1} \cup R_{n2} \cup \cdots \cup R_{n,n-1})) \end{pmatrix} \\
&= \left(\frac{1}{n}\right) (nm_1(P_1), \ldots, nm_n(P_n)) \\
&= (m_1(P_1), \ldots, m_n(P_n)) = p
\end{aligned}$$

Thus we see that p is a convex combination of $m(S^1), m(S^2), \ldots, m(S^n)$, and these points are elements of the IPS. As discussed earlier, we know that there are at least two non-p-equivalent S^i, and therefore the list $m(S^1), m(S^2), \ldots, m(S^n)$ of elements of the IPS contains at least two distinct points. Hence, p is a convex combination of at least two elements of the IPS and, since the coefficients used in this convex combination are all positive, it follows that p lies in the interior of a line segment contained in the IPS.

The proof that part c implies part a is similar to the corresponding proof in Theorem 2.6. Part c tells us that we can find partitions P and Q so that p is the midpoint of the line segment connecting $m(P)$ and $m(Q)$. We can imagine partition Q as arising from partition P by a collection of transfers between players. By Lemma 4.5, for each of these transfers there are an infinite number of different ways to complete what all players agree is half of the transfer (where "different" means "different by a set that has positive measure to any player who believes that the original transfer involves a set of positive measure"). Thus, there are an infinite number of different ways to complete the entire collection of such half-transfers. Each results in a partition that gets mapped by m to p. Hence, p is the image, under m, of infinitely many mutually non-s-equivalent partitions. This completes the proof of the theorem. □

As we did for Theorem 2.6, we may restate this theorem in terms of the composition of the relevant equivalence classes.

Theorem 4.4 – Equivalence Class Version *For any partition P, the following are equivalent:*

a. $[P]_p$ is the union of infinitely many s-classes.
b. $[P]_p$ is the union of at least two s-classes.
c. $m(P)$ lies in the interior of a line segment contained in the IPS.

For partitions P and Q, when is it the case that the p-equivalence of these two partitions implies their s-equivalence? By the theorem, this will be the case precisely when the point $m(P) = m(Q)$ in the IPS is not in the interior of any line segment contained in the IPS. This notion will be useful in Chapter 14 in our study of strong Pareto maximality.

As was the case for Theorem 2.6, Theorem 4.4 yields as immediate corollary.

Corollary 4.6 *Any point of the IPS that is not on the boundary of the IPS is the image, under m, of infinitely many mutually non-s-equivalent partitions.*

Corollary 4.6 – Equivalence Class Version *For any partition P, if $m(P)$ is not on the boundary of the IPS, then $[P]_p$ is the union of infinitely many s-classes.*

The only part of Theorems 2.2, 2.4, and 2.6 that we have not yet considered is part e of Theorem 2.4, which told us that in the two-player context, the IPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$. There are two natural generalizations of this statement to the n-player setting:

a. The IPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.
b. The IPS is symmetric about the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.

It turns out that each of these statements is, in general, false. To see that statement a can be false, suppose that the measures are all equal. Then, by part b of Theorem 4.2, the IPS is the simplex. Hence, since for $n > 2$, $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is not on the simplex, we see that the IPS need not be symmetric about $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

In Chapter 11, where we consider issues regarding the possibilities for the shape of the IPS, we shall give a specific example to show that statement b need not be true in general (see Theorem 11.5). For now, we give an informal perspective on why this is so.

When $n = 2$, the IPS is a subset of $[0, 1]^2$, the unit square together with its interior. The point $(\frac{1}{2}, \frac{1}{2})$ is at the center of this set, and so there is a certain naturalness in the IPS being symmetric about this point. However, when $n > 2$, the IPS is a subset of $[0, 1]^n$, and $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, not $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, is at the center of this set. Hence, it is not reasonable to expect the IPS to be symmetric about $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ when $n > 2$. As we shall see in Chapter 16, although the IPS is not symmetric about $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, a more general structure that contains the IPS is symmetric about $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

We can make the idea outlined in the previous paragraph somewhat more precise. Whether or not the measures are absolutely continuous with respect to each other, we may imagine that they concentrate on sets that are "almost" disjoint. Then the IPS will contain points close to $(1, 1, \ldots, 1)$. This certainly implies that, for $n > 2$, the IPS cannot be symmetric about $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, since reflecting points close to $(1, 1, \ldots, 1)$ about $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ yields points that have negative coordinates and, hence, cannot be in the IPS.

Does the IPS possess any sort of symmetry when there are more than two players? In searching for symmetry, let us consider how to generalize the simple proof of symmetry about $(\frac{1}{2}, \frac{1}{2})$ from the two-player context, as given in the proof of Lemma 2.3. In that case, we found symmetry by starting with some partition and then having the two players exchange pieces. Theorem 4.8, and its proof, can be viewed as a direct generalization of this idea where, instead of two players exchanging pieces, all players repeatedly "pass to the right." After stating and proving this result, we will examine what it tells us about the shape of the IPS. Before stating the theorem, we need to develop some terminology and notation.

For $q^1, q^2, \ldots, q^m \in \mathbf{R}^n$, we recall that $\mathrm{CH}(q^1, q^2, \ldots, q^m)$ denotes the convex hull of the set $\{q^1, q^2, \ldots, q^m\}$. By the convexity of the IPS, if $q^1, q^2, \ldots, q^m \in \mathrm{IPS}$, then $\mathrm{CH}(q^1, q^2, \ldots, q^m) \subseteq \mathrm{IPS}$.

We note that for two distinct points q^1 and q^2, $\mathrm{CH}(q^1, q^2)$ is a line segment; for three distinct non-collinear points q^1, q^2, and q^3, $\mathrm{CH}(q^1, q^2, q^3)$ is a triangle (including its interior); and for four distinct non-coplanar points q^1, q^2, q^3, and q^4, $\mathrm{CH}(q^1, q^2, q^3, q^4)$ is a tetrahedron (including its interior). We shall refer to any such object $\mathrm{CH}(q^1, q^2, \ldots, q^m)$ as an m-tetrahedron. (We do not assume that the m points $q^1, q^2, \ldots, q^m$ do not lie in a lower-dimensional subspace of $\mathbf{R}^n$. Thus, for example, if q^1, q^2, q^3, q^4 are coplanar but not collinear, then

$CH(q^1, q^2, q^3, q^4)$ is a triangle, but we shall still refer to it as a 4-tetrahedron in this case.)

We need to consider the centroid of an n-tetrahedron. Intuitively, the centroid of an object is its center. If the object is a physical object that is homogeneous (i.e., has constant density), then its centroid is its center of mass. Computing the centroid of an object is a problem studied in multivariable calculus, and formulas to compute the centroid for various standard objects have been worked out. We shall only be interested in the centroid of objects of the form $CH(q^1, q^2, \ldots, q^m)$, i.e., of m-tetrahedra.

Let $Cent(q^1, q^2, \ldots, q^m)$ denote the centroid of $CH(q^1, q^2, \ldots, q^m)$. Then, using the methods of multivariable calculus, it can be shown that $Cent(q^1, q^2, \ldots, q^m)$ is the coordinate average of $q^1, q^2, \ldots, q^m$. The reader may take the perspective that, for objects of the form $CH(q^1, q^2, \ldots, q^m)$, $Cent(q^1, q^2, \ldots, q^m)$ is, *by definition*, the coordinate average of $q^1, q^2, \ldots, q^m$. This perspective will suffice for our present purposes. It is easy to see that $Cent(q^1, q^2, \ldots, q^m) \in CH(q^1, q^2, \ldots, q^m)$.

The following lemma reveals some additions facts about the centroid of an m-tetrahedron and will be used in the proof of Corollary 4.9.

Lemma 4.7 *Suppose $q^1, q^2, \ldots, q^m$ are points in $\mathbf{R}^n$ and choose any $i = 1, 2, \ldots, m$.*

a. $Cent(q^1, q^2, \ldots, q^m)$ is on the line segment connecting q^i and $Cent(q^1, q^2, \ldots, q^{i-1}, q^{i+1}, \ldots, q^m)$.
b. The distance from $Cent(q^1, q^2, \ldots, q^m)$ to $Cent(q^1, q^2, \ldots, q^{i-1}, q^{i+1}, \ldots, q^m)$ is $\frac{1}{m-1}$ times the distance from $Cent(q^1, q^2, \ldots, q^m)$ to q^i.

Proof: Fix $q^1, q^2, \ldots, q^m \in \mathbf{R}^n$ and $i = 1, 2, \ldots, m$.

Claim $Cent(q^1, q^2, \ldots, q^m) = (\frac{1}{m})q^i + (\frac{m-1}{m})\ Cent(q^1, q^2, \ldots, q^{i-1}, q^{i+1}, \ldots, q^m)$.

Proof of Claim: Recalling that the centroid of an m-tetrahedron is the coordinate average of the vertices, we establish the claim as follows:

$$\begin{aligned}
&\text{Cent}\ (q^1, q^2, \ldots, q^m)\\
&= \left(\frac{1}{m}\right)(q^1 + q^2 + \cdots + q^m)\\
&= \left(\frac{1}{m}\right)q^i + \left(\frac{1}{m}\right)(q^1 + q^2 + \cdots + q^{i-1} + q^{i+1} + \cdots + q^m)
\end{aligned}$$

$$= \left(\frac{1}{m}\right) q^i + \left(\frac{m-1}{m}\right)\left(\frac{1}{m-1}\right)(q^1 + q^2 + \cdots + q^{i-1} + q^{i+1} + \cdots + q^m)$$
$$= \left(\frac{1}{m}\right) q^i + \left(\frac{m-1}{m}\right) \text{Cent}\,(q^1, q^2, \ldots, q^{i-1}, q^{i+1}, \ldots, q^m)$$

This establishes the claim.

The claim says that the point Cent($q^1, q^2, \ldots, q^m$) is a weighted average of the point q^i and the point Cent($q^1, q^2, \ldots, q^{i-1}, q^{i+1}, \ldots, q^m$), using weights $\frac{1}{m}$ and $\frac{m-1}{m}$, respectively. Since the weights sum to one, Cent($q^1, q^2, \ldots, q^m$) is on the line determined by q^i and Cent($q^1, q^2, \ldots, q^{i-1}, q^{i+1}, \ldots, q^m$) and, since the weights are both positive, Cent($q^1, q^2, \ldots, q^m$) is between q^i and Cent($q^1, q^2, \ldots, q^{i-1}, q^{i+1}, \ldots, q^m$) on this line. This establishes part a of the lemma. Part b follows since the weight of q^i (which is $\frac{1}{m}$) is $\frac{1}{m-1}$ times the weight of Cent($q^1, q^2, \ldots, q^{i-1}, q^{i+1}, \ldots, q^m$) (which is $\frac{m-1}{m}$). □

Let us examine the lemma in familiar, low-dimension situations. For two distinct points q^1 and q^2, in $\mathbf{R}^1$, $\mathbf{R}^2$, or $\mathbf{R}^3$, CH(q^1, q^2) is a line segment. The centroid of this line segment (i.e., the coordinate average of q^1 and q^2) is its midpoint. In this case, the truth of each part of the lemma is clear. (For part a, we note that the centroid of a point is itself.)

For three distinct non-coplanar points q^1, q^2, and q^3, in $\mathbf{R}^2$ or $\mathbf{R}^3$, CH(q^1, q^2, q^3) is a triangle. It is not obvious, but is a well-known geometric fact, that the three line segments that connect each vertex of a triangle with the midpoint of the opposite side intersect at a point. By part a of the lemma, the centroid must be on each of these three line segments. Hence the point of intersection of these line segments is the centroid. A standard theorem of plane geometry tells us that the distance from a vertex of a triangle to the centroid of the triangle is twice the distance from the centroid to the midpoint of the opposite side, thus illustrating part b of the lemma.

Theorem 4.8 *Suppose $p \in$ IPS. Then p is a vertex of an n-tetrahedron that lies in the IPS and has centroid $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. In other words, there are $n-1$ points $p^1, p^2, \ldots, p^{n-1} \in \mathbf{R}^n$ such that*

a. $p^1, p^2, \ldots, p^{n-1} \in$ IPS and hence, CH$(p, p^1, p^2, \ldots, p^{n-1}) \subseteq$ IPS, and
b. Cent$(p, p^1, p^2, \ldots, p^{n-1}) = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.

Proof: Assume that $p \in$ IPS. Then for some partition $\langle P_1, P_2, \ldots, P_n\rangle$ of C, $(m_1(P_1), m_2(P_2), \ldots, m_n(P_n)) = p$. We obtain $n-1$ new partitions by having each player repeatedly "pass to the right." More specifically, we consider the partitions $\langle P_n, P_1, P_2, \ldots, P_{n-2}, P_{n-1}\rangle$, $\langle P_{n-1}, P_n, P_1, \ldots, P_{n-3}, P_{n-2}\rangle, \ldots, \langle P_2, P_3, P_4, \ldots, P_n, P_1\rangle$.

Next, we let $p^1, p^2, \ldots, p^{n-1}$ be the corresponding points in the IPS. In other words,

$$\begin{aligned}
p^1 &= (m_1(P_n), m_2(P_1), m_3(P_2), \ldots, m_{n-1}(P_{n-2}), m_n(P_{n-1})),\\
p^2 &= (m_1(P_{n-1}), m_2(P_n), m_3(P_1), \ldots, m_{n-1}(P_{n-3}), m_n(P_{n-2})), \ldots\\
p^{n-1} &= (m_1(P_2), m_2(P_3), m_3(P_4), \ldots, m_{n-1}(P_n), m_n(P_1)).
\end{aligned}$$

Clearly, $p^1, p^2, \ldots, p^{n-1} \in$ IPS, and this implies that $\text{CH}(p, p^1, p^2, \ldots, p^{n-1}) \subseteq$ IPS. This establishes that $p^1, p^2, \ldots, p^{n-1}$ satisfy part a.

For part b, we note that since $\langle P_1, P_2, \ldots, P_n\rangle$ is a partition of C it follows that, for each $i = 1, 2, \ldots, n$,

$$m_i(P_1) + m_i(P_2) + \cdots + m_i(P_n) = m_i(P_1 \cup P_2 \cup \cdots \cup P_n) = m_i(C) = 1$$

This tells us that $p, p^1, p^2, \ldots, p^{n-1}$ have coordinate sum $(1, 1, \ldots, 1)$ and hence have coordinate average $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. Thus, $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is the centroid of $\text{CH}(p, p^1, p^2, \ldots, p^{n-1})$. □

To gain some perspective on what the theorem is telling us about the IPS, let us consider what it says for two players and for three players. When there are two players, the theorem is simply a restatement of part e of Theorem 2.4, which says that the IPS is symmetric about $(\frac{1}{2}, \frac{1}{2})$.

When there are three players, the theorem says that, given any $p \in$ IPS, there is a triangle T, having p as one vertex, that lies in the IPS and has centroid $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Hence, the theorem yields something resembling symmetry about $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Two examples of the type of information conveyed by this result for three players are as follows:

- Given any plane containing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the IPS contains points on one side of this plane if and only if it contains points on the other side of this plane. In particular, the IPS contains points on one side of the simplex if and only if it contains points on the other side of the simplex.
- Somewhat less formally, we see that if it is possible to move relatively far from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in one direction, then it must be possible to move at least moderately far in the opposite direction as well.

Of course, analogous facts are true for more than three players.

The idea of a "triangle" in the preceding discussion may be somewhat misleading. The triangle may be degenerate in the sense that two, or possibly all three, of the vertices may be identical. For example, it follows from Corollary 1.5 that there exists a partition P of C among the three players that all agree is a partition into three equal pieces. Setting $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in the statement of Theorem 4.8 and using partition P in the proof of the theorem, we find that the three vertices are all the same point, namely $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Of course, this sort of degeneracy applies to Theorem 4.8 generally. That is, the points $p, p^1, p^2, \ldots, p^{n-1}$ need not be distinct.

We now return to the general case of n players. The essential information on symmetry contained in Theorem 4.8 is given to us by the following corollary.

Corollary 4.9 *Suppose that $p = (p_1, p_2, \ldots, p_n) \in$ IPS. If q is the point such that p, $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, and q are collinear, with $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ between p and q, and the distance from $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ to q is $\frac{1}{n-1}$ times the distance from $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ to p, then*

$$q = \left(\frac{1-p_1}{n-1}, \frac{1-p_2}{n-1}, \ldots, \frac{1-p_n}{n-1}\right)$$

and $q \in$ IPS.

Proof: Given $p = (p_1, p_2, \ldots, p_n) \in$ IPS, let $p^1, p^2, \ldots, p^{n-1}$ be as in the statement of Theorem 4.8. Then $p^1, p^2, \ldots, p^{n-1} \in$ IPS and Cent$(p, p^1, p^2, \ldots, p^{n-1}) = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. Let q be as in the statement of the corollary. By Lemma 4.7, $q =$ Cent$(p^1, p^2, \ldots, p^{n-1})$. Since $p^1, p^2, \ldots, p^{n-1} \in$ IPS and the IPS is convex, it follows that CH$(p^1, p^2, \ldots, p^{n-1}) \subseteq$ IPS and, thus, $q =$ Cent$(p^1, p^2, \ldots, p^{n-1}) \in$ IPS.

It remains for us to show that

$$q = \left(\frac{1-p_1}{n-1}, \frac{1-p_2}{n-1}, \ldots, \frac{1-p_n}{n-1}\right).$$

We are given that

$$\left(q - \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)\right) = \left(\frac{1}{n-1}\right)\left(\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) - p\right).$$

Then,

$$\begin{aligned} q &= \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) + \left(\frac{1}{n-1}\right)\left(\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) - p\right) \\ &= \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) + \left(\frac{1}{n-1}\right)\left(\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) - (p_1, p_2, \ldots, p_n)\right) \\ &= \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) + \left(\frac{1}{n-1}\right)\left(\frac{1-p_1 n}{n}, \frac{1-p_2 n}{n}, \ldots, \frac{1-p_n n}{n}\right) \end{aligned}$$

$$= \left(\frac{n-1}{n(n-1)}, \frac{n-1}{n(n-1)}, \ldots, \frac{n-1}{n(n-1)} \right)$$
$$+ \left(\frac{1-p_1 n}{n(n-1)}, \frac{1-p_2 n}{n(n-1)}, \ldots, \frac{1-p_n n}{n(n-1)} \right)$$
$$= \left(\frac{n-p_1 n}{n(n-1)}, \frac{n-p_2 n}{n(n-1)}, \ldots, \frac{n-p_n n}{n(n-1)} \right)$$
$$= \left(\frac{1-p_1}{n-1}, \frac{1-p_2}{n-1}, \ldots, \frac{1-p_n}{n-1} \right).$$

□

We present an alternative proof of Corollary 4.9 that will be useful in Chapter 11 (see the proof of Theorem 11.5).

Alternate Proof of Corollary 4.9: Suppose that $p = (p_1, p_2, \ldots, p_n) \in \text{IPS}$ and $P = \langle P_1, P_2, \ldots, P_n \rangle$ is any partition corresponding to p (i.e., $m(P) = p$). It follows from Corollary 1.6 that each P_i can be divided into $n-1$ pieces in such a way that all n players believe these $n-1$ pieces are of equal size. For each $i = 1, 2, \ldots, n$, give one of these $n-1$ pieces to each of the other $n-1$ players and let $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ be the new partition obtained in this way.

For each $i = 1, 2, \ldots, n, m_i(P_i) = p_i$ and so $m_i(C \backslash P_i) = 1 - p_i$. Since Q_i is obtained by taking $\frac{1}{n-1}$ of each of the pieces of cake that make up $C \backslash P_i$, it follows that $m_i(Q_i) = \frac{1-p_i}{n-1}$. Hence,

$$m(Q) = \left(\frac{1-p_1}{n-1}, \frac{1-p_2}{n-1}, \ldots, \frac{1-p_n}{n-1} \right)$$

and so

$$\left(\frac{1-p_1}{n-1}, \frac{1-p_2}{n-1}, \ldots, \frac{1-p_n}{n-1} \right) \in \text{IPS}.$$

We must show that

$$\left(\frac{1-p_1}{n-1}, \frac{1-p_2}{n-1}, \ldots, \frac{1-p_n}{n-1} \right)$$

is the q of the corollary. This follows simply by noting that

$$\left(\frac{1}{n}\right)(p_1, p_2, \ldots, p_n) + \left(\frac{n-1}{n}\right)\left(\frac{1-p_1}{n-1}, \frac{1-p_2}{n-1}, \ldots, \frac{1-p_n}{n-1} \right)$$
$$= \left(\frac{1}{n}\right)(p_1, p_2, \ldots, p_n) + \left(\frac{1}{n}\right)(1-p_1, 1-p_2, \ldots, 1-p_n)$$
$$= \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right).$$

In other words, $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is the weighted average of $(p_1, p_2, \ldots, p_n)$ and

$$\left(\frac{1-p_1}{n-1}, \frac{1-p_2}{n-1}, \ldots, \frac{1-p_n}{n-1}\right)$$

using positive weights that sum to one, where the weight of $(p_1, p_2, \ldots, p_n)$ is $\frac{1}{n-1}$ times the weight of

$$\left(\frac{1-p_1}{n-1}, \frac{1-p_2}{n-1}, \ldots, \frac{1-p_n}{n-1}\right).$$

This establishes that q is as in the statement of the corollary. □

For $n = 2$, the corollary is just a restatement of part e of Theorem 2.4: symmetry of the IPS about $(\frac{1}{2}, \frac{1}{2})$. In what sense does the corollary generalize this result? One natural generalization (which, as we have discussed, is not true) would say that the IPS is symmetric about the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. Another way to state this is as follows: given any point in the IPS, if we travel from that point to the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ and then continue along the same line precisely the same distance, then we arrive at another point of the IPS. (Of course, by convexity, all points on this line segment would also be in the IPS.) Our present result is weaker. It says that given any point in the IPS, if we travel from that point to the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ and then continue along the same line precisely $\frac{1}{n-1}$ that distance, then we arrive at another point of the IPS.

Note that the corollary does not imply that the IPS is not symmetric about $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. As noted earlier in this section, the failure of symmetry about $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ for $n > 2$ will be established in Chapter 11, where we will show that, in a sense to be made precise, Corollary 4.9 is the best possible result of this sort.

4B. Why the IPS Does Not Suffice

We wish to consider generalizations to the n-player context of the fairness results developed in Chapter 3 for two players. We recall that in the two-player context, proportionality and envy-freeness correspond, as do strong proportionality, strong envy-freeness, and super envy-freeness. Also, for a given point p in the IPS, either all or none of the partitions in the p-class corresponding to p satisfy any given fairness property. More specifically, if $(p_1, p_2) \in$ IPS and $\langle P_1, P_2\rangle$ is any corresponding partition (i.e., $m(\langle P_1, P_2\rangle) = (p_1, p_2)$), then

$\langle P_1, P_2 \rangle$ is a proportional partition if and only if
$\langle P_1, P_2 \rangle$ is an envy-free partition if and only if
$p_1 \geq \frac{1}{2}$ and $p_2 \geq \frac{1}{2}$

and

$\langle P_1, P_2 \rangle$ is a strongly proportional partition if and only if
$\langle P_1, P_2 \rangle$ is an strongly envy-free partition if and only if
$\langle P_1, P_2 \rangle$ is a super envy-free partition if and only if
$p_1 > \frac{1}{2}$ and $p_2 > \frac{1}{2}$.

Analogous statements hold for chores fairness properties.

Thus, as we have seen, the location of a point in the IPS reveals whether or not that point corresponds to a partition that is proportional, envy-free, strongly proportional, strongly envy-free, or super envy-free. However, such is not the case when we consider more than two players.

The central point here is that in the two-player context, an element of the IPS contains all information about each player's evaluation of *every* piece of cake in any corresponding partition. In other words, if $(p_1, p_2) \in$ IPS and $\langle P_1, P_2 \rangle$ is any corresponding partition, then

$$\begin{aligned} m_1(P_1) &= p_1 & m_1(P_2) &= 1 - p_1 \\ m_2(P_1) &= 1 - p_2 & m_2(P_2) &= p_2 \end{aligned}$$

This is not the case when there are more than two players.

Consider the three-player context. Suppose that $(p_1, p_2, p_3) \in$ IPS and $\langle P_1, P_2, P_3 \rangle$ is any corresponding partition. Then certainly $\langle P_1, P_2, P_3 \rangle$ is proportional if and only if $p_1 \geq \frac{1}{3}$, $p_2 \geq \frac{1}{3}$, and $p_3 \geq \frac{1}{3}$, and is strongly proportional if and only if $p_1 > \frac{1}{3}$, $p_2 > \frac{1}{3}$, and $p_3 > \frac{1}{3}$. Thus, as in the case of two players, the location of a point in the IPS reveals whether or not that point corresponds to a partition that is proportional or strongly proportional. However, when there are more than two players, proportionality and envy-freeness need not correspond, and strong proportionality, strong envy-freeness, and super envy-freeness need not correspond. In particular, the position of the point (p_1, p_2, p_3) in the IPS does not contain enough information to tell us whether or not corresponding partitions are envy-free or strongly envy-free or super envy-free, since these notions involve the values of $m_1(P_2)$, $m_1(P_3)$, $m_2(P_1)$, $m_2(P_3)$, $m_3(P_1)$, and $m_3(P_2)$, and this information cannot be obtained from the values of p_1, p_2, and p_3, as can the analogous information in the two-player context.

To illustrate this point, suppose ε is some small positive number and $(\frac{1}{3} + 2\varepsilon, \frac{1}{3} + 5\varepsilon, \frac{1}{3} + 2\varepsilon) \in$ IPS. Clearly, any partition corresponding to this point is proportional and strongly proportional. Suppose that $P = \langle P_1, P_2, P_3\rangle$ is such a partition. Is P envy-free or strongly envy-free or super envy-free? The answer is "maybe and maybe not." We do not have enough information. The problem, as we discussed in the preceding paragraph, is that to answer such questions, we need to know the players' evaluations of all pieces of cake, not just their own. For two players, this information is implicit in each player's evaluation of his or her own piece of cake, but such is not the case for more than two players. For example, consider each of the following four situations:

a.
$$\begin{array}{lll} m_1(P_1) = \frac{1}{3} + 2\varepsilon & m_1(P_2) = \frac{1}{3} + 5\varepsilon & m_1(P_3) = \frac{1}{3} - 7\varepsilon \\ m_2(P_1) = \frac{1}{3} + 20\varepsilon & m_2(P_2) = \frac{1}{3} + 5\varepsilon & m_2(P_3) = \frac{1}{3} - 25\varepsilon \\ m_3(P_1) = \frac{1}{3} - 7\varepsilon & m_3(P_2) = \frac{1}{3} + 5\varepsilon & m_3(P_3) = \frac{1}{3} + 2\varepsilon \end{array}$$

b.
$$\begin{array}{lll} m_1(P_1) = \frac{1}{3} + 2\varepsilon & m_1(P_2) = \frac{1}{3} - 4\varepsilon & m_1(P_3) = \frac{1}{3} + 2\varepsilon \\ m_2(P_1) = \frac{1}{3} - 3\varepsilon & m_2(P_2) = \frac{1}{3} + 5\varepsilon & m_2(P_3) = \frac{1}{3} - 2\varepsilon \\ m_3(P_1) = \frac{1}{3} + 2\varepsilon & m_3(P_2) = \frac{1}{3} - 4\varepsilon & m_3(P_3) = \frac{1}{3} + 2\varepsilon \end{array}$$

c.
$$\begin{array}{lll} m_1(P_1) = \frac{1}{3} + 2\varepsilon & m_1(P_2) = \frac{1}{3} + \varepsilon & m_1(P_3) = \frac{1}{3} - 3\varepsilon \\ m_2(P_1) = \frac{1}{3} + 3\varepsilon & m_2(P_2) = \frac{1}{3} + 5\varepsilon & m_2(P_3) = \frac{1}{3} - 8\varepsilon \\ m_3(P_1) = \frac{1}{3} + \varepsilon & m_3(P_2) = \frac{1}{3} - 3\varepsilon & m_3(P_3) = \frac{1}{3} + 2\varepsilon \end{array}$$

d.
$$\begin{array}{lll} m_1(P_1) = \frac{1}{3} + 2\varepsilon & m_1(P_2) = \frac{1}{3} - \varepsilon & m_1(P_3) = \frac{1}{3} - \varepsilon \\ m_2(P_1) = \frac{1}{3} - 3\varepsilon & m_2(P_2) = \frac{1}{3} + 5\varepsilon & m_2(P_3) = \frac{1}{3} - 2\varepsilon \\ m_3(P_1) = \frac{1}{3} - \varepsilon & m_3(P_2) = \frac{1}{3} - \varepsilon & m_3(P_3) = \frac{1}{3} + 2\varepsilon \end{array}$$

In all four situations, the point in the IPS corresponding to partition P is $(\frac{1}{3} + 2\varepsilon, \frac{1}{3} + 5\varepsilon, \frac{1}{3} + 2\varepsilon)$ and thus, in all four situations, P is strongly proportional. However,

a. in situation a, P is not envy-free (since, for example, $m_1(P_1) < m_1(P_2)$).
b. in situation b, P is envy-free but not strongly envy-free (since, for example, $m_1(P_1) = m_1(P_3)$).
c. in situation c, P is strongly envy-free but not super envy-free (since, for example, $m_1(P_2) \geq \frac{1}{3}$).
d. in situation d, P is super envy-free.

Note that because C has measure one to each player, in each aforementioned situation each player's values for the three pieces sum to one.

Are each of the situations above possible? That is, for each situation, is there a cake C, measures m_1, m_2, and m_3, and a partition $\langle P_1, P_2, P_3\rangle$ so that the given conditions are satisfied? This question will be answered in the affirmative in Section 5E (see Example 5.53). Thus we see that, in general (in contrast with the two-player context), the location of a point in the IPS does not tell us whether corresponding partitions have a particular fairness property. And (also in contrast with the two-player context) it is not the case that all partitions corresponding to a point in the IPS have the exact same fairness properties.

The various possibilities illustrated in this section show that the IPS is not a complete enough structure to capture all information in which we may be interested. We address this problem in the next section.

4C. Geometric Object #1c: The FIPS

In this section, we broaden our perspective to include the additional information necessary to evaluate envy-freeness, strong envy-freeness, super envy-freeness, and the corresponding chores properties.

Definition 4.10 For any partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of C, let $m_F(P)$ be the $n \times n$ matrix $[m_i(P_j)]_{i,j \leq n}$. The *Full Individual Pieces Set*, or *FIPS*, is the set $\{m_F(P) : P \in \text{Part}\}$.

The FIPS is the subject of Dvoretsky, Wald, and Wolfovitz's theorem (Theorem 1.4). This result tells us that the FIPS is closed and convex. We observe that the IPS consists of the set of all n-tuples that are diagonals of matrices in the FIPS.

In general, one can consider m players and partitions of C into n pieces, where m need not equal n. Then the FIPS would consist of $m \times n$ matrices. We have chosen not to consider this more general setting, in keeping with our theme of considering partitions of C in which each player receives a piece of cake. Thus, we shall always have the number of players equal to the number of pieces in the partition, and so all of the matrices will be square.

In contrast with the aforementioned IPS limitations, we see that the location of a point in the FIPS does tell us all relevant facts about fairness properties of associated partitions. Suppose that $p = [p_{ij}]_{i,j \leq n} \in$ FIPS. Then, for any partition P corresponding to p (i.e., for any partition P with $m_F(P) = p$), P is

a. proportional if and only if, for all $i = 1, 2, \ldots, n$, $p_{ii} \geq \frac{1}{n}$.
b. strongly proportional if and only if, for all $i = 1, 2, \ldots, n$, $p_{ii} > \frac{1}{n}$.
c. envy-free if and only if, for all $i, j = 1, 2, \ldots, n$, $p_{ii} \geq p_{ij}$.
d. strongly envy-free if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $p_{ii} > p_{ij}$.

e. super envy-free if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $p_{ii} > \frac{1}{n}$ and $p_{ij} < \frac{1}{n}$.

Or, equivalently, a partition P is

a. proportional if and only if each diagonal entry of $m_F(P)$ is at least $\frac{1}{n}$.
b. strongly proportional if and only if each diagonal entry of $m_F(P)$ is greater than $\frac{1}{n}$.
c. envy-free if and only if each diagonal entry of $m_F(P)$ is at least as large as every other entry in its row.
d. strongly envy-free if and only if each diagonal entry of $m_F(P)$ is greater than every other entry in its row.
e. super envy-free if and only if each diagonal entry of $m_F(P)$ is greater than $\frac{1}{n}$ and each non-diagonal entry is less than $\frac{1}{n}$.

In the preceding analysis, we have not mentioned chores fairness properties. The correspondence between partitions having chores fairness properties and points in the FIPS is entirely analogous.

We observe that proportionality and strong proportionality depend only upon the diagonal entries of $m_F(P)$, i.e., the values of $m(P)$, whereas envy-freeness, strong envy-freeness, and super envy-freeness also depend on non-diagonal entries and, hence, require more information than is provided by $m(P)$. This is why the IPS suffices when discussing proportionality or strong proportionality, but the FIPS is needed to discuss envy-freeness, strong envy-freeness, or super envy-freeness. Or, to put it another way, players need not look at other players' pieces to decide about proportionality and strong proportionality, but must do so to decide on envy-freeness, strong envy-freeness, and super envy-freeness. The IPS only provides information on players' views of their own pieces but the FIPS provides information on all players' views of all pieces.

In analogy with what was done previously, we can refer to points in the FIPS as having fairness properties.

Definition 4.11 Suppose $p = [p_{ij}]_{i,j \leq n} \in$ FIPS.

a. p is a *proportional point* if and only if, for each $i = 1, 2, \ldots, n$, $p_{ii} \geq \frac{1}{n}$.
b. p is a *strongly proportional point* if and only if, for each $i = 1, 2, \ldots, n$, $p_{ii} > \frac{1}{n}$.
c. p is an *envy-free point* if and only if, for all $i, j = 1, 2, \ldots, n$, $p_{ii} \geq p_{ij}$.
d. p is a *strongly envy-free point* if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $p_{ii} > p_{ij}$.
e. p is a *super envy-free point* if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $p_{ii} > \frac{1}{n}$ and $p_{ij} < \frac{1}{n}$.

The corresponding definitions for points with chores fairness properties are the following.

Definition 4.12 Suppose $p = [p_{ij}]_{i,j \le n} \in$ FIPS.

a. p is a *c-proportional point* if and only if, for each $i = 1, 2, \ldots, n$, $p_{ii} \le \frac{1}{n}$.
b. p is a *strongly c-proportional point* if and only if, for each $i = 1, 2, \ldots, n$, $p_{ii} < \frac{1}{n}$.
c. p is a *c-envy-free point* if and only if, for all $i, j = 1, 2, \ldots, n$, $p_{ii} \le p_{ij}$.
d. p is a *strongly c-envy-free point* if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $p_{ii} < p_{ij}$.
e. p is a *super c-envy-free point* if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $p_{ii} < \frac{1}{n}$ and $p_{ij} > \frac{1}{n}$.

Consider the following two questions that arise naturally in shifting our focus from the IPS to the FIPS.

Question 1: What are the appropriate generalizations of our two equivalence relations, s-equivalence and p-equivalence?
Question 2: What is the appropriate generalization of Theorem 4.4?

Regarding Question 1, we recall that our definitions of the s-equivalence and the p-equivalence of two partitions concern each player's view of only his or her own piece of cake in these partitions. This is consistent with the definition of the IPS, where players' evaluations of only their own piece of cake is relevant. However, in now considering the FIPS, we wish to consider players' evaluations of all pieces of cake in a partition, not just their own. This leads us naturally to the following definitions.

Definition 4.13 Fix partitions $P = \langle P_1, P_2, \ldots, P_n \rangle$ and $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$.

a. P and Q are *fs-equivalent* if and only if, for all $i, j = 1, 2, \ldots, n$, $m_i(P_j \Delta Q_j) = 0$.
b. P and Q are *fp-equivalent* if and only if, for all $i, j = 1, 2, \ldots, n$, $m_i(P_j) = m_i(Q_j)$. Or, equivalently, P and Q are *fp-equivalent* if and only if $m_F(P) = m_F(Q)$.

It is easy to see that fs-equivalence and fp-equivalence are each equivalence relations. We shall follow our previous terminological convention and refer to the associated equivalence classes as *fs-classes* and *fp-classes*, and we shall let $[P]_{fs}$ and $[P]_{fp}$ denote the fs-class and the fp-class, respectively, of partition P.

Clearly, fs-equivalence implies s-equivalence, and fp-equivalence implies p-equivalence. Each converse is true only in the two-player context.

We have added "f" to "s-equivalent" and "p-equivalent" to get "fs-equivalent" and "fp-equivalent," respectively, in order to correspond to going from "IPS" to "FIPS" and going from "$m(P)$" to "$m_F(P)$." As was the case with s-equivalence and p-equivalence and the function m, it is clear that the function m_F from Part to the FIPS respects fs-equivalence and fp-equivalence in the sense that if partitions P and Q are fs-equivalent or fp-equivalent, then $m_F(P) = m_F(Q)$. It follows that m_F induces a function from the set of fs-classes to the FIPS and a function from the set of fp-classes to the FIPS. Note that this induced function from the set of fp-classes to the FIPS is a bijection.

Since fs-equivalence and fp-equivalence clearly respect our various fairness and efficiency notions, we can extend our previous terminological conventions and, for example, refer to "strongly envy-free fs-classes" or "Pareto maximal fp-classes."

Next, we consider Question 2. As we shall see, Theorem 4.4 is true with "IPS" changed to "FIPS" and "m" changed to "m_F," once we understand what "line segment" and "interior of a line segment" mean in the space of $n \times n$ matrices. Of course, we do not have the same geometric intuition here that we had when we were working in the two-player or the three-player context and considered line segments in $\mathbf{R}^2$ or $\mathbf{R}^3$. However, these notions can be viewed algebraically and generalize easily to our present setting.

Definition 4.14 Suppose that p and q are $n \times n$ matrices. The *line segment* between p and q is the set of all $n \times n$ matrices of the form $\lambda p + (1 - \lambda)q$ with $0 \leq \lambda \leq 1$. A matrix r is in the *interior* of this line segment if and only if $r = \lambda p + (1 - \lambda)q$ for some λ with $0 < \lambda < 1$.

We note that if $p, q \in$ FIPS then, by convexity, it follows that the line segment between p and q is contained in the FIPS.

We can now state the natural generalization of Theorem 4.4.

Theorem 4.15 *Let p be a point of the FIPS. The following are equivalent:*

a. p is the image, under m_F, of infinitely many mutually non-fs-equivalent partitions.
b. p is the image, under m_F, of at least two non-fs-equivalent partitions.
c. p lies in the interior of a line segment contained in the FIPS.

The proof is a direct generalization of the proof of Theorem 4.4, and we omit it.

As was the case for Theorem 4.4, we may restate this result in terms of equivalence classes.

Theorem 4.15 – Equivalence Class Version *For any partition P, the following are equivalent:*

a. $[P]_{fp}$ is the union of infinitely many fs-classes.
b. $[P]_{fp}$ is the union of at least two fs-classes.
c. $m_F(P)$ lies on the interior of a line segment contained in the FIPS.

4D. A Theorem on the Possibilities for the FIPS

The main result of this section, Theorem 4.18, provides a general framework for showing that various sorts of partitions exist. Before stating the theorem, we give an informal description and discuss some obvious restrictions.

Suppose first that $(p_1, p_2, \ldots, p_n)$ is an element of the simplex with all positive coordinates. By Corollary 1.5, we know that there exists a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ such that all players believe that P_1 has measure p_1, all players believe that P_2 has measure p_2, etc. Then, $m_F(P) = [p_j]_{i,j \le n}$, the $n \times n$ matrix with all p_1s in the first column, all p_2s in the second column, etc., and hence $[p_j]_{i,j \le n} \in$ FIPS. We wish to consider the following question:

> In what ways can an element of the FIPS have entries that deviate from $[p_j]_{i,j \le n}$?

We illustrate with an example in the three-player context. Consider the point $(\frac{1}{7}, \frac{2}{7}, \frac{4}{7})$. Note that this point is in the simplex. Then, as described in the preceding paragraph,

$$\begin{bmatrix} \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \end{bmatrix} \in \text{FIPS}.$$

Or, in other words, there is a partition $P = \langle P_1, P_2, P_3 \rangle$ such that

$$m_F(P) = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \end{bmatrix}.$$

Can we, for example, find a partition $Q = \langle Q_1, Q_2, Q_3 \rangle$ such that:

$$\begin{array}{lll} m_1(Q_1) > \frac{1}{7} & m_1(Q_2) = \frac{2}{7} & m_1(Q_3) < \frac{4}{7} \\ m_2(Q_1) < \frac{1}{7} & m_2(Q_2) < \frac{2}{7} & m_2(Q_3) > \frac{4}{7} \\ m_3(Q_1) = \frac{1}{7} & m_3(Q_2) < \frac{2}{7} & m_3(Q_3) > \frac{4}{7} \end{array}$$

Or, equivalently, is there a $[p_{ij}]_{i,j\le 3} \in$ FIPS such that:

$$\begin{array}{lll} p_{11} > \frac{1}{7} & p_{12} = \frac{2}{7} & p_{13} < \frac{4}{7} \\ p_{21} < \frac{1}{7} & p_{22} < \frac{2}{7} & p_{23} > \frac{4}{7} \\ p_{31} = \frac{1}{7} & p_{32} < \frac{2}{7} & p_{33} > \frac{4}{7} \end{array}$$

Theorem 4.18 gives a precise criterion for determining when this is possible.

There are two sorts of restrictions on what partitions or, equivalently, what elements of the FIPS, of the type just described, can exist. The first suggests a necessary condition on rows of a matrix in the FIPS, and the second suggests a necessary condition on columns of a matrix in the FIPS.

- Each player believes that the entire cake has measure one. Thus, the values that each player gives to the pieces of cake in a partition of C must sum to one. Referring to the preceding example, if we change "$m_2(Q_3) > \frac{4}{7}$" to "$m_2(Q_3) < \frac{4}{7}$" (or, equivalently, if we change "$p_{23} > \frac{4}{7}$" to "$p_{23} < \frac{4}{7}$"), then there would be no such partition (or equivalently, there would be no such an element of the FIPS) since, if there were, it would follow that
$$1 = m_2(C) = m_2(Q_1) + m_2(Q_2) + m_2(Q_3) < \frac{1}{7} + \frac{2}{7} + \frac{4}{7} = 1$$
which is a contradiction.
- There may be linear dependence relationships between the measures. Referring again to the preceding example, suppose that $m_1 = \frac{1}{2}m_2 + \frac{1}{2}m_3$. Then there would be no partition $Q = \langle Q_1, Q_2, Q_3 \rangle$ with $m_1(Q_2) = \frac{2}{7}$, $m_2(Q_2) < \frac{2}{7}$, and $m_3(Q_2) < \frac{2}{7}$ (or, equivalently, there would be no $[p_{ij}]_{i,j\le 3} \in$ FIPS with $p_{12} = \frac{2}{7}$, $p_{22} < \frac{2}{7}$, and $p_{32} < \frac{2}{7}$) since, if there were, it would follow that
$$\frac{2}{7} = m_1(Q_2) = \frac{1}{2}m_2(Q_2) + \frac{1}{2}m_3(Q_2) < \left(\frac{1}{2}\right)\left(\frac{2}{7}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{7}\right) = \frac{2}{7}$$
which is a contradiction.

We can state these two ideas in terms of the FIPS as follows:

- Any row of any element of the FIPS must sum to one.
- Any column of any element of the FIPS must be consistent with all linear dependence relationships between the measures.

These conditions will be formalized as part of the notion of "proper matrix" in Definition 4.17.

As we shall see in a precise way in Theorem 4.18, these are the only such restrictions. Before stating the theorem, we need some definitions.

Definition 4.16 Let *DEP* denote the set of all true linear equations involving any of the measures $m_1, m_2, \ldots, m_n$.

For example, if $m_1(A) = \frac{1}{2}m_2(A) + \frac{1}{2}m_3(A)$ for all $A \subseteq C$, then "$m_1 = \frac{1}{2}m_2 + \frac{1}{2}m_3$" is one of the equations in DEP. Of course, DEP stands for dependence.

Definition 4.17 Suppose that $q = [q_{ij}]_{i,j \leq n}$ is a $n \times n$ matrix of real numbers; q is a *proper matrix* if and only if the following two conditions hold:

a. Each row of q sums to zero. That is, for each $i = 1, 2, \ldots, n$, $q_{i1} + q_{i2} + \cdots + q_{in} = 0$.
b. Each column of q is consistent with the equations in DEP. That is, for each equation in DEP and each $j = 1, 2, \ldots, n$, the given equation holds with $m_1, m_2, \ldots, m_n$ replaced by $q_{1j}, q_{2j}, \ldots, q_{nj}$, respectively.

We are now ready to state our main result of this section.

Theorem 4.18 *Suppose that $r = (r_1, r_2, \ldots, r_n)$ is an element of the simplex with all positive coordinates and $q = [q_{ij}]_{i,j \leq n}$ is a proper matrix. Then, for some $\lambda > 0$, $[r_j + \lambda q_{ij}]_{i,j \leq n} \in$ FIPS.*

The theorem tells us that, starting with the matrix $[r_j]_{i,j \leq n}$ (which we know, by Corollary 1.5, is a member of the FIPS), we can move some positive distance in the direction given by the matrix $[q_{ij}]_{i,j \leq n}$ and stay within the FIPS. The two conditions of Definition 4.17 guarantee that this movement does not violate the two previously discussed restrictions. The conclusion to the theorem tells us that, for some $\lambda > 0$ and some partition $P = \langle P_1, P_2, \ldots, P_n \rangle$, $m_i(P_j) = r_j + \lambda q_{ij}$ for all $i, j = 1, 2, \ldots, n$.

The proof of Theorem 4.18 involves considering a new geometric object that arises naturally in our present context. For each $A \subseteq C$, we consider the vector of values obtained by applying each player's measure to A. We then consider the set of all vectors obtained in this way.

Definition 4.19 The *One Piece Set*, or *OPS*, is the set $\{(m_1(A), m_2(A), \ldots, m_n(A)) : A \subseteq C\}$.

We use the term One Piece Set to contrast with the Individual Pieces Set. Note that the OPS is simply the set of all possible first (or second, or third, etc.) columns of matrices in the FIPS. (This is in contrast with the IPS, which is the set of all diagonals of matrices in the FIPS.) This set is sometimes called a zonoid. It has been studied by E. D. Bolker [14] and A. Neyman [33] and is the set that is the subject of Lyapounov's theorem (Theorem 1.3).

We observe that OPS $\subseteq \mathbf{R}^n$ and, by Lyapounov's theorem, we know that the OPS is closed and convex. Also, by letting $A = \emptyset$, we see that $(0, 0, \ldots, 0) \in$ OPS. Similarly, by letting $A = C$, we see that $(1, 1, \ldots, 1) \in$ OPS. By convexity, it follows that the line segment connecting $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ is a subset of the OPS. It is easy to see that the OPS consists precisely of this line segment if and only if the measures are all equal.

We claim that the OPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. To see this, suppose that $(p_1, p_2, \ldots, p_n) \in$ OPS. Then, for some $A \subseteq C$, $(m_1(A), m_2(A), \ldots, m_n(A)) = (p_1, p_2, \ldots, p_n)$. But then $(m_1(C \backslash A), m_2(C \backslash A), \ldots, m_n(C \backslash A)) = (1 - m_1(A), 1 - m_2(A), \ldots, 1 - m_n(A)) = (1 - p_1, 1 - p_2, \ldots, 1 - p_n)$, and so $(1 - p_1, 1 - p_2, \ldots, 1 - p_n) \in$ OPS. Since $(1 - p_1, 1 - p_2, \ldots, 1 - p_n)$ is the reflection of $(p_1, p_2, \ldots, p_n)$ about $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, it follows that the OPS is symmetric about $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

The following lemma will be used in the proof of Theorem 4.18.

Lemma 4.20 *If the measures are linearly independent then, for any real number κ with $0 < \kappa < 1$, the point $(\kappa, \kappa, \ldots, \kappa)$ is an interior point of the OPS.*

We remind the reader that, by definition, measures $m_1, m_2, \ldots, m_n$ are *linearly independent* if and only if, for any real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n = 0$ implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Equivalently, $m_1, m_2, \ldots, m_n$ are linearly independent if and only if for no $i = 1, 2, \ldots, n$ are there constants $\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_n$ such that $m_i = \beta_1 m_1 + \beta_2 m_2 + \cdots + \beta_{i-1} m_{i-1} + \beta_{i+1} m_{i+1} + \cdots + \beta_n m_n$. If, for some such i and constants $\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_n$, $m_i = \beta_1 m_1 + \beta_2 m_2 + \cdots + \beta_{i-1} m_{i-1} + \beta_{i+1} m_{i+1} + \cdots + \beta_n m_n$, we say that m_i is a *linear combination* of $m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n$. Thus, $m_1, m_2, \ldots, m_n$ are linearly independent if and only if no one of these measures is a linear combination of the others. If the measures are not linearly independent, then they are said to be *linearly dependent*.

The proof of the lemma will use the following result, which is a basic and well-known theorem in the field of convexity theory (see, for example, [25]):

> Given any convex set $G \subseteq \mathbf{R}^n$ and any point p on the boundary of G, there is a hyperplane H that includes p and is such that G is contained in one of the closed half-spaces of $\mathbf{R}^n$ determined by H.

A *hyperplane* in $\mathbf{R}^n$ is given by an equation of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = k$ for some constants $\alpha_1, \alpha_2, \ldots, \alpha_n, k$, where not all of the α_i are equal to zero.

Proof of Lemma 4.20: Fix some κ with $0 < \kappa < 1$ and assume that $(\kappa, \kappa, \ldots, \kappa)$ is not an interior point of the OPS. We must show that measures $m_1, m_2, \ldots, m_n$ are linearly dependent.

Since $(\kappa, \kappa, \ldots, \kappa)$ is not an interior point of the OPS, it is on the boundary of the OPS. Since the OPS is convex, the aforementioned result implies that there is a hyperplane H in $\mathbf{R}^n$ such that $(\kappa, \kappa, \ldots, \kappa) \in H$ and the OPS is a subset of one of the closed half-spaces determined by H. In fact, we shall show that OPS $\subseteq H$.

We claim that $(0, 0, \ldots, 0) \in H$ and $(1, 1, \ldots, 1) \in H$. Since $(\kappa, \kappa, \ldots, \kappa) \in H$ and $(\kappa, \kappa, \ldots, \kappa)$ is strictly between $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$, it follows that if $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ are not both in H, then they are on opposite sides of H. This is impossible, since both of these points are in the OPS and the OPS is a subset of one of the closed half-spaces determined by H. Hence $(0, 0, \ldots, 0) \in H$ and $(1, 1, \ldots, 1) \in H$.

Suppose that H is given by $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = k$. Since $(0, 0, \ldots, 0) \in H$, it follows that $k = 0$ and, hence, H is given by $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$. Also, since $(1, 1, \ldots, 1) \in H$, we know that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0$.

Since H is given by $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$ and the OPS is a subset of one of the closed half-spaces determined by H, we know that either $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \leq 0$ for every $(x_1, x_2, \ldots, x_n) \in$ OPS, or else $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \geq 0$ for every $(x_1, x_2, \ldots, x_n) \in$ OPS. Without loss of generality, we assume that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \leq 0$ for every $(x_1, x_2, \ldots, x_n) \in$ OPS.

To show that OPS $\subseteq H$, let us suppose that $(p_1, p_2, \ldots, p_n)$ is an arbitrary element of the OPS. This implies that $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n \leq 0$. By the symmetry of the OPS about the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, we know that $(1 - p_1, 1 - p_2, \ldots, 1 - p_n) \in$ OPS and, hence, $\alpha_1(1 - p_1) + \alpha_2(1 - p_2) + \cdots + \alpha_n(1 - p_n) \leq 0$.

Next, we note that

$$\begin{aligned}
&\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n \\
&\quad = (\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n) - (\alpha_1 + \alpha_2 + \cdots + \alpha_n) \\
&\quad = \alpha_1(p_1 - 1) + \alpha_2(p_2 - 1) + \cdots + \alpha_n(p_n - 1) \\
&\quad = -[\alpha_1(1 - p_1) + \alpha_2(1 - p_2) + \cdots + \alpha_n(1 - p_n)] \geq 0.
\end{aligned}$$

Since $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n \leq 0$ and $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n \geq 0$, it follows that $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n = 0$ and, thus, $(p_1, p_2, \ldots, p_n) \in H$. Since $(p_1, p_2, \ldots, p_n)$ was an arbitrary element of the OPS, this establishes that OPS $\subseteq H$.

Finally, we observe that, for any $A \subseteq C$, $(m_1(A), m_2(A), \ldots, m_n(A)) \in$ OPS and therefore, since OPS $\subseteq H$, $(m_1(A), m_2(A), \ldots, m_n(A)) \in H$. It follows that $\alpha_1 m_1(A) + \alpha_2 m_2(A) + \cdots + \alpha_n m_n(A) = 0$, and hence, since $A \subseteq C$ was arbitrary, we conclude that the measures $m_1, m_2, \ldots, m_n$ are linearly dependent. □

We are almost ready to begin the proof of Theorem 4.18. First, we give an informal overview of our method. We shall define $n-1$ elements of the FIPS, $p^1, p^2, \ldots, p^{n-1}$. The importance of p^j, the jth of these new matrices, will be contained in its jth column. The jth column of the desired matrix, $[r_j + \lambda q_{ij}]_{i,j \le n}$, will be some positive scalar multiple of the jth column of p^j. By taking an appropriate convex combination of these $n-1$ elements of the FIPS, we shall get this desired element of the FIPS. The reason we need not define a matrix p^n is that if the first $n-1$ columns do what is required, then so will the last column. The is so because the entries in each row sum to one and hence the last column is determined by the previous columns.

Proof of Theorem 4.18: Assume that $r = (r_1, r_2, \ldots, r_n)$ is an element of the simplex with all positive coordinates and $q = [q_{ij}]_{i,j \le n}$ is a proper matrix. We must show that for some $\lambda > 0$, $[r_j + \lambda q_{ij}]_{i,j \le n} \in$ FIPS.

By renumbering, if necessary, we may assume that, for some $s = 1, 2, \ldots, n$, the measures $m_1, m_2, \ldots, m_s$ are linearly independent and each of the measures $m_{s+1}, m_{s+2}, \ldots, m_n$ can be expressed as a linear combination of these measures. (We are *not* assuming that the measures are linearly dependent. If they are linearly independent, then $s = n$.)

For the remainder of the proof, we shall let OPS denote the OPS corresponding to the cake C with just the measures $m_1, m_2, \ldots, m_s$ rather than all measures $m_1, m_2, \ldots, m_n$. In other words, OPS $= \{(m_1(A), m_2(A), \ldots, m_s(A)) : A \subseteq C\}$. Notice that $0 < r_n < 1$ and, hence, $0 < 1 - r_n < 1$. It follows from Lemma 4.20 that $(1 - r_n, 1 - r_n, \ldots, 1 - r_n)$ is an interior point of the OPS.

Let $\varepsilon > 0$ be such that there exists a neighborhood of $(1 - r_n, 1 - r_n, \ldots, 1 - r_n)$ in $\mathbf{R}^s$ of radius ε that lies completely in the OPS. Fix some $\lambda > 0$ such that

$$\max\{\lambda(\frac{1 - r_n}{r_j})|(q_{1j}, q_{2j}, \ldots, q_{sj})| : j \le n - 1\} < \varepsilon$$

where "$|(q_{1j}, q_{2j}, \ldots, q_{sj})|$" denotes the magnitude of vector $(q_{1j}, q_{2j}, \ldots, q_{sj})$.

Our choice of ε and then of λ implies that, for each $j = 1, 2, \ldots, n - 1$,

$$(1 - r_n, 1 - r_n, \ldots, 1 - r_n) + \lambda \left(\frac{1 - r_n}{r_j}\right)(q_{1j}, q_{2j}, \ldots, q_{sj}) \in \text{OPS}.$$

Hence, for each such j, there is a set $A_j \subseteq C$ such that

$$m_i(A_j) = (1 - r_n) + \lambda \left(\frac{1 - r_n}{r_j}\right) q_{ij} \text{ for each } i = 1, 2, \ldots, s.$$

Claim For each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n - 1, m_i(A_j) = (1 - r_n) + \lambda(\frac{1-r_n}{r_j}) q_{ij}$.

Proof of Claim: We have just shown that the claim is true for $i = 1, 2, \ldots, s$. If $s = n$, then we are done. Suppose then that $s < n$ and fix some $i = s + 1, s + 2, \ldots, n$. By assumption, measure m_i is a linear combination on the measures $m_1, m_2, \ldots, m_s$. Then, for some constants $c_1, c_2, \ldots, c_s$, we have $m_i = c_1 m_1 + c_2 m_2 + \cdots + c_s m_s$. Note that $c_1 + c_2 + \cdots + c_s = c_1 m_1(C) + c_2 m_2(C) + \cdots + c_s m_s(C) = m_i(C) = 1$.

Next, we observe that "$m_i = c_1 m_1 + c_2 m_2 + \cdots + c_s m_s$" is one of the equations in DEP. Since each column of q is consistent with the equations in DEP, it follows that for each $j = 1, 2, \ldots, n - 1, q_{ij} = c_1 q_{1j} + c_2 q_{2j} + \cdots + c_s q_{sj}$. Hence, for each such j,

$$\begin{aligned} m_i(A_j) &= c_1 m_1(A_j) + c_2 m_2(A_j) + \cdots + c_s m_s(A_j) \\ &= c_1 \left[(1 - r_n) + \lambda \left(\frac{1 - r_n}{r_j}\right) q_{1j}\right] \\ &\quad + c_2 \left[(1 - r_n) + \lambda \left(\frac{1 - r_n}{r_j}\right) q_{2j}\right] + \cdots \\ &\quad \cdots + c_s \left[(1 - r_n) + \lambda \left(\frac{1 - r_n}{r_j}\right) q_{sj}\right] \\ &= (c_1 + c_2 + \cdots + c_s)(1 - r_n) \\ &\quad + \lambda \left(\frac{1 - r_n}{r_j}\right)(c_1 q_{1j} + c_2 q_{2j} + \cdots + c_s q_{sj}) \\ &= (1 - r_n) + \lambda \left(\frac{1 - r_n}{r_j}\right) q_{ij} . \end{aligned}$$

This establishes the claim.

Next, for each $j = 1, 2, \ldots, n - 1$, we define an $n \times n$ matrix p^j as follows:

a. The jth column of p^j is given by $[m_i(A_j)]_{i \leq n}$.
b. The nth column of p^j is given by $[m_i(C \backslash A_j)]_{i \leq n}$.
c. All other columns of p^j consist of all zeros.

It is easy to see that each such p^j is in the FIPS, because it arises from the partition $\langle \emptyset, \ldots, \emptyset, A_j, \emptyset, \ldots, \emptyset, C \backslash A_j \rangle$, where the A_j is in the jth position. We will take a convex combination of the p^j to get the desired member of the FIPS.

Consider the numbers $\frac{r_1}{1-r_n}, \frac{r_2}{1-r_n}, \ldots, \frac{r_{n-1}}{1-r_n}$. These numbers are all positive and, recalling that $r_1 + r_2 + \cdots + r_n = 1$, we have

$$\frac{r_1}{1-r_n} + \frac{r_2}{1-r_n} + \cdots + \frac{r_{n-1}}{1-r_n} = \frac{r_1 + r_2 + \cdots + r_{n-1}}{1-r_n} = \frac{1-r_n}{1-r_n} = 1\,.$$

These are the coefficients we shall use to define a new matrix p. Thus, we define $p = (\frac{r_1}{1-r_n})p^1 + (\frac{r_2}{1-r_n})p^2 + \cdots + (\frac{r_{n-1}}{1-r_n})p^{n-1}$. Since each p^j is in the FIPS, and $\frac{r_1}{1-r_n}, \frac{r_2}{1-r_n}, \ldots, \frac{r_{n-1}}{1-r_n}$ are positive numbers that sum to one, the convexity of the FIPS implies that $p \in$ FIPS. To prove the theorem, we show that $p = [r_j + \lambda q_{ij}]_{i,j \le n}$.

Since $p \in$ FIPS, we know that each row of matrix p sums to one. And, since $r_1 + r_2 + \cdots + r_n = 1$ and each row of q sums to zero, it follows that each row of the matrix $[r_j + \lambda q_{ij}]_{i,j \le n}$ sums to one. Thus, it suffices to show that the first $n-1$ columns of these two matrices agree. Fix some $j = 1, 2, \ldots, n-1$. We must show that column j of matrix p is equal to column j of matrix $[r_j + \lambda q_{ij}]_{i,j \le n}$.

Concerning column j of matrix p, we recall that column j of matrix p^j is given by $[m_i(A_j)]_{i \le n}$ and column j of matrix p^k for $k \ne j$ consists of all zeros. Hence, column j of matrix p is given by $(\frac{r_j}{1-r_n})[m_i(A_j)]_{i \le n}$. Then, by the claim, it follows that

$$\begin{aligned} \text{column } j \text{ of matrix } p &= \left(\frac{r_j}{1-r_n}\right)[m_i(A_j)]_{i \le n} \\ &= \left(\frac{r_j}{1-r_n}\right)\left[(1-r_n) + \lambda\left(\frac{1-r_n}{r_j}\right)q_{ij}\right]_{i \le n} \\ &= [r_j + \lambda q_{ij}]_{i \le n}. \end{aligned}$$

This establishes that column j of matrix p is equal to column j of matrix $[r_j + \lambda q_{ij}]_{i,j \le n}$ and, hence, completes the proof of the theorem. □

The theorem will be an important tool for us in Chapter 5.

We close this section by noting that a slightly more general form of Theorem 4.18 is true. As we mentioned following the definition of the FIPS (Definition 4.10), one can consider more general FIPSs in which the number of players need not equal the number of pieces in partitions of C. Then matrices in the FIPS need not be square. Theorem 4.18 and its proof easily generalize to this setting.

5

What the IPS and the FIPS Tell Us About Fairness and Efficiency in the General *n*-Player Context

In this chapter, we consider generalizations of our fairness and efficiency results from Chapter 3 to the general *n*-player context. In Section 5A, we consider fairness; in Section 5B, we consider efficiency; and in Section 5C, we consider fairness and efficiency together. In these sections, we assume that the measures are absolutely continuous with respect to each other. In Section 5D, we consider the situation when absolute continuity fails. In Section 5E, where we consider examples and open questions, absolute continuity will sometimes hold and sometimes fail.

5A. Fairness

We begin by recalling that, when there are two players, proportionality and envy-freeness correspond, as do strong proportionality, strong envy-freeness, and super envy-freeness. The following two facts are implied by Corollary 3.3 for the two-player context.

Fact 1: There exist infinitely many mutually non-*s*-equivalent partitions that are proportional and envy-free.

Fact 2: There exist partitions that are strongly proportional, strongly envy-free, and super envy-free if and only if the measures are not equal (and, in this case, there are infinitely many mutually non-*p*-equivalent such partitions).

Analogous facts for chores fairness properties are implied by Corollary 3.5.

The natural guess for the generalization of Fact 1 when there are more than two players is obvious and turns out to be true.

How to generalize Fact 2 is more difficult. As we discussed in Chapter 4, strong proportionality, strong envy-freeness, and super envy-freeness are not equivalent when there are more than two players, and we will need three generalizations of Fact 2, one for each of these three properties. We not only claim that there are three such generalizations of Fact 2, but that these generalizations are all quite natural. The three generalizations come from three different ways that the statement "the measures are not equal" can be generalized from the two-player context to the n-player context. Consider the following three statements:

a. The measures are not all equal.
b. No two of the measures are equal.
c. The measures are linearly independent.

Clearly these three statements are equivalent when there are exactly two measures. (Regarding statement c, notice that two measures are linearly dependent if and only if one is a scalar multiple of the other and, since any measure assigns value one to C, this scalar must be one.) If there are more than two measures, then this is not the case. Certainly, statement c implies statement b, and statement b implies statement a, but neither of the reverse implications holds. It turns out that these three generalizations of "the measures are not equal" correspond precisely to our three fairness properties. More specifically, Theorems 5.1 and 5.5 shall establish that

- there exist strongly proportional partitions if and only if the measures are not all equal.
- there exist strongly envy-free partitions if and only if no two of the measures are equal.
- there exist super envy-free partitions if and only if the measures are linearly independent.

As we have seen, the IPS is the appropriate structure for considering proportionality and strong proportionality (and the analogous chores properties), whereas the FIPS is the appropriate structure for considering envy-freeness, strong envy-freeness, and super envy-freeness (and the analogous chores properties). Accordingly, when discussing proportionality or strong proportionality, we shall be concerned about the numbers of mutually non-p-equivalent partitions satisfying these properties (or, equivalently, the number of proportional or strongly proportional p-classes), and when discussing envy-freeness, strong envy-freeness, or super envy-freeness, we shall be concerned about the number of mutually non-fp-equivalent partitions satisfying these properties (or, equivalently, the number of envy-free, strongly envy-free, or super envy-free fp-classes). We shall make an analogous distinction for the corresponding chores properties. We will show that, for all three of the preceding statements, if at least one partition of the given type exists then there are infinitely many such partitions that are mutually non-*p*-equivalent or mutually non-*fp*-equivalent, whichever equivalence is appropriate.

The results on proportionality and strong proportionality using the IPS will be easy to establish in much the same manner as in the two-player context (as in Theorem 3.2).

Theorem 5.1

a. *If the measures are all equal, then*
   i. *the IPS has exactly one proportional point, and that point is* $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.
   ii. *the IPS has no strongly proportional points.*

b. *If the measures are not all equal, then*
   i. *the IPS has infinitely many proportional points. In particular, for any* $q \in \mathbf{R}^n$ *with all non-negative coordinates and at least one positive coordinate, there are infinitely many* $\lambda > 0$ *such that* $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ *is a proportional point.*
   ii. *the IPS has infinitely many strongly proportional points. In particular, for any* $q \in \mathbf{R}^n$ *with all positive coordinates, there are infinitely many* $\lambda > 0$ *such that* $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ *is a strongly proportional point.*

Proof: The proof of part a is similar to the proof of part a of Theorem 3.2, with $(\frac{1}{2}, \frac{1}{2})$ replaced by $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, the use of Theorem 2.2 replaced by Theorem 4.2, and the line segment between $(1, 0)$ and $(0, 1)$ (i.e., the one-simplex) replaced by the $(n-1)$-simplex.

The proof of part bi is only slightly different from the proof of part bi of Theorem 3.2. The proof of this part of Theorem 3.2 used the fact that, when $n = 2$, the IPS is symmetric about $(\frac{1}{2}, \frac{1}{2})$. As we have discussed, there is no analogous result for general n. However, Corollary 4.9 gives us what we need. This result implies that if the simplex is a proper subset of the IPS then there are points of the IPS on both sides of the simplex. The rest of the proof of part bi is similar to the proof of part bi of Theorem 3.2.

The proof for part bii is the same, except that we must now insist that the coefficients of q be positive. □

We can think of q as pointing from the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ in a direction of non-increasing coordinates (for part bi) or strictly increasing coordinates (for part bii). The theorem tells us that we encounter infinitely many proportional or strongly proportional points as we move from $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ in the direction given by q.

The following corollary is the obvious generalization of Corollary 3.3. The proof is identical, except for the obvious changes ($(\frac{1}{2}, \frac{1}{2})$ replaced by

$(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, the line segment from $(1, 0)$ to $(0, 1)$ replaced by the $(n-1)$-simplex, and the use of Theorem 2.6 replaced by the use of Theorem 4.4).

Corollary 5.2

a. If the measures are all equal, then

- *i. there are infinitely many mutually non-s-equivalent proportional partitions.*
- *ii. all proportional partitions are p-equivalent.*
- *iii. there are no strongly proportional partitions.*

b. If the measures are not all equal, then

- *i. there are infinitely many mutually non-s-equivalent proportional partitions.*
- *ii. there are infinitely many mutually non-p-equivalent proportional partitions.*
- *iii. there are infinitely many mutually non-s-equivalent strongly proportional partitions.*
- *iv. there are infinitely many mutually non-p-equivalent strongly proportional partitions.*

Corollary 5.2 – Equivalence Class Version

a. If the measures are all equal, then

- *i. there are infinitely many proportional s-classes.*
- *ii. there is exactly one proportional p-class.*
- *iii. there are no strongly proportional s-classes or p-classes.*

b. If the measures are not all equal, then

- *i. there are infinitely many proportional s-classes*
- *ii. there are infinitely many proportional p-classes.*
- *iii. there are infinitely many strongly proportional s-classes.*
- *iv. there are infinitely many strongly proportional p-classes.*

Next, we give the chores versions of Theorem 5.1 and Corollary 5.2. The proofs are analogous and we omit them.

Theorem 5.3

a. If the measures are all equal, then

- *i. the IPS has exactly one c-proportional point, and that point is* $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.
- *ii. the IPS has no strongly c-proportional points.*

b. If the measures are not all equal, then

- *i. the IPS has infinitely many c-proportional points. In particular, for any* $q \in \mathbf{R}^n$ *with all non-positive coordinates and at least one negative* *coordinate, there are infinitely many* $\lambda > 0$ *such that* $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ *is a c-proportional point.*

ii. *the IPS has infinitely many strongly c-proportional points. In particular, for any* $q \in \mathbf{R}^n$ *with all negative coordinates, there are infinitely many* $\lambda > 0$ *such that* $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ *is a strongly c-proportional point.*

Corollary 5.4

a. *If the measures are all equal, then*
   i. *there are infinitely many mutually non-s-equivalent c-proportional partitions.*
   ii. *all c-proportional partitions are p-equivalent.*
   iii. *there are no strongly c-proportional partitions.*

b. *If the measures are not all equal, then*
   i. *there are infinitely many mutually non-s-equivalent c-proportional partitions.*
   ii. *there are infinitely many mutually non-p-equivalent c-proportional partitions.*
   iii. *there are infinitely many mutually non-s-equivalent strongly c-proportional partitions.*
   iv. *there are infinitely many mutually non-p-equivalent strongly c-proportional partitions.*

Corollary 5.4 – Equivalence Class Version

a. *If the measures are all equal, then*
   i. *there are infinitely many c-proportional s-classes.*
   ii. *there is exactly one c-proportional p-class.*
   iii. *there are no strongly c-proportional s-classes or p-classes.*

b. *If the measures are not all equal, then*
   i. *there are infinitely many c-proportional s-classes*
   ii. *there are infinitely many c-proportional p-classes.*
   iii. *there are infinitely many strongly c-proportional s-classes.*
   iv. *there are infinitely many strongly c-proportional p-classes.*

We now switch our focus from the IPS to the FIPS in order to discuss envy-freeness, strong envy-freeness, and super-envy-freeness. Theorem 4.18 will be a central tool. We recall that a point $p = [p_{ij}]_{i,j \leq n}$ of the FIPS is

- envy-free if and only if, for all $i, j = 1, 2, \ldots, n$, $p_{ii} \geq p_{ij}$ (i.e., each diagonal entry is at least as large as every other entry in its row).
- strongly envy-free if and only if, for all $i, j = 1, 2, \ldots, n$, $p_{ii} > p_{ij}$ (i.e., each diagonal entry is greater than every other entry in its row).

- super envy-free if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $p_{ii} > \frac{1}{n}$ and $p_{ij} < \frac{1}{n}$ (i.e., each diagonal entry is greater than $\frac{1}{n}$ and each non-diagonal entry is less than $\frac{1}{n}$).

Theorem 5.5

a. (Envy-freeness)

i. If the measures are all equal, then the FIPS has exactly one envy-free point, and that point is $[\frac{1}{n}]_{i,j \le n}$, i.e., the $n \times n$ matrix with all entries $\frac{1}{n}$.

ii. If the measures are not all equal, then the FIPS has infinitely many envy-free points.

b. (Strong envy-freeness)

The following are equivalent:

i. No two of the measures are equal.

ii. The FIPS has at least one strongly envy-free point.

iii. The FIPS has infinitely many strongly envy-free points.

c. (Super envy-freeness)

The following are equivalent:

i. The measures are linearly independent.

ii. The FIPS has at least one super envy-free point.

iii. The FIPS has infinitely many super envy-free points.

Before beginning the proof of Theorem 5.5, we state and prove a lemma. The lemma will be used in conjunction with Theorem 4.18.

Lemma 5.6 *There exists a proper matrix $q = [q_{ij}]_{i,j \le n}$ such that, for all $i, j = 1, 2, \ldots, n$, $q_{ii} \ge q_{ij}$, with equality holding if and only if $m_i = m_j$. (For the definition of proper matrix, see Definition 4.17.)*

Proof: We first define a matrix q''. We will use q'' to define a matrix q' and then we will use q' to define q.

By renumbering, if necessary, we may assume that, for some $s = 1, 2, \ldots, n$, the measures $m_1, m_2, \ldots, m_s$ are linearly independent and each of the measures $m_{s+1}, m_{s+2}, \ldots, m_n$ can be expressed as a linear combination of these measures.

For each $j = s+1, s+2, \ldots, n$, let $c^j = \langle c_1^j, c_2^j, \ldots, c_s^j \rangle$ be such that $m_j = c_1^j m_1 + c_2^j m_2 + \cdots + c_s^j m_s$. Then, for each such j, $c_1^j + c_2^j + \cdots + c_s^j = c_1^j m_1(C) + c_2^j m_2(C) + \cdots + c_s^j m_s(C) = m_j(C) = 1$. For each $j = 1, 2, \ldots, s$, let d^j be the vector in $\mathbf{R}^s$ that has a one in position j and zeros elsewhere, and for each $j = s+1, s+2, \ldots, n$, let $d^j = \frac{c^j}{|c^j|}$ (where $|c^j|$ denotes the magnitude of vector c^j). Notice that, for each $j = 1, 2, \ldots, n$, $|d^j| = 1$.

Define an $s \times n$ matrix q'' by declaring that, for each $j = 1, 2, \ldots, n$, column j of q'' is the vector d^j. Thus, the matrix consisting of the first s columns of q'' is simply the $s \times s$ identity matrix and, for each i and j with $i = 1, 2, \ldots, s$ and $j = s + 1, s + 2, \ldots, n$, the ij entry of q'' is $\frac{c_i^j}{|c^j|}$.

Next, we define an $n \times n$ matrix q'. For each $i = 1, 2, \ldots, s$, row i of q' is defined to be row i of q''. In other words, we are defining q' by starting with q'' on top and then defining the bottom $n-s$ rows of q'.

For each i with $i = s + 1, s + 2, \ldots, n$, row i of q' is defined to be the dot product of c^i and the corresponding column of q''. In other words, for each i and j with $i = s + 1, s + 2, \ldots, n$ and $j = 1, 2, \ldots, n$, the ij entry of q' is $c^i \bullet d^j = \sum_{k=1}^{s} c_k^i q'_{kj}$.

Our construction certainly implies that the columns of q' are consistent with all equations in DEP (see Definition 4.16) that involve the c^i. It follows that the columns in q' are consistent with *all* of the equations in DEP. (In fact, once we defined q'', our desire to have the columns of q' be consistent with the equations in DEP forced us to define the bottom $n-s$ rows of q' precisely as we did.)

There is no reason to believe that each row of q' sums to zero. Hence, q' need not be proper. We shall return to this issue shortly.

Claim q' satisfies the condition given in the statement of the lemma.

Proof of Claim: We must show that, for all $i, j = 1, 2, \ldots, n$, $q'_{ii} \geq q'_{ij}$, with equality holding if and only if $m_i = m_j$. We consider the following three cases.

Case 1: $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, s$. Then, $q'_{ii} = 1$ and, for $i \neq j$, $q'_{ij} = 0$. Then certainly the given condition is satisfied since, for the given values of i and j, we know that $m_i \neq m_j$, since $m_1, m_2, \ldots, m_s$ are linearly independent.

Case 2: $i = 1, 2, \ldots, s$ and $j = s + 1, s + 2, \ldots, n$. Then $q'_{ii} = 1$ and $q'_{ij} = \frac{c_i^j}{|c^j|}$. Since c_i^j is one component of the vector c^j, it follows that $c_i^j \leq |c^j|$ and, hence, $q'_{ij} = \frac{c_i^j}{|c^j|} \leq 1 = q'_{ii}$. Furthermore, $\frac{c_i^j}{|c^j|} = 1$ if and only if c^j is the vector with one in the ith position and zeros everywhere else, and this is so if and only if $m_i = m_j$. This establishes that $q'_{ii} \geq q'_{ij}$, with equality holding if and only if $m_i = m_j$, as desired.

Case 3: $i = s + 1, s + 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Recall that the entries in row i are the dot products of c^i with $d^1, d^2, \ldots, d^n$ and that each d^j has magnitude one. Also, $d^j = \frac{c^j}{|c^j|}$ has the same direction as c^j. Hence, $q'_{ii} = c^i \bullet d^i \geq c^i \bullet d^j = q'_{ij}$. Equality holds if and only if $d^i = d^j$, and this holds if and only if $m_i = m_j$. This establishes the claim.

We now focus on the need to have each row sum to zero. Define $q = [q_{ij}]_{i,j \leq n}$ to be the matrix obtained from q' by subtracting from each entry in q' the average of the entries in that entry's row. In other words, for all i, $j = 1, 2, \ldots, n$, $q_{ij} = q'_{ij} - (\frac{1}{n})\sum_{j=1}^{n} q'_{ij}$.

Clearly, each row of q sums to zero. We claim that the columns of q satisfy the equations of DEP. To establish this, we need only show that q satisfies all equations corresponding to the c^i.

Fix i and j with $i = s+1, s+2, \ldots, n$ and $j = 1, 2, \ldots, n$. We must show that $c_1^i q_{1j} + c_2^i q_{2j} + \cdots + c_s^i q_{sj} = q_{ij}$. We establish this as follows:

$$\begin{aligned}
&c_1^i q_{1j} + c_2^i q_{2j} + \cdots + c_s^i q_{sj} \\
&= c_1^i \left[q'_{1j} - \left(\frac{1}{n}\right) \sum_{j=1}^{n} q'_{1j} \right] + c_2^i \left[q'_{2j} - \left(\frac{1}{n}\right) \sum_{j=1}^{n} q'_{2j} \right] \\
&\quad + \cdots + c_s^i \left[q'_{sj} - \left(\frac{1}{n}\right) \sum_{j=1}^{n} q'_{sj} \right] \\
&= \left[c_1^i q'_{1j} + c_2^i q'_{2j} + \cdots + c_s^i q'_{sj} \right] \\
&\quad - \left(\frac{1}{n}\right) \left[c_1^i \sum_{j=1}^{n} q'_{1j} + c_2^i \sum_{j=1}^{n} q'_{2j} + \cdots + c_s^i \sum_{j=1}^{n} q'_{sj} \right] \\
&= \sum_{k=1}^{s} c_k^i q'_{kj} - \left(\frac{1}{n}\right) \left[\sum_{k=1}^{s} c_k^i q'_{k1} + \sum_{k=1}^{s} c_k^i q'_{k2} + \cdots + \sum_{k=1}^{s} c_k^i q'_{kn} \right] \\
&= (c^i \bullet d^j) - \left(\frac{1}{n}\right) [(c^i \bullet d^1) + (c^i \bullet d^2) + \cdots + (c^i \bullet d^n)] \\
&= q'_{ij} - \left(\frac{1}{n}\right) (q'_{i1} + q'_{i2} + \cdots + q'_{in}) \\
&= q'_{ij} - \left(\frac{1}{n}\right) \sum_{j=1}^{n} q'_{ij} = q_{ij}
\end{aligned}$$

This establishes that q is a proper matrix. All that remains is for us to show that q satisfies the condition given in the statement of the lemma. But this is trivial since, in passing from q' to q, the same number was subtracted from each entry in any given row and hence, since q' satisfies the given condition, so does q. This establishes the lemma. □

Proof of Theorem 5.5: For part a, we first note that Corollary 1.5 implies that the matrix $[\frac{1}{n}]_{i,j \le n}$ is in the FIPS. This point is clearly envy-free. For part ai, we assume that the measures are all equal and we must show that there are no envy-free points in the FIPS besides $[\frac{1}{n}]_{i,j \le n}$. Since the measures are all equal, all entries in any one column of a matrix in the FIPS must equal each other. Since the entries in any row must sum to one, it follows that, for any matrix in the FIPS other than $[\frac{1}{n}]_{i,j\le n}$, some column's entries must be less than $\frac{1}{n}$. Therefore, this matrix has a diagonal entry that is less than $\frac{1}{n}$. It follows that such a matrix is not envy-free. Hence, $[\frac{1}{n}]_{ij\le n}$ is the only envy-free point in the FIPS.

For part aii, we assume that the measures are not all equal. Let $q = [q_{ij}]_{i,j\le n}$ be as in Lemma 5.6. By Theorem 4.18 with this q and with $r = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, we know that for some $\lambda > 0$, $[\frac{1}{n} + \lambda q_{ij}]_{i,j\le n}$ ∈FIPS. It is easy to see that this matrix is envy-free since, for any $i, j = 1, 2, \dots, n$, $q_{ii} \ge q_{ij}$ and, hence, since $\lambda > 0$, $\frac{1}{n} + \lambda q_{ii} \ge \frac{1}{n} + \lambda q_{ij}$. In other words, the matrix $[\frac{1}{n} + \lambda q_{ij}]_{i,j\le n}$ is envy-free since each diagonal entry of this matrix is greater than or equal to any other entry in its row.

As discussed earlier, the matrix $[\frac{1}{n}]_{i,j\le n}$ is an envy-free member of the FIPS. We claim that $[\frac{1}{n} + \lambda q_{ij}]_{i,j\le n} \neq [\frac{1}{n}]_{i,j\le n}$. Since $\lambda \neq 0$, it suffices to show that not all of the q_{ij} are equal to zero. This follows easily from that fact that the measures are not all equal and, if $m_i \neq m_j$, then $q_{ii} > q_{ij}$. Thus, there are at least two distinct envy-free points in the FIPS. By the convexity of the FIPS, any point on the line segment between these two points is in the FIPS. (For the definition of "line segment" in this context, see Definition 4.14.) It is easy to see that any point on this line segment is envy-free. Therefore, there are infinitely many envy-free points in the FIPS. This establishes part a of the theorem.

For part b, we shall show that condition bi implies condition bii, condition bii implies condition bi, condition bii implies condition biii, and condition biii implies condition bii.

To show that condition bi implies condition bii, we assume that no two of the measures are equal. We must show that the FIPS has at least one strongly envy-free point. We use Lemma 5.6 and Theorem 4.18 in a manner similar to our preceding use of these results. Let $q = [q_{ij}]_{i,j\le n}$ be as in the lemma. Since no two of the measures are equal, we know that for distinct $i, j = 1, 2, \dots, n$, $q_{ii} > q_{ij}$. Theorem 4.18 with $r = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ implies that, for some $\lambda > 0$, $[\frac{1}{n} + \lambda q_{ij}]_{i,j\le n} \in$ FIPS. Clearly, $[\frac{1}{n} + \lambda q_{ij}]_{i,j\le n}$ is strongly envy-free since $\lambda > 0$ and, for distinct $i, j = 1, 2, \dots, n$, $q_{ii} > q_{ij}$, and hence $\frac{1}{n} + \lambda q_{ii} > \frac{1}{n} + \lambda q_{ij}$. In other words, the matrix $[\frac{1}{n} + \lambda q_{ij}]_{i,j\le n}$ is strongly envy-free because each diagonal entry of this matrix is greater than any other entry in its row.

To show that condition bii implies condition bi, we assume that $p = [p_{ij}]_{i,j\le n} \in$ FIPS is strongly-envy free and we assume, by way of contradiction, that for some distinct $i, j = 1, 2, \dots, n$, $m_i = m_j$. Since p is strongly envy-free, it follows that $p_{ii} > p_{ij}$ and $p_{jj} > p_{ji}$. But, $m_i = m_j$ implies that row i and row j of p are equal. Hence, $p_{ii} > p_{ij} = p_{jj} > p_{ji} = p_{ii}$. This is a contradiction. Hence, no two of the measures are equal.

Next, we must show that condition bii implies condition biii. We assume that p is a strongly envy-free point we and let $p' = [\frac{1}{n}]_{i,j \le n}$. We know that $p' \in$ FIPS. Clearly, p' is not strongly envy-free and so $p \neq p'$. By convexity, every point on the line segment between p and p' is in the FIPS. It is easy to see that every point on this line segment, with the exception of p', is strongly envy-free. Hence, there are infinitely many strongly envy-free points in the FIPS.

Condition biii trivially implies condition bii. This establishes part b of the theorem.

For part c, we shall show that condition ci implies condition cii, condition cii implies condition ci, condition cii implies condition ciii, and condition ciii implies condition cii.

To show that condition ci implies condition cii, we assume that the measures are linearly independent. Let $r = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ and let

$$q = [q_{ij}]_{i,j \le n} = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{n-1} & \text{if } i \neq j. \end{cases}$$

In other words, q is the $n \times n$ matrix with all diagonal entries equal to one and all non-diagonal entries equal to $-\frac{1}{n-1}$. Then each row of q sums to one. Also, since the measures are linearly independent, DEP $= \emptyset$. Therefore, q is a proper matrix. By Theorem 4.18, let $\lambda > 0$ be such that $[\frac{1}{n} + \lambda q_{ij}]_{i,j \le n} \in$ FIPS. Since $\lambda > 0$, it follows that, for each $i = 1, 2, \ldots, n$, $m_i(P_i) = \frac{1}{n} + \lambda q_{ii} = \frac{1}{n} + \lambda > \frac{1}{n}$ and, for distinct $i, j = 1, 2, \ldots, n$, $m_i(P_j) = \frac{1}{n} + \lambda q_{ij} = \frac{1}{n} - \frac{\lambda}{n-1} < \frac{1}{n}$. In other words, the matrix $[\frac{1}{n} + \lambda q_{ij}]_{i,j \le n}$ has diagonal entries that are each greater than $\frac{1}{n}$ and non-diagonal entries that are each less than $\frac{1}{n}$. This establishes that $[\frac{1}{n} + \lambda q_{ij}]_{i,j \le n}$ is super envy-free.

To show that condition cii implies condition ci, we assume that the FIPS has a point that is super envy-free. In particular, let us assume that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a super envy-free partition and suppose, by way of contradiction, that the measures $m_1, m_2, \ldots, m_n$ are linearly dependent. Then, for some constants $k_1, k_2, \ldots, k_n$, not all zero, $k_1 m_1 + k_2 m_2 + \cdots + k_n m_n = 0$. This implies that there are positive constants $\alpha_1, \alpha_2, \ldots, \alpha_s, \beta_1, \beta_2, \ldots, \beta_t$, and disjoint subsets $\{a_1, a_2, \ldots, a_s\}$ and $\{b_1, b_2, \ldots, b_t\}$ of $\{1, 2, \ldots, n\}$, where $s, t > 0$, such that $\alpha_1 m_{a_1} + \alpha_2 m_{a_2} + \cdots + \alpha_s m_{a_s} = \beta_1 m_{b_1} + \beta_2 m_{b_2} + \cdots + \beta_t m_{b_t}$.

We note that, since each measure assigns value one to C,

$$\begin{aligned} \alpha_1 + \alpha_2 + \cdots + \alpha_s &= \alpha_1 m_{a_1}(C) + \alpha_2 m_{a_2}(C) + \cdots + \alpha_s m_{a_s}(C) \\ &= \beta_1 m_{b_1}(C) + \beta_2 m_{b_2}(C) + \cdots + \beta_t m_{b_t}(C) \\ &= \beta_1 + \beta_2 + \cdots + \beta_t. \end{aligned}$$

Hence, $\alpha_1 + \alpha_2 + \cdots + \alpha_s = \beta_1 + \beta_2 + \cdots + \beta_t$. Call this common value λ.

We claim that, for every $j = 1, 2, \ldots, n$, $\alpha_1 m_{a_1}(P_j) + \alpha_2 m_{a_2}(P_j) + \cdots + \alpha_s m_{a_s}(P_j) < \frac{\lambda}{n}$. Fix such a j and suppose first that $j \notin \{a_1, a_2, \ldots, a_s\}$. For every $i = 1, 2, \ldots, s$, our super envy-freeness assumption implies that $m_{a_i}(P_j) < \frac{1}{n}$. It follows that

$$\alpha_1 m_{a_1}(P_j) + \alpha_2 m_{a_2}(P_j) + \cdots + \alpha_s m_{a_s}(P_j)$$
$$< (\alpha_1 + \alpha_2 + \cdots + \alpha_s)\left(\frac{1}{n}\right) = \frac{\lambda}{n}.$$

Suppose now that $j \in \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$. Then, $j \notin \{b_1, b_2, \ldots, b_t\}$. Arguing precisely as above, it follows that $\beta_1 m_{b_1}(P_j) + \beta_2 m_{b_2}(P_j) + \cdots + \beta_t m_{b_t}(P_j) < \frac{\lambda}{n}$. But, $\alpha_1 m_{a_1} + \alpha_2 m_{a_2} + \cdots + \alpha_s m_{a_s} = \beta_1 m_{b_1} + \beta_2 m_{b_2} + \cdots + \beta_t m_{b_t}$ and, thus, $\alpha_1 m_{a_1}(P_j) + \alpha_2 m_{a_2}(P_j) + \cdots + \alpha_s m_{a_s}(P_j) < \frac{\lambda}{n}$. This establishes that, for *every* $j = 1, 2, \ldots, n$,

$$\alpha_1 m_{a_1}(P_j) + \alpha_2 m_{a_2}(P_j) + \cdots + \alpha_s m_{a_s}(P_j) < \frac{\lambda}{n}.$$

Then we have

$$\begin{aligned}
\lambda &= \alpha_1 + \alpha_2 + \cdots + \alpha_s \\
&= \alpha_1 m_{a_1}(C) + \alpha_2 m_{a_2}(C) + \cdots + \alpha_s m_{a_s}(C) \\
&= \alpha_1 \sum_{j=1}^{n} m_{a_1}(P_j) + \alpha_2 \sum_{j=1}^{n} m_{a_2}(P_j) + \cdots + \alpha_s \sum_{j=1}^{n} m_{a_s}(P_j) \\
&= \sum_{j=1}^{n} [\alpha_1 m_{a_1}(P_j) + \alpha_2 m_{a_2}(P_j) + \cdots + \alpha_s m_{a_s}(P_j)] < \sum_{j=1}^{n} \frac{\lambda}{n} = \lambda.
\end{aligned}$$

This is a contradiction; thus, we have established that the measures are linearly independent.

The proof that condition cii implies condition ciii is analogous to the proof that condition bii implies condition biii.

Condition ciii trivially implies condition cii. This completes the proof of the theorem. □

Corollary 5.7

a. (Envy-freeness)

i. If the measures are all equal then

- *there are infinitely many mutually non-fs-equivalent envy-free partitions.*
- *all envy-free partitions are fp-equivalent.*

ii. *If the measures are not all equal, then*
  - *there are infinitely many mutually non-fs-equivalent envy-free partitions.*
  - *there are infinitely many mutually non-fp-equivalent envy-free partitions.*

b. *(Strong envy-freeness) The following are equivalent:*
  i. *No two of the measures are equal.*
  ii. *There is at least one strongly envy-free partition.*
  iii. *There are infinitely many non-fs-equivalent strongly envy-free partitions.*
  iv. *There are infinitely many non-fp-equivalent strongly envy-free partitions.*

c. *(Super envy-freeness) The following are equivalent:*
  i. *The measures are linearly independent.*
  ii. *There is at least one super envy-free partition.*
  iii. *There are infinitely many non-fs-equivalent super envy-free partitions.*
  iv. *There are infinitely many non-fp-equivalent super envy-free partitions.*

Proof: For part ai, we assume that the measures are all equal and we set $p = [\frac{1}{n}]_{i,j\leq n}$, the $n \times n$ matrix with all entries $\frac{1}{n}$. Then, by part ai of the theorem, $p \in$ FIPS and p is envy-free. Let P be a partition such that $m_F(P) = p$. Then P is an envy-free partition.

We claim that p is an interior point of a line segment that is contained in the FIPS. By Corollary 1.5, there exists a partition $Q = \langle Q_1, Q_2, \ldots, Q_n\rangle$ such that, for each $i = 1, 2, \ldots, n$, $m_i(Q_1) = \frac{2}{n}$, $m_i(Q_2) = 0$, and, for each j with $j = 3, 4, \ldots, n$, $m_i(Q_j) = \frac{1}{n}$. Similarly, there exists a partition $R = \langle R_1, R_2, \ldots, R_n\rangle$ such that, for each $i = 1, 2, \ldots, n$, $m_i(R_1) = 0$, $m_i(R_2) = \frac{2}{n}$, and, for each j with $j = 3, 4, \ldots, n$, $m_i(R_j) = \frac{1}{n}$. Then $p = \frac{1}{2}m_F(Q) + \frac{1}{2}m_F(R)$, and so p is an interior point of the line segment connecting $m_F(Q)$ and $m_F(R)$. By convexity, this line segment is contained in the FIPS.

Theorem 4.15 tells us that p is the image, under m_F, of infinitely many mutually non-*fs*-equivalent partitions. Hence, there are infinitely many mutually non-*fs*-equivalent partitions that are *fp*-equivalent to P. Clearly, any partition that is *fp*-equivalent to P is envy-free. This establishes that there are infinitely many mutually non-*fs*-equivalent envy-free partitions.

We must show that if the measures are all equal, then all envy-free partitions are *fp*-equivalent. This follows immediately from part ai of the theorem. This establishes part ai.

For part aii, we assume that the measures are not all equal and we note that the first statement follows from the second, since non-*fp*-equivalence implies non-*fs*-equivalence. Hence, we must show that there are infinitely many non-*fp*-equivalent envy-free partitions. This is implied by part aii of the theorem and the fact that distinct points of the FIPS are the image, under m_F, of non-*fp*-equivalent partitions. This establishes part aii.

For part b, we claim that condition bi implies condition bii, condition bii implies condition biv, condition biv implies condition biii, and condition biii implies condition bi. The first and second of these implications follow immediately from part b of the theorem. The third implication is trivial, since non-*fp*-equivalence implies non-*fs*-equivalence. For the fourth implication, we simply note that if there are infinitely many non-*fs*-equivalent strongly envy-free partitions, then certainly the FIPS has at least one strongly envy-free point and so, by part b of the theorem, no two of the measures are equal. This establishes part b.

Part c follows from part c of the theorem in the same manner that part b followed from part b of the theorem. □

Corollary 5.7 – Equivalence Class Version

a. (Envy-freeness)

i. If the measures are all equal then
- *there are infinitely many envy-free fs-classes.*
- *there is exactly one envy-free fp-class.*

ii. If the measures are not all equal, then
- *there are infinitely many envy-free fs-classes.*
- *there are infinitely many envy-free fp-classes.*

b. (Strong envy-freeness) The following are equivalent:

i. No two of the measures are equal.
ii. There is at least one strongly envy-free fs-class.
iii. There is at least one strongly envy-free fp-class.
iv. There are infinitely many strongly envy-free fs-classes.
v. There are infinitely many strongly envy-free fp-classes.

c. (Super envy-freeness) The following are equivalent:

i. The measures are linearly independent.
ii. There is at least one super envy-free fs-class.
iii. There is at least one super envy-free fp-class.
iv. There are infinitely many super envy-free fs-classes.
v. There are infinitely many super envy-free fp-classes.

The chores versions of Theorems 5.5 and Corollary 5.7 are the following. The proofs are similar to the proofs of Theorem 5.5 and Corollary 5.7, respectively, and we omit them except to note that each time Lemma 5.6 is used to obtain matrix q, we use $-q$ rather than q when applying Theorem 4.18.

Theorem 5.8

a. (c-Envy-freeness)

i. If the measures are all equal, then the FIPS has exactly one c-envy-free point, and that point is $[\frac{1}{n}]_{i,j\leq n}$, *i.e., the* $n \times n$ *matrix with all entries* $\frac{1}{n}$.

ii. If the measures are not all equal, then the FIPS has infinitely many c-envy-free points.

b. (Strong c-envy-freeness) The following are equivalent:

i. No two of the measures are equal.

ii. The FIPS has at least one strongly c-envy-free point.

iii. The FIPS has infinitely many strongly c-envy-free points.

c. (Super c-envy-freeness) The following are equivalent:

i. The measures are linearly independent.

ii. The FIPS has at least one super c-envy-free point.

iii. The FIPS has infinitely many super c-envy-free points.

Corollary 5.9

a. (c-Envy-freeness)

i. If the measures are all equal then

- *there are infinitely many mutually non-fs-equivalent c-envy-free partitions.*
- *all c-envy-free partitions are fp-equivalent.*

ii. If the measures are not all equal, then

- *there are infinitely many mutually non-fs-equivalent c-envy-free partitions.*
- *there are infinitely many mutually non-fp-equivalent c-envy-free partitions.*

b. (Strong c-envy-freeness) The following are equivalent:

i. No two of the measures are equal.

ii. There is at least one strongly c-envy-free partition.

iii. There are infinitely many non-fs-equivalent strongly c-envy-free partitions.

iv. There are infinitely many non-fp-equivalent strongly c-envy-free partitions.

c. (Super c-envy-freeness) The following are equivalent:

i. The measures are linearly independent.

ii. There is at least one super c-envy-free partition.

iii. There are infinitely many non-fs-equivalent super c-envy-free partitions.

iv. There are infinitely many non-fp-equivalent super c-envy-free partitions.

Corollary 5.9 – Equivalence Class Version

a. (c-Envy-freeness)

i. If the measures are all equal then
- *there are infinitely many c-envy-free fs-classes.*
- *there is exactly one c-envy-free fp-class.*

ii. If the measures are not all equal, then
- *there are infinitely many c-envy-free fs-classes.*
- *there are infinitely many c-envy-free fp-classes.*

b. (Strong c-envy-freeness) The following are equivalent:

i. No two of the measures are equal.
ii. There is at least one strongly c-envy-free fs-class.
iii. There is at least one strongly c-envy-free fp-class.
iv. There are infinitely many strongly c-envy-free fs-classes.
v. There are infinitely many strongly c-envy-free fp-classes.

c. (Super c-envy-freeness) The following are equivalent:

i. The measures are linearly independent.
ii. There is at least one super c-envy-free fs-class.
iii. There is at least one super c-envy-free fp-class.
iv. There are infinitely many super c-envy-free fs-classes.
v. There are infinitely many super c-envy-free fp-classes.

We close this section by noting that the set of all points in the IPS or the FIPS (whichever is appropriate) having any given fairness property is convex. In other words:

- The set of all proportional points in the IPS is a convex subset of the IPS.
- The set of all strongly proportional points in the IPS is a convex subset of the IPS.
- The set of all envy-free points in the FIPS is a convex subset of the FIPS.
- The set of all strongly envy-free points in the FIPS is a convex subset of the FIPS.
- The set of all super envy-free points in the FIPS is a convex subset of the FIPS.

Analogous facts hold for the chores fairness properties. The proofs are straightforward and we omit them.

5B. Efficiency

We now turn from fairness issues to efficiency issues for the general case of n players. We examined efficiency in the two-player context in Section 3B. Our presentation here parallels that of Section 3B. All of the relevant definitions from the two-player context generalize in a natural way to the n-player context. We shall see that this is not true of all theorems. For convenience, we state the generalizations of Definitions 3.6 and 3.7.

Definition 5.10 Suppose $p = (p_1, p_2, \ldots, p_n) \in \text{IPS}$.

a. p is a *Pareto maximal point* if there is no $q = (q_1, q_2, \ldots, q_n) \in \text{IPS}$ such that, for each $i = 1, 2, \ldots, n$, $q_i \geq p_i$, with at least one of these inequalities being strict.
b. p is a *Pareto minimal point* if there is no $q = (q_1, q_2, \ldots, q_n) \in \text{IPS}$ such that, for each $i = 1, 2, \ldots, n$, $q_i \leq p_i$, with at least one of these inequalities being strict.

Just as we say that partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is Pareto bigger than partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ if and only if, for each $i = 1, 2, \ldots, n$, $m_i(P_i) \geq m_i(Q_i)$, with at least one of these inequalities being strict, we shall say that point $p = (p_1, p_2, \ldots, p_n)$ is *Pareto bigger* than point $q = (q_1, q_2, \ldots, q_n)$ if and only if, for each $i = 1, 2, \ldots, n$, $p_i \geq q_i$, with at least one of these inequalities being strict. Similarly, we shall say that point $p = (p_1, p_2, \ldots, p_n)$ is *Pareto smaller* than point $q = (q_1, q_2, \ldots, q_n)$ if and only if, for each $i = 1, 2, \ldots, n$, $p_i \leq q_i$, with at least one of these inequalities being strict. Then, a point p is Pareto maximal if and only if no point is Pareto bigger than p, and a point p is Pareto minimal if and only if no point is Pareto smaller than p.

For any $(p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$, define $B^+(p_1, p_2, \ldots, p_n) = \{(q_1, q_2, \ldots, q_n) \in \mathbf{R}^n\colon \text{ for each } i = 1, 2, \ldots, n, q_i \geq p_i\}$ and $B^-(p_1, p_2, \ldots, p_n) = \{(q_1, q_2, \ldots, q_n) \in \mathbf{R}^n\colon \text{for each } i = 1, 2, \ldots, n, q_i \leq p_i\}$.

Definition 5.11

a. The *outer boundary* of the IPS consists of all points $(p_1, p_2, \ldots, p_n)$ on the boundary of the IPS for which $p_1 + p_2 + \cdots + p_n \geq 1$, and the *inner boundary* of the IPS consists of all points $(p_1, p_2, \ldots, p_n)$ on the boundary of the IPS for which $p_1 + p_2 + \cdots + p_n \leq 1$.
b. The *outer Pareto boundary* of the IPS consists of all points $(p_1, p_2, \ldots, p_n) \in \text{IPS}$ for which $B^+(p_1, p_2, \ldots, p_n) \cap \text{IPS} = \{(p_1, p_2, \ldots, p_n)\}$, and the *inner Pareto boundary* of the IPS consists of all points $(p_1, p_2, \ldots, p_n) \in \text{IPS}$ for which $B^-(p_1, p_2, \ldots, p_n) \cap \text{IPS} = \{(p_1, p_2, \ldots, p_n)\}$. The *Pareto boundary* is the union of the outer Pareto boundary and the inner Pareto boundary,

Just as in the two-player context, the definitions of outer and inner Pareto boundary make it easy to describe which points on the IPS are Pareto maximal and which are Pareto minimal.

Theorem 5.12

a. *The outer Pareto boundary of the IPS consists precisely of the set of all Pareto maximal points of the IPS.*
b. *The inner Pareto boundary of the IPS consists precisely of the set of all Pareto minimal points of the IPS.*

Theorem 3.9 marks our point of departure from Section 3B. This result told us that when there are two players the boundary of the IPS is equal to the Pareto boundary of the IPS. In other words, every Pareto maximal point and every Pareto minimal point is on the boundary of the IPS, and every point on the boundary of the IPS is either Pareto maximal or Pareto minimal. In Section 3D, we saw (see Theorem 3.22) that this result is false if the measures are not absolutely continuous with respect to each other. Now we shall see that, even with absolute continuity, this result does not hold generally if there are more than two players. It is easy to see that the Pareto boundary of the IPS is always a subset of the boundary of the IPS. The following theorem tells us that when there are more than two players, the reverse inclusion holds only in a special case.

Theorem 5.13 *Assume that there are more than two players.*

a. *The outer Pareto boundary of the IPS is a proper subset of the outer boundary of the IPS unless the measures are all equal, in which case the outer Pareto boundary is equal to the outer boundary.*
b. *The inner Pareto boundary of the IPS is a proper subset of the inner boundary of the IPS unless the measures are all equal, in which case the inner Pareto boundary is equal to the inner boundary.*

Before proving the theorem, we present an example that will be useful in the proof and will also be used later in this chapter.

Example 5.14 We assume that there are three players, Player 1, Player 2, and Player 3, with measures m_1, m_2, and m_3, respectively, where $m_1 \neq m_2$.

Let $\text{IPS}_{12} = \{m(P) : P = \langle P_1, P_2, \emptyset \rangle$ is a partition of $C\}$. In other words, IPS_{12} is the subset of the IPS associated with the set of partitions of C among the three players in which Player 3 gets nothing. Then IPS_{12} is a closed and convex subset of the xy plane. In particular, IPS_{12} is the intersection of the IPS

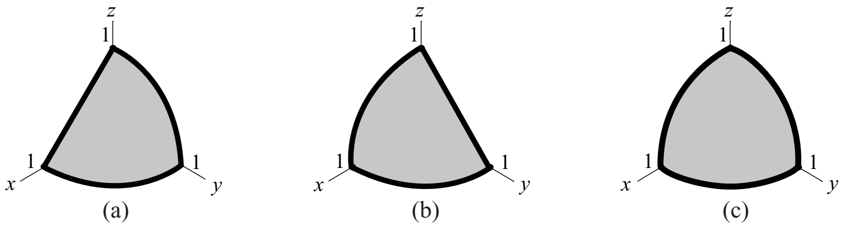


Figure 5.1

and the xy plane. Similarly, let $\text{IPS}_{13} = \{m(P) : P = \langle P_1, \emptyset, P_3 \rangle$ is a partition of $C\}$ and $\text{IPS}_{23} = \{m(P) : P = \langle \emptyset, P_2, P_3 \rangle$ is a partition of $C\}$.

Figure 5.1 illustrates three possibilities for the IPS. In Figure 5.1a we have assumed that $m_1 = m_3$, in Figure 5.1b we have assumed that $m_2 = m_3$, and in Figure 5.1c we have assumed that $m_1 \neq m_3$ and $m_2 \neq m_3$. In all three figures, we have highlighted the intersection of the outer boundary of the IPS with each of the coordinate planes (or, at least, the part of the intersection that is visible, given the perspective in the figures). In Figure 5.1a, we see that, since $m_1 = m_3$, IPS_{13} is the line segment between the points $(1, 0, 0)$ and $(0, 0, 1)$, and so the intersection of the outer boundary of the IPS with the xz plane consists of just this line segment. Since $m_1 \neq m_2$ and $m_2 \neq m_3$, IPS_{12} and IPS_{23} include more that just the line segment between $(1, 0, 0)$ and $(0, 1, 0)$ and the line segment between $(0, 1, 0)$ and $(0, 0, 1)$, respectively. Therefore, the intersection of the outer boundary of the IPS with the xy plane and with the yz plane includes points that are farther from the origin than these line segments. In Figure 5.1b, the situation is similar, with the roles of Player 1 and Player 2 reversed. In Figure 5.1c, none of the measures are equal and thus the intersection of the outer boundary of the IPS with each of the coordinate planes includes points that are farther from the origin than is the line segment connecting the vertices of the two players corresponding to that coordinate plane.

Proof of Theorem 5.13: We will prove both parts together. First, we assume that the measures are all equal. It follows (by Theorem 4.2) that the IPS consists precisely of the simplex. In this case, it is clear that the outer Pareto boundary, the inner Pareto boundary, the outer boundary, and the inner boundary are each equal to the simplex. Hence, the outer Pareto boundary is equal to the outer boundary, and the inner Pareto boundary is equal to the inner boundary.

Next, we assume that the measures are not all equal. We know that the outer Pareto boundary is a subset of the outer boundary and the inner Pareto boundary is a subset of the inner boundary. We must show that each of these inclusions is proper.

For convenience, we assume that there are three players, Player 1, Player 2, and Player 3, with measures m_1, m_2, and m_3, respectively. It will be clear that our proof will work for any number of players greater than two. Let us also assume, by renumbering, if necessary, that $m_1 \neq m_2$.

Consider IPS_{12}, as given in Example 5.14 and illustrated in (any of the three parts of) Figure 5.1. Fix any point $(p_1, p_2, 0)$ of IPS_{12} that is an interior point of the line segment between $(1, 0, 0)$ and $(0, 1, 0)$. Then certainly $(p_1, p_2, 0) \in$ IPS and, since no point of the IPS has a negative third coordinate, we know that $(p_1, p_2, 0)$ is on the boundary of the IPS. Also, since $p_1 + p_2 = 1$, it follows that $(p_1, p_2, 0)$ is on both the outer and the inner boundary of the IPS. However, it is clear that $(p_1, p_2, 0)$ is on neither the outer nor the inner Pareto boundary of the IPS, since $B^+(p_1, p_2, 0)$ and $B^-(p_1, p_2, 0)$ each contain many points of IPS_{12} and hence many points of the IPS. Thus, $(p_1, p_2, 0)$ is not on the Pareto boundary. This establishes that the outer Pareto boundary is a proper subset the outer boundary and the inner Pareto boundary is a proper subset the inner boundary. □

Then (in sharp contrast to the situation when there are two players) we have the following corollary.

Corollary 5.15 *There are points on the boundary of the IPS that do not correspond to Pareto maximal or to Pareto minimal partitions.*

The proof of Theorem 3.9 relied on Lemma 3.10, which told us that the only points of the IPS that lie on the unit square are $(1, 0)$ and $(0, 1)$. Thus, one perspective we may take is that the failure of Theorem 3.9 to generalize to the context of n players rests on the failure of Lemma 3.10 to generalize. As we saw in the proof of Theorem 5.13 (and illustrated in Figure 5.1), when $n = 3$ and the measures are not all equal, it is not the case that the only points of the IPS that lie on the unit cube are the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The next result shows that the natural generalization of Theorem 3.9 does hold, except possibly at such points. In other words, Theorem 3.9 does generalize if we restrict our attention to points in the interior of the unit hypercube.

Theorem 5.16 *Suppose that $p = (p_1, p_2, \ldots, p_n) \in$ IPS and p is an interior point of the unit hypercube (i.e., for all $i = 1, 2, \ldots, n$, $0 < p_i < 1$). Then*

a. p is on the outer boundary of the IPS if and only if it is on the outer Pareto boundary of the IPS and

b. p is on the inner boundary of the IPS if and only if it is on the inner Pareto boundary of the IPS.

The proof will use the following simple observations. The convexity of the IPS implies that any line through the origin that intersects the IPS does so in a single line segment (where we consider a single point to be a line segment of length zero). Each of the two endpoints of the line segment is on the boundary of the IPS, where the point farther from the origin is on the outer boundary and the point closer to the origin is on the inner boundary. The interior points of the line segment are on the boundary of the IPS if and only if the line segment lies on the surface of the unit hypercube.

Proof of Theorem 5.16: Fix $p = (p_1, p_2, \ldots, p_n)$ as in the statement of the theorem.

For the forward direction of part a, let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition such that $m(P) = p$ and suppose that p is not on the outer Pareto boundary. Then p is not a Pareto maximal point and P is not a Pareto maximal partition. Let $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ be a partition that is Pareto bigger than P. Then, for each $i = 1, 2, \ldots, n$, $m_i(Q_i) \geq m_i(P_i)$, where at least one of these inequalities is strict. By Corollary 1.7, for each such i, we may choose $R_i \subseteq Q_i$ such that $m_i(R_i) = m_i(P_i)$. Define a partition $S = \langle S_1, S_2, \ldots, S_n \rangle$ of C as follows: for each $i = 1, 2, \ldots, n$,

$$S_i = \begin{cases} R_i \bigcup \left(\bigcup_{j \leq n} (Q_j \backslash R_j) \right) & \text{if } i = 1 \\ R_i & \text{if } i \neq 1 \end{cases}$$

We may view partition S as arising from partition Q by having all players give to Player 1 an amount of cake equal to the excess of cake beyond the amount that they would have received with partition P. Notice that for at least one j, $Q_j \backslash R_j$ has positive measure and, hence, $m_1(S_1) > m_1(R_1)$.

For each $i = 2, 3, \ldots, n$, $m_i(S_i) = m_i(R_i) = m_i(P_i)$ and $m_1(S_1) > m_1(R_1) = m_1(P_1)$. Let $s = m(S)$. Then $s \in$ IPS, and s and p agree in all coordinates except the first, where s is bigger. Hence, for some $\varepsilon_1 > 0$, $s = (p_1 + \varepsilon_1, p_2, \ldots, p_n)$. In a similar manner, we can find $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n > 0$ such that $(p_1, p_2 + \varepsilon_2, p_3, p_4, \ldots p_{n-1}, p_n), (p_1, p_2, p_3 + \varepsilon_3, p_4, \ldots, p_{n-1}, p_n), \ldots, (p_1, p_2, p_3, p_4, \ldots p_{n-1}, p_n + \varepsilon_n) \in$IPS. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. The convexity of the IPS implies that $(p_1 + \varepsilon, p_2, p_3, \ldots, p_{n-1}, p_n)$, $(p_1, p_2 + \varepsilon, p_3, \ldots, p_{n-1}, p_n), \ldots, (p_1, p_2, p_3, \ldots, p_{n-1}, p_n + \varepsilon_n) \in$ IPS.

Consider the n numbers $\frac{p_1}{p_1+p_2+\cdots+p_n}, \frac{p_2}{p_1+p_2+\cdots+p_n}, \ldots, \frac{p_n}{p_1+p_2+\cdots+p_n}$. These numbers are all positive and they sum to one. Let $t = (\frac{p_1}{p_1+p_2+\cdots+p_n})(p_1 + \varepsilon, p_2, p_3, \ldots, p_{n-1}, p_n) + (\frac{p_2}{p_1+p_2+\cdots+p_n})(p_1, p_2 + \varepsilon, p_3, \ldots, p_{n-1}, p_n) + \cdots + (\frac{p_n}{p_1+p_2+\cdots+p_n})(p_1, p_2, p_3, \ldots, p_{n-1}, p_n + \varepsilon)$. This is a convex combination of elements of the IPS and hence $t \in$ IPS. Straightforward simplification tells us that $t = (1 + \frac{\varepsilon}{p_1+p_2+\cdots+p_n})p$. This implies that p, t, and the origin are collinear. Let ℓ be the line containing these points. Since $p_1 + p_2 + \cdots + p_n > 0$ and $\varepsilon > 0$, it follows that $(1 + \frac{\varepsilon}{p_1+p_2+\cdots+p_n}) > 1$. Hence, we see that p is strictly between t and the origin. By our preceding observation that $t \in$ IPS, this implies that p is not on the outer boundary of the IPS.

The reverse direction of part a is immediate, since we know (by Theorem 5.13) that the outer Pareto boundary is a subset of the outer boundary. This establishes part a. The proof for part b is similar. □

The theorem immediately yields the following corollary.

Corollary 5.17 *If p is on the boundary but not on the Pareto boundary of the IPS, then p lies on the surface of the unit hypercube.*

We continue to parallel the discussion of Section 3B. The next result in that section, Theorem 3.11, states that when there are two players there are infinitely many Pareto maximal points and infinitely many Pareto minimal points in the IPS. In particular, this result told us that we hit a Pareto maximal point and a Pareto minimal point when we move from the origin in any direction into the first quadrant. The analogous result for the general case of n players is the following.

Theorem 5.18 *The IPS has infinitely many Pareto maximal points and infinitely many Pareto minimal points. In particular, for any $p \in \mathbf{R}^n$ with all non-negative coordinates and at least one positive coordinate, there are positive numbers λ_1 and λ_2 such that $\lambda_1 p$ is a Pareto maximal point and $\lambda_2 p$ is a Pareto minimal point.*

We can think of p as pointing in a direction from the origin into the sector of $\mathbf{R}^n$ in which all coordinates are non-negative. The theorem tells us that there is a Pareto maximal point and a Pareto minimal point in any such direction.

Before beginning the proof, we discuss an issue raised in Chapter 3. In that chapter, we commented on the fact that, in the statement of Theorem 3.11, we could have focused our attention on lines through the point $(\frac{1}{2}, \frac{1}{2})$ (as we did in earlier theorems in that chapter) instead of on lines through the origin, making the appropriate adjustments in the range of the slope κ. (See the third paragraph after the proof of the theorem.) As we pointed out, the reason we chose to focus on lines through the origin was that this version generalizes to the context of more than two players, whereas the other version does not. We are now in a position to see why this is so.

Suppose that there are three players and consider the set IPS_{12} given in Example 5.14 and illustrated in (any of the three parts of) Figure 5.1. This set lies in the xy plane and, as we saw in the proof of Theorem 5.13, there are many points in this set that are on the boundary, but not on the Pareto boundary, of the IPS. It is easy to see that there are lines that go through the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and intersect the boundary of the IPS at such points in IPS_{12}. Such a point of intersection is neither Pareto maximal nor Pareto minimal (although it may be that the other point of intersection of such a line with the boundary of the IPS is a Pareto maximal point or a Pareto minimal point). This tells us that considering lines through the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is problematic, because it is not at all clear how to restrict the lines to be considered so that such lines intersect the boundary at a Pareto maximal point and at a Pareto minimal point.

On the other hand, lines of the form $\{\lambda p : \lambda \in \mathbf{R}\}$, where $p \in \mathbf{R}^n$ has all positive coordinates, do not intersect the boundary of the IPS at any of the "bad" points described in the preceding paragraph. This is so since each of these bad points is an interior point of one of the two-dimensional regions that is the intersection of the IPS and one of the coordinate planes. Such a line only intersects each coordinate plane at the origin. And, lines of the form $\{\lambda p : \lambda \in \mathbf{R}\}$, where $p \in \mathbf{R}^n$ has all non-negative but not necessarily all positive coordinates (but where at least one of the coordinates is positive), may contain some bad points but will also contain a Pareto maximal and a Pareto minimal point. To see this, consider, for example (any of the parts of), Figure 5.1, and imagine moving away from the origin along the line segment corresponding to $(\frac{1}{2}, \frac{1}{2}, 0)$. As we do so, we hit the IPS at a Pareto minimal point, then go through an open interval of bad points, and then leave the IPS at a Pareto maximal point.

Before proving Theorem 5.18, we prove a lemma that tells us something about these bad points and which will be useful in the proof of Theorem 5.18.

Lemma 5.19 *If point q is on the surface of the unit hypercube and is in the IPS, then either*

a. none of the coordinates of q is equal to one or
b. one of the coordinates of q is equal to one and the other coordinates are all equal to zero.

The bad points just described (i.e., the interior points of IPS_{12}) satisfy condition a of the lemma. A point satisfies condition b if it corresponds to partitions in which one player receives all of the cake. The lemma implies that any point that is on the surface of the unit hypercube and is in the IPS lies on one of the faces of the unit hypercube that includes the origin.

Proof of Lemma 5.19: Let $q = (q_1, q_2, \ldots, q_n)$ be a point that is on the surface of the unit hypercube and is in the IPS and suppose that, for some

$i = 1, 2, \ldots, n$, $q_i = 1$. We must show that, for all $j = 1, 2, \ldots, n$ with $j \neq i$, $q_j = 0$.

Fix such a j and let $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ be a partition such that $m(Q) = q$. Then $m_i(Q_i) = 1$. This implies that $m_i(Q_j) = 0$ and thus, since m_j is absolutely continuous with respect to m_i, it follows that $m_j(Q_j) = 0$. Hence, $q_j = 0$. □

It will be convenient to let $1^i \in \mathbf{R}^n$ be the point with ith coordinate equal to one and zeros elsewhere. Then condition b of Lemma 5.19 says that, for some $i = 1, 2, \ldots, n$, $q = 1^i$.

Proof of Theorem 5.18: The first statement clearly follows from the second. For the second statement, fix some $p = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n$ that has all non-negative coordinates and at least one positive coordinate.

Suppose first that all of the coordinates of p are positive and consider the set $G = \{\lambda > 0 : \lambda p \in \text{IPS}\}$. This set is clearly non-empty (since, for example, λp is in the simplex for some $\lambda > 0$ and every point of the simplex is in the IPS). Let $\lambda_1 = \sup(G)$. Since the IPS is closed, $\lambda_1 p \in$IPS. It is clear that $\lambda_1 p$ is on the outer boundary of the IPS.

Since λ_1 and all of the coordinates of p are positive, all of the coordinates of $\lambda_1 p$ are positive. Hence, $\lambda_1 p$ is not on a face of the hypercube that includes the origin. This, together with Lemma 5.19, implies that $\lambda_1 p$ does not lie on the surface of the unit hypercube. Then, since $\lambda_1 p$ is on the outer boundary of the IPS, it follows from Theorem 5.16 that $\lambda_1 p$ is on the outer Pareto boundary of the IPS. Hence, $\lambda_1 p$ is a Pareto maximal point.

Suppose that not all of the coordinates of p are positive. Let $\delta = \{i \leq n : p_i > 0\}$ and let $p' = (p_i : i \in \delta)$. In other words, p' is the point in $\mathbf{R}^{|\delta|}$ obtained by eliminating all of the zeros from p. Since p' has all positive coordinates, we can apply the preceding argument, with p' in place of p, to obtain a $\lambda_1 > 0$ such that $\lambda_1 p'$ is a Pareto maximal point in the IPS that is associated with the set of measures $\{m_i : i \in \delta\}$. Let $P = \langle P_i : i \in \delta \rangle$ be a partition that is associated with $\lambda_1 p'$. Then P is a Pareto maximal partition among the players named by δ. Define a partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ as follows: for each $i = 1, 2, \ldots, n$,

$$Q_i = \begin{cases} P_i & \text{if } i \in \delta \\ \emptyset & \text{if } i \notin \delta \end{cases}$$

It is straightforward to verify that Q is a Pareto maximal partition and that $m(Q) = \lambda_1 p$. Hence, $\lambda_1 p$ is a Pareto maximal point.

The proof that there is a positive number λ_2 such that $\lambda_2 p$ is a Pareto minimal point is similar. We obtain λ_2 by setting $\lambda_2 = \inf(G)$, where G is as above. □

The following corollary is the same as Corollary 3.12, which applied to the two-player situation.

Corollary 5.20

a. *There are infinitely many mutually non-s-equivalent Pareto maximal partitions and infinitely many mutually non-s-equivalent Pareto minimal partitions.*

b. *There are infinitely many mutually non-p-equivalent Pareto maximal partitions and infinitely many mutually non-p-equivalent Pareto minimal partitions.*

Proof: Part a follows immediately from part b. Part b follows from the theorem and the fact that distinct points of the IPS are the image, under m, of non-p-equivalent partitions. □

Corollary 5.20 – Equivalence Class Version

a. *There are infinitely many Pareto maximal s-classes and infinitely many Pareto minimal s-classes.*

b. *There are infinitely many Pareto maximal p-classes and infinitely many Pareto minimal p-classes.*

If P is a partition that is not Pareto maximal, there is a partition Q that is Pareto bigger than P. The following easy corollary to Theorem 5.18 tells us that there is such a partition Q that is Pareto maximal. It also gives us the analogous result for Pareto minimality.

Corollary 5.21 *Fix any partition P.*

a. *If P is not Pareto maximal, then there is a Pareto maximal partition Q that is Pareto bigger than P.*

b. *If P is not Pareto minimal, then there is a Pareto minimal partition R that is Pareto smaller than P.*

Proof: For part a, we observe that $m(P) \in \mathbf{R}^n$ has all non-negative coordinates and at least one positive coordinate. By the theorem, there is a positive number λ_1 such that $\lambda_1 m(P)$ is a Pareto maximal point. Then there is a Pareto maximal partition Q such that $m(Q) = \lambda_1 m(P)$. Since Q is Pareto maximal and P is not, it must be that $\lambda_1 > 1$. This implies that Q is Pareto bigger than P. The proof for part b is similar. □

Notice that the proof established slightly more than what the corollary stated. It showed that if P is not Pareto maximal then there is a Pareto maximal partition

Q such that $m(Q)$ is a multiple of $m(P)$. Then $m(Q)$ is on the line determined by the origin and $m(P)$ and is farther from the origin than is $m(P)$.

Finally, we turn to Theorem 3.13, which told us that if a partition P is neither Pareto maximal nor Pareto minimal, then P is p-equivalent to infinitely many non-s-equivalent partitions. It is tempting to guess that the result does not generalize to our present setting, since its proof rests firmly on Theorem 3.9 which, as we have seen, does not generalize to the context of more than two players. However, it turns out that this is not the case. Theorem 3.13 does generalize, but with a rather different proof.

Theorem 5.22 *If partition P is neither Pareto maximal nor Pareto minimal, then P is p-equivalent to infinitely many mutually non-s-equivalent partitions.*

Proof: Suppose that P is neither Pareto maximal nor Pareto minimal. We will show that $m(P)$ is an interior point of a line segment contained in the IPS and we will then apply Theorem 4.4.

Let P be a partition that is neither Pareto maximal nor Pareto minimal. By Theorem 5.18, we obtain a Pareto maximal partition Q, a Pareto minimal partition R, and positive numbers λ_1 and λ_2, such that $m(Q) = \lambda_1 m(P)$ and $m(R) = \lambda_2 m(P)$. Since Q is Pareto maximal and P is not, it follows that $\lambda_1 > 1$. Similarly, since R is Pareto minimal and P is not, it follows that $\lambda_2 < 1$. This implies that $m(P)$ is an interior point of the line segment connecting $m(Q)$ and $m(R)$. By the convexity of the IPS, this line segment lies completely in the IPS. Then Theorem 4.4 implies that $m(P)$ is the image, under m, of infinitely many non-s-equivalent partitions, and this implies that P is p-equivalent to infinitely many mutually non-s-equivalent partitions. □

Theorem 5.22 – Equivalence Class Version *If partition P is neither Pareto maximal nor Pareto minimal, then* $[P]_p$ *is the union of infinitely many s-classes.*

In our study of strong Pareto maximality and strong Pareto minimality in Chapter 14, we shall see that that the converse to this theorem is false. We will show that a Pareto maximal partition P may be p-equivalent to infinitely many mutually non-s-equivalent partitions, or it may be p-equivalent to no partitions to which it is not s-equivalent. In other words, we will show that there are partitions P that are Pareto maximal for which the equivalence class $[P]_p$ is the union of infinitely many s-classes, and other partitions P that are Pareto maximal for which $[P]_p$ is a single s-class. (This will follow from the equivalence of parts a and e of Theorem 14.4, the equivalence of parts a and e of Corollary 14.5, Theorem 14.14, and Theorem 4.4.) A similar statement holds for Pareto minimal partitions.

We close this section by noting that, in contrast with our remarks at the end of the previous section concerning fairness properties, the set of points in the IPS that are Pareto maximal or Pareto minimal is generally not convex. More specifically:

- the set of all Pareto maximal points in the IPS is convex if and only if the measures are all equal.
- the set of all Pareto minimal points in the IPS is convex if and only if the measures are all equal.

We omit the proofs other than to note that the reverse direction of these statements is trivial, since if the measures are all equal, then the IPS, the set of all Pareto maximal points, and the set of all Pareto minimal points are all equal to the simplex, and the simplex is convex.

5C. Fairness and Efficiency Together: Part 1b

We wish to study the existence of partitions that have both fairness and efficiency properties as we did in Section 3C for two players. In this section, we shall only combine proportionality and strong proportionality (and the corresponding chores properties) with efficiency. We shall consider combining the other fairness notions with efficiency in Section12E. We will see that these combinations are considerably more difficult to obtain than those we study in this section.

All of the results in this section are direct generalizations of the results for two players presented in Section 3C. The proofs are similar (using previous results from this chapter) and we omit them. We note that all relevant information concerning proportionality, strong proportionality, Pareto maximality, and Pareto minimality is contained in the IPS. Hence, as in the previous section, the relevant equivalence classes for this section are *s*-equivalence and *p*-equivalence, rather than *fs*-equivalence and *fp*-equivalence.

Theorem 5.23

a. If the measures are all equal, then

i. the IPS has exactly one point that is both proportional and Pareto maximal, and that point is $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.

ii. the IPS has no point that is both strongly proportional and Pareto maximal.

b. If the measures are not all equal, then

i. the IPS has infinitely many points that are both proportional and Pareto maximal. In particular, for any $q \in \mathbf{R}^n$ *with all non-negative coordinates* *and at least one positive coordinate, there is a positive number λ such that $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ is both proportional and Pareto maximal.*

ii. *the IPS has infinitely many points that are both strongly proportional and Pareto maximal. In particular, for any $q \in \mathbf{R}^n$ with all positive coordinates, there is a positive number λ such that $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ is both strongly proportional and Pareto maximal.*

Corollary 5.24

a. *If the measures are all equal, then*
   i. *there are infinitely many mutually non-s-equivalent partitions that are both proportional and Pareto maximal.*
   ii. *all partitions that are both proportional and Pareto maximal are p-equivalent.*
   iii. *there are no partitions that are both strongly proportional and Pareto maximal.*
b. *If the measures are not all equal, then*
   i. *there are infinitely many mutually non-s-equivalent partitions that are both proportional and Pareto maximal.*
   ii. *there are infinitely many mutually non-p-equivalent partitions that are both proportional and Pareto maximal.*
   iii. *there are infinitely many mutually non-s-equivalent partitions that are both strongly proportional and Pareto maximal.*
   iv. *there are infinitely many mutually non-p-equivalent partitions that are both strongly proportional and Pareto maximal.*

Corollary 5.24 – Equivalence Class Version

a. *If the measures are all equal, then*
   i. *there are infinitely many s-classes that are both proportional and Pareto maximal.*
   ii. *there is exactly one p-class that is both proportional and Pareto maximal.*
   iii. *there are no s-classes and no p-classes that are both strongly proportional and Pareto maximal.*
b. *If the measures are not all equal, then*
   i. *there are infinitely many s-classes that are both proportional and Pareto maximal.*
   ii. *there are infinitely many p-classes that are both proportional and Pareto maximal.*
   iii. *there are infinitely many s-classes that are both strongly proportional and Pareto maximal.*
   iv. *there are infinitely many p-classes that are both strongly proportional and Pareto maximal.*

The chores versions of Theorem 5.23 and Corollary 5.24 are the following. The proofs are similar and we omit them.

Theorem 5.25

a. If the measures are all equal, then

i. the IPS has exactly one point that is both c-proportional and Pareto minimal, and that point is $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.

ii. the IPS has no point that is both strongly c-proportional and Pareto minimal.

b. If the measures are not all equal, then

i. the IPS has infinitely many points that are both c-proportional and Pareto minimal. In particular, for any $q \in \mathbf{R}^n$ *with all non-positive coordinates and at least one negative coordinate, there is a positive number* λ *such that* $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ *is both c-proportional and Pareto minimal.*

ii. the IPS has infinitely many points that are both strongly c-proportional and Pareto minimal. In particular, for any $q \in \mathbf{R}^n$ *with all negative coordinates, there is a positive number* λ *such that* $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ *is both strongly c-proportional and Pareto minimal.*

Corollary 5.26

a. If the measures are all equal, then

i. there are infinitely many mutually non-s-equivalent partitions that are both c-proportional and Pareto minimal.

ii. all partitions that are both c-proportional and Pareto minimal are p-equivalent.

iii. there are no partitions that are both strongly c-proportional and Pareto minimal.

b. If the measures are not all equal, then

i. there are infinitely many mutually non-s-equivalent partitions that are both c-proportional and Pareto minimal.

ii. there are infinitely many mutually non-p-equivalent partitions that are both c-proportional and Pareto minimal.

iii. there are infinitely many mutually non-s-equivalent partitions that are both strongly c-proportional and Pareto minimal.

iv. there are infinitely many mutually non-p-equivalent partitions that are both strongly c-proportional and Pareto minimal.

Corollary 5.26 – Equivalence Class Version

a. If the measures are all equal, then

i. there are infinitely many s-classes that are both c-proportional and Pareto minimal.

ii. *there is exactly one p-class that is both c-proportional and Pareto minimal.*
iii. *there are no s-classes and no p-classes that are both strongly c-proportional and Pareto minimal.*

b. *If the measures are not all equal, then*
i. *there are infinitely many s-classes that are both c-proportional and Pareto minimal.*
ii. *there are infinitely many p-classes that are both c-proportional and Pareto minimal.*
iii. *there are infinitely many s-classes that are both strongly c-proportional and Pareto minimal.*
iv. *there are infinitely many p-classes that are both strongly c-proportional and Pareto minimal.*

We close this section by briefly considering a different property of partitions. We shall say that a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is *egalitarian* if and only if $m_1(P_1) = m_2(P_2) = \cdots = m_n(P_n)$. (Some authors call such a partition "equitable.") Informally, the idea here is that a partition is egalitarian if and only if it makes each player equally happy (where a player's happiness is given by that player's view of the size of his or her piece of cake.)

Notice that, given the perspectives on fairness and efficiency properties presented in Chapter 1 (see the discussion preceding Definitions 1.8 and 1.10), egalitarianism fits into neither category. Because it is defined in terms of an equality involving different measures, there is no natural way to define what it means for a single player to think that a partition is egalitarian. Hence, we cannot make sense of the idea that a partition is egalitarian if and only if every player thinks it is egalitarian. This tells us that egalitarianism is not a fairness property. Egalitarianism is not an efficiency property, because it does not involve comparisons between different partitions.

Clearly, egalitarianism respects s-equivalence and p-equivalence. Since egalitarianism respects p-equivalence, we can speak of egalitarian points of the IPS.

It is easy to see that a point of the IPS is egalitarian if and only if it lies on the line $x_1 = x_2 = \cdots = x_n$. Call this line ℓ. One point of intersection of ℓ and the IPS is the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. Since this point is a proportional point, it follows that there exists a point of the IPS that is egalitarian and proportional. Also, since $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ certainly lies in the interior of a line segment contained in the IPS, Theorem 4.4 implies that there are infinitely many mutually non-s-equivalent partitions that are egalitarian and proportional. If all of the measures are equal, then $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is also Pareto maximal and Pareto minimal. Hence, in this case, there are infinitely many mutually non-s-equivalent partitions that are egalitarian, proportional, Pareto maximal, and Pareto minimal.

Suppose that the measures are not all equal. Since the IPS is convex, we know that the intersection of ℓ with the IPS is a line segment, and since the IPS is closed, this line segment is closed. Let p and q be the endpoints of this line segment, where p is the point farthest from the origin and q is the point closest to the origin. Then p is strongly proportional and Pareto maximal, and q is strongly c-proportional and Pareto minimal. (These can be seen as examples of parts bii of Theorems 5.23 and 5.25, with $q = (1, 1, \ldots, 1)$ and $q = (-1, -1, \ldots, -1)$, respectively.) Thus, we have established that, if the measures are not all equal, then there exists a point (and therefore a partition) that is egalitarian, strongly proportional, and Pareto maximal, and a point (and therefore a partition) that is egalitarian, strongly c-proportional, and Pareto minimal. Also, if r is any point on the open line segment between p and $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then r is egalitarian and strongly proportional. Since there are infinitely many such r, there are infinitely many mutually non-p-equivalent partitions that are egalitarian and strongly proportional. Similarly, by considering points on the open line segment between q and $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, we see that there are infinitely many mutually non-p-equivalent partitions that are egalitarian and strongly c-proportional.

5D. The Situation Without Absolute Continuity

We now drop our assumption of absolute continuity. As we did in Section 3D in the case of two players, we explicitly assume throughout this section that the measures are not all absolutely continuous with respect to each other, and we consider how this assumption affects the previous results in this chapter.

As in the two-player context, generalizing our fairness results is quite easy, since none of the results from Section 5A use absolute continuity and, hence, all hold in our present context. The failure of absolute continuity certainly implies that the measures are not all equal. Then the results on proportionality and strong proportionality are the following.

Theorem 5.27

a. *The IPS has infinitely many proportional points. In particular, for any $q \in \mathbf{R}^n$ with all non-negative coordinates and at least one positive coordinate, there are infinitely many $\lambda > 0$ such that $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ is a proportional point.*
b. *The IPS has infinitely many strongly proportional points. In particular, for any $q \in \mathbf{R}^n$ with all positive coordinates, there are infinitely many $\lambda > 0$ such that $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ is a strongly proportional point.*

Corollary 5.28

a. There are infinitely many mutually non-s-equivalent proportional partitions.
b. There are infinitely many mutually non-p-equivalent proportional partitions.
c. There are infinitely many mutually non-s-equivalent strongly proportional partitions.
d. There are infinitely many mutually non-p-equivalent strongly proportional partitions.

Corollary 5.28 – Equivalence Class Version

a. There are infinitely many proportional s-classes.
b. There are infinitely many proportional p-classes.
c. There are infinitely many strongly proportional s-classes.
d. There are infinitely many strongly proportional p-classes.

The corresponding chores results are the following.

Theorem 5.29

a. The IPS has infinitely many c-proportional points. In particular, for any $q \in \mathbf{R}^n$ with all non-positive coordinates and at least one negative coordinate, there are infinitely many $\lambda > 0$ such that $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ is a c-proportional point.
b. The IPS has infinitely many strongly c-proportional points. In particular, for any $q \in \mathbf{R}^n$ with all negative coordinates, there are infinitely many $\lambda > 0$ such that $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) + \lambda q$ is a strongly c-proportional point.

Corollary 5.30

a. There are infinitely many mutually non-s-equivalent c-proportional partitions.
b. There are infinitely many mutually non-p-equivalent c-proportional partitions.
c. There are infinitely many mutually non-s-equivalent strongly c-proportional partitions.
d. There are infinitely many mutually non-p-equivalent strongly c-proportional partitions.

Corollary 5.30 – Equivalence Class Version

a. There are infinitely many c-proportional s-classes.
b. There are infinitely many c-proportional p-classes.
c. There are infinitely many strongly c-proportional s-classes.
d. There are infinitely many strongly c-proportional p-classes.

The results on envy-freeness, strong envy-freeness, and super envy-freeness are the following.

Theorem 5.31
a. (Envy-freeness) The FIPS has infinitely many envy-free points.
b. (Strong envy-freeness) The following are equivalent:
i. No two of the measures are equal.
ii. The FIPS has at least one strongly envy-free point.
iii. The FIPS has infinitely many strongly envy-free points.
c. (Super envy-freeness) The following are equivalent:
i. The measures are linearly independent.
ii. The FIPS has at least one super envy-free point.
iii. The FIPS has infinitely many super envy-free points.

Corollary 5.32
a. (Envy-freeness)
i. There are infinitely many mutually non-fs-equivalent envy-free partitions.
ii. There are infinitely many mutually non-fp-equivalent envy-free partitions.
b. (Strong envy-freeness) The following are equivalent:
i. No two of the measures are equal.
ii. There is at least one strongly envy-free partition.
iii. There are infinitely many non-fs-equivalent strongly envy-free partitions.
iv. There are infinitely many non-fp-equivalent strongly envy-free partitions.
c. (Super envy-freeness) The following are equivalent:
i. The measures are linearly independent.
ii. There is at least one super envy-free partition.
iii. There are infinitely many non-fs-equivalent super envy-free partitions.
iv. There are infinitely many non-fp-equivalent super envy-free partitions.

Corollary 5.32 – Equivalence Class Version
a. (Envy-freeness)
i. there are infinitely many envy-free fs-classes.
ii. there are infinitely many envy-free fp-classes.
b. (Strong envy-freeness) The following are equivalent:
i. No two of the measures are equal.
ii. There is at least one strongly envy-free fs-class.
iii. There is at least one strongly envy-free fp-class.
iv. There are infinitely many strongly envy-free fs-classes.
v. There are infinitely many strongly envy-free fp-classes.

c. (Super envy-freeness) The following are equivalent:
  i. The measures are linearly independent.
  ii. There is at least one super envy-free fs-class.
  iii. There is at least one super envy-free fp-class.
  iv. There are infinitely many super envy-free fs-classes.
  v. There are infinitely many super envy-free fp-classes.

The corresponding chores results are the following.

Theorem 5.33

a. (c-Envy-freeness) The FIPS has infinitely many c-envy-free points.
b. (Strong c-envy-freeness) The following are equivalent:
  i. No two of the measures are equal.
  ii. The FIPS has at least one strongly c-envy-free point.
  iii. The FIPS has infinitely many strongly c-envy-free points.
c. (Super c-envy-freeness) The following are equivalent:
  i. The measures are linearly independent.
  ii. The FIPS has at least one super c-envy-free point.
  iii. The FIPS has infinitely many super c-envy-free points.

Corollary 5.34

a. (c-Envy-freeness)
  i. There are infinitely many mutually non-fs-equivalent c-envy-free partitions.
  ii. There are infinitely many mutually non-fp-equivalent c-envy-free partitions.
b. (Strong c-envy-freeness) The following are equivalent:
  i. No two of the measures are equal.
  ii. There is at least one strongly c-envy-free partition.
  iii. There are infinitely many non-fs-equivalent strongly c-envy-free partitions.
  iv. There are infinitely many non-fp-equivalent strongly c-envy-free partitions.
c. (Super c-envy-freeness) The following are equivalent:
  i. The measures are linearly independent.
  ii. There is at least one super c-envy-free partition.
  iii. There are infinitely many non-fs-equivalent super c-envy-free partitions.
  iv. There are infinitely many non-fp-equivalent super c-envy-free partitions.

Corollary 5.34 – Equivalence Class Version

a. (c-Envy-freeness)

i. There are infinitely many c-envy-free fs-classes.

ii. There are infinitely many c-envy-free fp-classes.

b. (Strong c-envy-freeness) The following are equivalent:

i. No two of the measures are equal.

ii. There is at least one strongly c-envy-free fs-class.

iii. There is at least one strongly c-envy-free fp-class.

iv. There are infinitely many strongly c-envy-free fs-classes.

v. There are infinitely many strongly c-envy-free fp-classes.

c. (Super c-envy-freeness) The following are equivalent:

i. The measures are linearly independent.

ii. There is at least one super c-envy-free fs-class.

iii. There is at least one super c-envy-free fp-class.

iv. There are infinitely many super c-envy-free fs-classes.

v. There are infinitely many super c-envy-free fp-classes.

Next, we consider efficiency. Our presentation parallels our discussion of these subjects for the two-player context in Section 3D.

As was the case in the two-player context, the corresponding definitions trivially imply that the union of the outer and inner boundaries of the IPS is the boundary of the IPS, and that Theorem 5.12 holds regardless of any assumptions about absolute continuity. What is the relationship between the Pareto boundary of the IPS and the boundary of the IPS in our present context? When there are two players, we saw in Chapter 3 (see Theorems 3.9 and 3.22) that the Pareto boundary is equal to the boundary if and only if the two measures are absolutely continuous with respect to each other. Then, in Section 5B (see Theorem 5.13), we saw that if there are more than two players and the measures are all absolutely continuous with respect to each other, then the Pareto boundary is equal to the boundary if and only if the measures are all equal. The relationship in our present context, as expected, is the following.

Theorem 5.35

a. The outer Pareto boundary of the IPS is a proper subset of the outer boundary of the IPS.

b. The inner Pareto boundary of the IPS is a proper subset of the inner boundary of the IPS.

We omit the proof, since it is similar to the two-player proof of Theorem 3.22. We can also see the truth of Theorem 5.35 by noting that the failure of absolute continuity implies that the measures are not all equal and, hence, since the proof of Theorem 5.13 did not use absolute continuity, this proof also implies the truth of Theorem 5.35.

As was the case for Theorem 5.13, Theorem 5.35 immediately yields the following corollary.

Corollary 5.36 *There are points on the boundary of the IPS that do not correspond to Pareto maximal or Pareto minimal partitions.*

In Chapter 3 (see Corollary 3.25), we saw that when absolute continuity fails and there are two players, any points that are on the boundary of the IPS but not on the Pareto boundary of the IPS (i.e., the points whose existence is implied by Theorem 3.22) are on the unit square. In Section 5B, we saw that the natural generalization of this result holds when there are more than two players and absolute continuity holds (see Corollary 5.17). Curiously, this result does not hold when there are more than two players and absolute continuity fails.

Theorem 5.37 *Fix $n > 2$. There is a cake C, corresponding measures m_1, $m_2, \ldots, m_n$, and a point $p = (p_1, p_2, \ldots, p_n)$ that is on the boundary but not on the Pareto boundary of the associated IPS and is in the interior of the unit hypercube.*

It will be convenient to delay the proof of Theorem 5.37 until later in this section.

In continuing to parallel our discussion of Section 3D, we should next consider the question of how many points of the IPS are Pareto maximal and how many are Pareto minimal. In other words, the question is how the first statement in Theorem 5.18 adjusts to our present context. It will be convenient to delay this investigation slightly, for the following reason. Corollary 5.21, which followed easily from Theorem 5.18, tells us that given any partition P that is not Pareto maximal there is a Pareto maximal partition Q that is Pareto bigger than P. (It also told us the analogous chores fact.) This corollary does hold in our present context, but it requires a completely different proof. We will find this result to be useful in our proof of the appropriate adjustment of Theorem 5.18 to our present context. Therefore we shall reverse the order of presentation of these two ideas in this section.

Theorem 5.38 *Fix any partition P.*

a. If P is not Pareto maximal, then there is a Pareto maximal partition Q that is Pareto bigger than P.
b. If P is not Pareto minimal, then there is a Pareto minimal partition R that is Pareto smaller than P.

Proof: For part a, assume that the partition P is not Pareto maximal and let $m(P) = p = (p_1, p_2, \ldots, p_n)$. Let $r_1 = \sup\{q_1 : (q_1, q_2, \ldots, q_n) \in \text{IPS}$ is Pareto bigger than $p\}$. Since P is not a Pareto maximal partition, and thus p is not a Pareto maximal point, we know that r_1 is the supremum of a non-empty set. And, since the IPS is closed, it follows that there are numbers $q_2, q_3, \ldots, q_n$ such that $(r_1, q_2, \ldots, q_n) \in \text{IPS}$ is Pareto bigger that p. Let $r_2 = \sup\{q_2 : (r_1, q_2, \ldots, q_n) \in \text{IPS}$ is Pareto bigger than $p\}$. Again, we know that r_2 is the supremum of a non-empty set and, since the IPS is closed, there are numbers $q_3, q_4, \ldots, q_n$ such that $(r_1, r_2, q_3, \ldots, q_n) \in \text{IPS}$ is Pareto bigger that p. Continuing in this manner, we obtain a point $r = (r_1, r_2, \ldots, r_n) \in \text{IPS}$ that is Pareto bigger than p. Notice that any point Pareto bigger than r is Pareto bigger than p. Hence, the existence of such a point would contradict the definition of one of the r_i. It follows that r is Pareto maximal. Let Q be any partition with $m(Q) = r$. Then Q is a Pareto maximal partition that is Pareto bigger than P. This completes the proof of part a. The proof for part b is similar. □

We are now ready to consider how Theorem 5.18 adjusts to our present setting. As was the case in Chapter 3 (in going from Theorem 3.11 to Theorem 3.27), this theorem is still true, with one exception for Pareto maximality and one for Pareto minimality. The exception for Pareto maximality involves the concentration of the measures on disjoint sets, and the exception for Pareto minimality involves the concentration of the measures on the complements of disjoint sets. These two notions are equivalent when there are two players, but not when there are more than two players.

Definition 5.39

a. Measures $m_1, m_2, \ldots, m_n$ *concentrate on disjoint sets* if and only if there is a partition $\langle P_1, P_2, \ldots, P_n \rangle$ of C with $m_1(P_1) = 1, m_2(P_2) = 1, \ldots, m_n(P_n) = 1$.
b. Measures $m_1, m_2, \ldots, m_n$ *concentrate on the complements of disjoint sets* if and only if there is a partition $\langle Q_1, Q_2, \ldots, Q_n \rangle$ of C with $m_1(C \backslash Q_1) = 1, m_2(C \backslash Q_2) = 1, \ldots, m_n(C \backslash Q_n) = 1$ (or, equivalently, with $m_1(Q_1) = 0, m_2(Q_2) = 0, \ldots, m_n(Q_n) = 0$).

If $P_1, P_2, \ldots, P_n$ are as in the definition, then we shall say that the measures $m_1, m_2, \ldots, m_n$ concentrate on the disjoint sets $P_1, P_2, \ldots, P_n$, respectively. Similarly, if $Q_1, Q_2, \ldots, Q_n$ are as in the definition, then we shall say that the measures $m_1, m_2, \ldots, m_n$ concentrate on the complements of the disjoint sets $Q_1, Q_2, \ldots, Q_n$, respectively. As was the case in the two-player context (see Definition 3.26 and the comments following the definition), our assumption in part a that $\langle P_1, P_2, \ldots, P_n \rangle$ is a partition of C is not really necessary. It is only necessary that $P_1, P_2, \ldots, P_n$ be pairwise disjoint and that $m_1(P_1) = 1$, $m_2(P_2) = 1, \ldots, m_n(P_n) = 1$. If $\cup_{i \le n} P_i \ne C$, then for any $j = 1, 2, \ldots, n$ we could simply replace P_j by $P_j \cup (C \setminus \cup_{i \le n} P_i)$ to make it so. On the other hand, our assumption that $\langle Q_1, Q_2, \ldots, Q_n \rangle$ is a partition of C in part b of the definition is necessary, since otherwise we could, for example, set $Q_1 = Q_2 = \cdots = Q_n = \emptyset$ and conclude that the measures concentrate on the complements of disjoint sets.

We also note that if $m_1, m_2, \ldots, m_n$ concentrate on disjoint sets, then these measures also concentrate on the complements of disjoint sets. To see this, suppose that $P_1, P_2, \ldots, P_n$ are as in part a of the definition and set $Q_1 = P_2, Q_2 = P_3, \ldots, Q_{n-1} = P_n, Q_n = P_1$. Then $Q_1, Q_2, \ldots, Q_n$ satisfy part b of the definition. It is not hard to see that, when there are more than two players, the converse of this statement is not true. In other words, it is possible that the measures concentrate on the complements of disjoint sets but do not concentrate on disjoint sets.

If $m_1, m_2, \ldots, m_n$ and $P_1, P_2, \ldots, P_n$ are as in part a of the definition, then $m(\langle P_1, P_2, \ldots, P_n \rangle) = (1, 1, \ldots, 1)$ and, hence, $(1, 1, \ldots, 1) \in$ IPS. Conversely, if $(1, 1, \ldots, 1) \in$ IPS, then the measures concentrate on disjoint sets. Similarly, if $m_1, m_2, \ldots, m_n$ and $Q_1, Q_2, \ldots, Q_n$ are as in part b of the definition, then $m(\langle Q_1, Q_2, \ldots, Q_n \rangle) = (0, 0, \ldots, 0)$ and, hence, $(0, 0, \ldots, 0) \in$ IPS. Conversely, if $(0, 0, \ldots, 0) \in$ IPS, then the measures concentrate on the complements of disjoint sets.

Recall that if the measures are identical then the IPS is equal to the simplex, which is the smallest possible IPS, having (n-dimensional) volume zero. And, in the most extreme case of disagreement of the measures (i.e., when the measures concentrate on disjoint sets), the IPS is equal to the unit hypercube, which is the largest possible IPS, having volume one. We may think of the size of the IPS as corresponding to the degree of disagreement of the measures.

Our adjusted version of Theorem 5.18 is the following.

Theorem 5.40

a. *If the measures concentrate on disjoint sets, then the IPS has exactly one Pareto maximal point, and that point is* $(1, 1, \ldots, 1)$.
b. *If the measures do not concentrate on disjoint sets, then the IPS has infinitely many Pareto maximal points.*
c. *If the measures concentrate on the complements of disjoint sets, then the IPS has exactly one Pareto minimal point, and that point is* $(0, 0, \ldots, 0)$.
d. *If the measures do not concentrate on the complements of disjoint sets, then the IPS has infinitely many Pareto minimal points.*

Proof: The proof of part a is straightforward and is similar to the proof of part a of Theorem 3.27.

For part b, we assume that the measures do not concentrate on disjoint sets. We first show that there are at least two Pareto maximal points and then we use this fact to show that there are infinitely many.

It follows from Theorem 5.38 that there is at least one Pareto maximal partition and, hence, at least one Pareto maximal point. Suppose then that the point $p = (p_1, p_2, \ldots, p_n)$ is Pareto maximal. Since the measures do not concentrate on disjoint sets, we know that $(1, 1, \ldots, 1) \notin$ IPS. Hence, for at least one $i = 1, 2, \ldots, n$, $p_i < 1$. Fix such an i, choose q_i such that $p_i < q_i < 1$, and let q be any point of the IPS with ith coordinate q_i. (We know there are such points because there are certainly such points in the simplex, and every point in the simplex is in the IPS.) If q is not Pareto maximal then, by Theorem 5.38, let $r = (r_1, r_2, \ldots, r_n)$ be a Pareto maximal point that is Pareto bigger than q. If q is Pareto maximal, let $r = q$. Since $r_i \geq q_i > p_i$, it follows that $r \neq p$. Thus, there are at least two Pareto maximal points.

For any $s \in$ IPS, let *Sum(s)* denote the sum of the coordinates of s.

Assume, by way of contradiction, that there are only finitely many Pareto maximal points. Suppose there are exactly m such points and these points are $p^1, p^2, \ldots, p^m$, where, for each $j = 1, 2, \ldots, m$, we set $p^j = (p_1^j, p_2^j, \ldots, p_n^j)$. Since we have just shown that $m \geq 2$, we may fix distinct $k, k' = 1, 2, \ldots, m$ such that $\text{Sum}(p^k)$ and $\text{Sum}(p^{k'})$ are the largest possible. In other words, fix such a k and k' so that, for every $j = 1, 2, \ldots, n$ with $j \neq k$ and $j \neq k'$, $p_1^j + p_2^j + \cdots + p_n^j \leq p_1^k + p_2^k + \cdots + p_n^k$ and $p_1^j + p_2^j + \cdots + p_n^j \leq p_1^{k'} + p_2^{k'} + \cdots + p_n^{k'}$. Let s be the midpoint of the line segment connecting p^k and $p^{k'}$. Then, $s = (\frac{p_1^k + p_1^{k'}}{2}, \frac{p_2^k + p_2^{k'}}{2}, \ldots, \frac{p_n^k + p_n^{k'}}{2})$ and $\text{Sum}(s)$ is the average of $\text{Sum}(p^k)$ and $\text{Sum}(p^{k'})$. We claim that for no $j = 1, 2, \ldots, n$ is p^j Pareto bigger than s.

First, consider $j = k$ and $j = k'$. Since $p^k \neq p^{k'}$ and neither of these points is Pareto bigger than the other, it follows that, for some distinct $i, i' = 1, 2, \ldots, n$, $p_i^{k'} > p_i^k$ and $p_{i'}^k > p_{i'}^{k'}$. This implies that $\frac{p_i^k + p_i^{k'}}{2} > p_i^k$ and $\frac{p_{i'}^k + p_{i'}^{k'}}{2} > p_{i'}^{k'}$. Hence, neither p^knor $p^{k'}$ is Pareto bigger than s.

Next, consider any $j = 1, 2, \ldots, m$ with $j \neq k$ and $j \neq k'$ and suppose, by way of contradiction, that p^j is Pareto bigger than s. This implies that $\text{Sum}(p^j) > \text{Sum}(s)$. But then, since $\text{Sum}(s)$ is the average of $\text{Sum}(p^k)$ and $\text{Sum}(p^{k'})$, it follows that either $\text{Sum}(p^j) > \text{Sum}(p^k)$ or $\text{Sum}(p^j) > \text{Sum}(p^{k'})$ (or both). This contradicts our choice of p^k or our choice of $p^{k'}$ (or both). Hence, no p^j Pareto bigger than s.

Theorem 5.38 implies that there is a Pareto maximal point t that is Pareto bigger than s. Since no p^j is Pareto bigger than s, we know that $t \neq p^j$ for any $j = 1, 2, \ldots, m$. This shows that there are at least $m + 1$ Pareto maximal points and contradicts our assumption that there are exactly m such points. We conclude that there are infinitely many Pareto maximal points.

The proof of part c is straightforward and the proof of part d is analogous to the proof of part b. □

The second statement of Theorem 5.18 certainly does not hold if absolute continuity fails. This statement tells us that if the measures are absolutely continuous with respect to each other and we move along a straight line in any direction from the origin into the quadrant in which all coordinates are non-negative, then we will hit a Pareto maximal point. As in the two-player context (see Figure 3.8 and the comments following the proof of Theorem 3.27), such is not the case if absolute continuity fails.

The following corollary is the n-player version of Corollary 3.28. The proof is similar and we omit it.

Corollary 5.41

a. *If the measures $m_1, m_2, \ldots, m_n$ concentrate on the disjoint sets $P_1, P_2, \ldots, P_n$, respectively, then*
   i. *all Pareto maximal partitions are s-equivalent (and are s-equivalent to $\langle P_1, P_2, \ldots, P_n \rangle$).*
   ii. *all Pareto maximal partitions are p-equivalent (and are p-equivalent to $\langle P_1, P_2, \ldots, P_n \rangle$).*

b. *If the measures do not concentrate on disjoint sets then*
   i. *there are infinitely many mutually non-s-equivalent Pareto maximal partitions.*
   ii. *there are infinitely many mutually non-p-equivalent Pareto maximal partitions.*

c. *If the measures $m_1, m_2, \ldots, m_n$ concentrate on the complements of the disjoint sets $Q_1, Q_2, \ldots, Q_n$, respectively, then*
   i. *all Pareto minimal partitions are s-equivalent (and are s-equivalent to $\langle Q_1, Q_2, \ldots, Q_n \rangle$).*
   ii. *all Pareto minimal partitions are p-equivalent (and are p-equivalent to $\langle Q_1, Q_2, \ldots, Q_n \rangle$).*

d. *If the measures do not concentrate on the complements of disjoint sets then*
   i. *there are infinitely many mutually non-s-equivalent Pareto minimal partitions.*
   ii. *there are infinitely many mutually non-p-equivalent Pareto minimal partitions.*

Corollary 5.41 – Equivalence Class Version

a. *If the measures* $m_1, m_2, \ldots, m_n$ *concentrate on the disjoint sets* $P_1, P_2, \ldots, P_n$*, respectively, then*
   i. *there is exactly one Pareto maximal s-class (and that class is* $[\langle P_1, P_2, \ldots, P_n\rangle]_s$*).*
   ii. *there is exactly one Pareto maximal p-class (and that class is* $[\langle P_1, P_2, \ldots, P_n\rangle]_p$*).*

b. *If the measures do not concentrate on disjoint sets then*
   i. *there are infinitely many Pareto maximal s-classes.*
   ii. *there are infinitely many Pareto maximal p-classes.*

c. *If the measures* $m_1, m_2, \ldots, m_n$ *concentrate on the complements of the disjoint sets* $Q_1, Q_2, \ldots, Q_n$*, respectively, then*
   i. *there is exactly one Pareto minimal s-class (and that class is* $[\langle Q_1, Q_2, \ldots, Q_n\rangle]_s$*).*
   ii. *there is exactly one Pareto minimal p-class (and that class is* $[\langle Q_1, Q_2, \ldots, Q_n\rangle]_p$*).*

d. *If the measures do not concentrate on disjoint sets then*
   i. *there are infinitely many Pareto minimal s-classes.*
   ii. *there are infinitely many Pareto minimal p-classes.*

Continuing to parallel our discussion in Section 3D, we consider generalizing Theorem 3.30. We recall that this result examined the following question: given a partition P that is neither Pareto maximal nor Pareto minimal, to how many non-s-equivalent partitions is P p-equivalent? The result used Lemma 3.29, which distinguished between the points $(1, 0)$ and $(0, 1)$, on the one hand, and all other points of the IPS, on the other. It also considered whether each measure fails to be absolutely continuous with respect to the other, or whether exactly one of the two measures fails to be absolutely continuous with respect to the other. We will obtain a partial generalization of this lemma to the n-player context and will give an example to show that the full generalization fails. In particular, we will show that the natural generalization of the lemma's statement about the points $(1, 0)$ and $(0, 1)$ does generalize, but the statement about other points does not.

Before stating this generalization of part of Lemma 3.29, we review the geometric perspective on the failure of absolute continuity, which we discussed in the two-player context in Section 3D and illustrated in Figure 3.6. We saw that the failure of absolute continuity corresponds to a vertical or a horizontal line segment on the boundary of the IPS, and any such line segment has one endpoint at $(1, 0)$ or at $(0, 1)$. In Figure 3.6a, m_1 is not absolutely continuous with respect to m_2, but m_2 is absolutely continuous with respect to m_1. In this case, the point $(1, 0)$ is Pareto maximal but not Pareto minimal, and the point $(0, 1)$ is Pareto minimal but not Pareto maximal. In Figure 3.6b, the situation is reversed. In Figure 3.6c, neither m_1 nor m_2 is absolutely continuous with respect to the other, and hence the points $(1, 0)$ and $(0, 1)$ are each neither Pareto maximal nor Pareto minimal.

As we did in Section 5B, we let $1^i \in \mathbf{R}^n$ be the point with ith coordinate equal to one and zeros elsewhere. In addition, for distinct i and j, we let 1^{ij} be the point in $\mathbf{R}^n$ with ith and jth coordinates each equal to one and zeros elsewhere. The generalization of part of Lemma 3.29 to the n-player context is parts c and d of the following.

Lemma 5.42 *Fix any $i = 1, 2, \ldots, n$.*

a. *For any $j = 1, 2, \ldots, n$ with $j \neq i$, m_i fails to be absolutely continuous with respect to m_j if and only if the IPS contains a line segment with one endpoint at 1^j that extends some positive distance toward 1^{ij}.*
b. *m_i fails to be absolutely continuous with respect to m_j for some $j = 1, 2, \ldots, n$ if and only if the IPS contains a line segment with one endpoint at 1^i that extends some positive distance toward the origin.*
c. *1^i is Pareto maximal if and only if, for every $j = 1, 2, \ldots, n$, m_j is absolutely continuous with respect to m_i.*
d. *1^i is Pareto minimal if and only if, for every $j = 1, 2, \ldots, n$, m_i is absolutely continuous with respect to m_j.*

Proof: Fix some $i = 1, 2, \ldots, n$. For part a, fix $j = 1, 2, \ldots, n$ with $j \neq i$ and, for the forward direction, assume that m_i fails to be absolutely continuous with respect to m_j. Then, for some $A \subseteq C$, $m_i(A) > 0$ and $m_j(A) = 0$. Define a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ as follows: for each $k = 1, 2, \ldots, n$,

$$P_k = \begin{cases} A & \text{if } k = i \\ C \backslash A & \text{if } k = j \\ \emptyset & \text{if } k \neq i, k \neq j \end{cases}$$

Then

$$m_i(P_i) = m_i(A) > 0$$
$$m_j(P_j) = m_j(C \backslash A) = m_j(C) - m_j(A) = 1$$

and

$$m_k(P_k) = m_k(\emptyset) = 0 \quad \text{for every } k \text{ with } k \neq i \text{ and } k \neq j.$$

Thus, $m(P)$ is a point of the IPS that is on the line segment between 1^j and 1^{ij}. Since $m_i(P_i) > 0$, we know that $m(P) \neq 1^j$. (It may be that $m(P) = 1^{ij}$.

This occurs if and only if m_i and m_j concentrate on the disjoint sets A and $C \backslash A$, respectively.) Since $1^j \in$ IPS, the convexity of the IPS implies that the IPS contains a line segment with one endpoint at 1^j that extends some positive distance toward 1^{ij}.

For the reverse direction of part a, we assume that the IPS contains a line segment with one endpoint at 1^j that extends some positive distance toward 1^{ij}. This implies that there is a point p in the IPS that is between 1^j and 1^{ij}. Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ such that $m(P) = p$. Then $m_i(P_i) > 0$ and $m_j(P_j) = 1$. Since $m_j(P_j) = 1$, we know that $m_j(P_i) = 0$. This implies that m_i fails to be absolutely continuous with respect to m_j.

For the forward direction of part b, we assume that m_i fails to be absolutely continuous with respect to m_j for some $j = 1, 2, \ldots, n$. Fix such a j. Then, for some $A \subseteq C, m_i(A) > 0$ and $m_j(A) = 0$. Define a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ as follows: for each $k = 1, 2, \ldots, n$,

$$P_k = \begin{cases} C \backslash A & \text{if } k = i \\ A & \text{if } k = j \\ \emptyset & \text{if } k \neq i, k \neq j \end{cases}$$

Then,

$$\begin{aligned} m_i(P_i) &= m_i(C \backslash A) = m_i(C) - m_i(A) = 1 - m_i(A) < 1 \\ m_j(P_j) &= m_j(A) = 0 \end{aligned}$$

and

$$m_k(P_k) = m_k(\emptyset) = 0 \quad \text{for every } k \text{ with } k \neq i \text{ and } k \neq j.$$

Thus, $m(P)$ is a point of the IPS that is on the line segment between 1^i and the origin. Since $m_i(P_i) < 1$, we know that $m(P) \neq 1^i$. Hence, since $1^i \in$ IPS, the convexity of the IPS implies that the IPS contains a line segment with one endpoint at 1^i that extends some positive distance toward the origin.

For the reverse direction of part b, we assume that the IPS contains a line segment with one endpoint at 1^i that extends some positive distance toward the origin. This implies that there is a point p in the IPS between 1^i and the origin. Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ such that $m(P) = p$. Then $m_i(P_i) < 1$ and $m_j(P_j) = 0$ for every $j = 1, 2, \ldots, n$ with $j \neq i$. But $m_i(P_i) < 1$ implies that for some such $j, m_i(P_j) > 0$. This implies that m_i fails to be absolutely continuous with respect to m_j.

For the forward direction of part c, we assume that, for some $j = 1, 2, \ldots, n, m_j$ is not absolutely continuous with respect to m_i. By part a (with the roles of i and j reversed), the IPS contains a line segment, with one endpoint at 1^i, that extends some positive distance toward 1^{ij}. This implies that 1^i is not Pareto maximal.

For the reverse direction of part c, we assume that 1^i is not Pareto maximal. Then, for some partition $P = \langle P_1, P_2, \ldots, P_n \rangle$, $m_i(P_i) = 1$ and, for some $j \neq i$, $m_j(P_j) > 0$. Define a new partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ as follows:

$$Q_k = \begin{cases} \bigcup_{k \neq j} P_k & \text{if } k = i \\ P_j & \text{if } k = j \\ \emptyset & \text{if } k \neq i, k \neq j \end{cases}$$

We may view Q as arising from P by having all players other than Player j give their piece of cake to Player i. Then,

$$m_i(Q_i) = m_i\left(\bigcup_{k \neq j} P_k\right) \geq m_i(P_i) = 1$$

and, hence,

$$m_i(Q_i) = 1$$
$$m_j(Q_j) = m_j(P_j) > 0$$

and

$$m_k(Q_k) = m_k(\emptyset) = 0 \quad \text{for every } k \text{ with } k \neq i \text{ and } k \neq j.$$

Hence, $m(Q)$ is a point of the IPS that is on the line segment between 1^i and 1^{ij} and is not equal to 1^i. Then, since $1^i \in$IPS, the convexity of the IPS implies that the IPS contains a line segment with one endpoint at 1^i that extends some positive distance toward 1^{ij}. By part a (again, with the roles of i and j reversed), this tells us that m_j fails to be absolutely continuous with respect to m_i. This establishes part c of the lemma.

For the forward direction of part d, assume that, for some j, m_i is not absolutely continuous with respect to m_j. Part b tells us that the IPS contains a line segment with one endpoint at 1^i that extends some positive distance toward the origin. This implies that 1^i is not Pareto minimal.

For the reverse direction of part d, suppose that 1^i is not Pareto minimal. Then, for some partition $P = \langle P_1, P_2, \ldots, P_n \rangle$, $m_i(P_i) < 1$ and, for all $j \neq i$, $m_j(P_j) = 0$. It follows that $m(P)$ is on the line segment between 1^i and the origin and, since $m_i(P_i) < 1$, we know that $m(P) \neq 1^i$. By the convexity of the IPS, this implies that the IPS contains a line segment with one endpoint at 1^i that extends some positive distance toward the origin. By part b, it follows that,

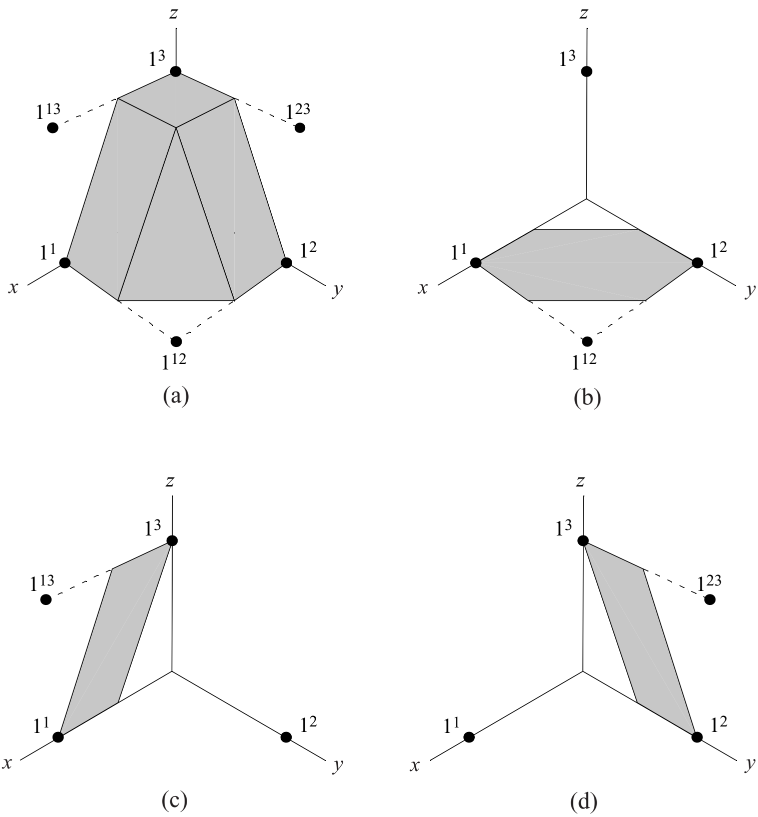


Figure 5.2

for some $j = 1, 2, \ldots, n$, m_i fails to be absolutely continuous with respect to m_j. This completes the proof of the lemma. □

The lemma is illustrated, for the case of three players, in Figures 5.2 and 5.3. Figures 5.2a and 5.3a show an IPS for some cake and corresponding measures. Figures 5.2b and 5.3b, 5.2c and 5.3c, and 5.2d and 5.3d show the intersection on each IPS with the *xy*, the *xz*, and the *yz* plane, respectively. In Figure 5.2,

- m_1 and m_2 each fail to be absolutely continuous with respect to the other and with respect to m_3, and
- m_3 is absolutely continuous with respect to both m_1 and m_2;

and in Figure 5.3,

- m_1 and m_2 are each absolutely continuous with respect each other and with respect to m_3, and
- m_3 fails to be absolutely continuous with respect to both m_1 and m_2.

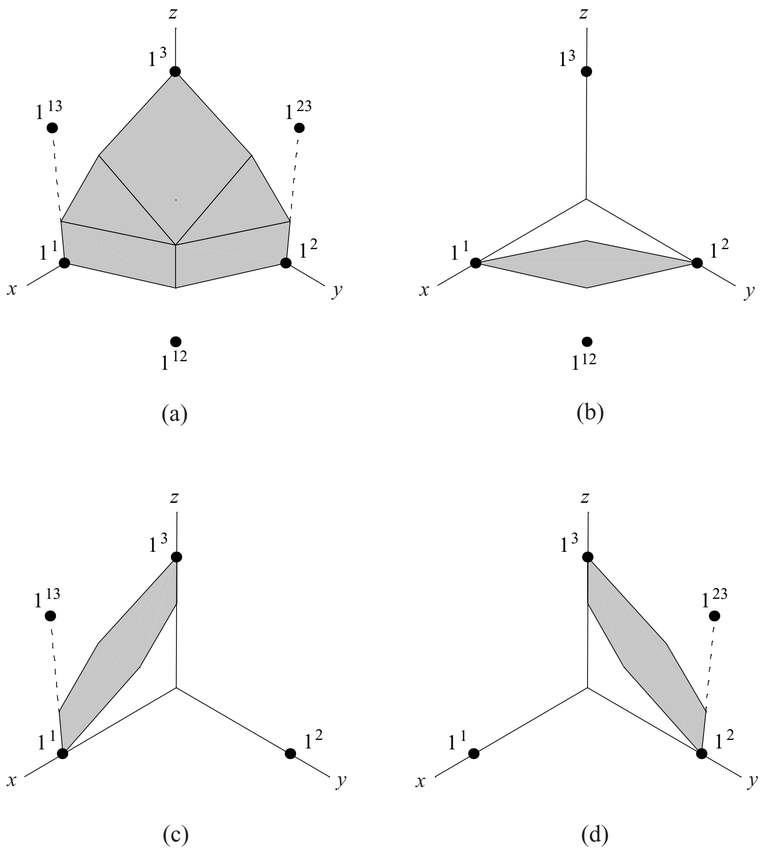


Figure 5.3

Then, in Figure 5.2,

- the IPS contains a line segment with
  - one endpoint at 1^1 that extends some positive distance toward 1^{12}.
  - one endpoint at 1^2 that extends some positive distance toward 1^{12}.
  - one endpoint at 1^3 that extends some positive distance toward 1^{13}.
  - one endpoint at 1^3 that extends some positive distance toward 1^{23}.
  - one endpoint at 1^1 that extends some positive distance toward the origin.
  - one endpoint at 1^2 that extends some positive distance toward the origin.
- the IPS does not contain a line segment with
  - one endpoint at 1^1 that extends some positive distance toward 1^{13}.
  - one endpoint at 1^2 that extends some positive distance toward 1^{23}.
  - one endpoint at 1^3 that extends some positive distance toward the origin.

- the points 1^1 and 1^2 are each neither Pareto maximal nor Pareto minimal.
- the point 1^3 is Pareto minimal but not Pareto maximal.

And, in Figure 5.3,

- the IPS contains a line segment with
  - one endpoint at 1^1 that extends some positive distance toward 1^{13}.
  - one endpoint at 1^2 that extends some positive distance toward 1^{23}.
  - one endpoint at 1^3 that extends some positive distance toward the origin.
- the IPS does not contain a line segment with
  - one endpoint at 1^1 that extends some positive distance toward 1^{12}.
  - one endpoint at 1^2 that extends some positive distance toward 1^{12}.
  - one endpoint at 1^3 that extends some positive distance toward 1^{13}.
  - one endpoint at 1^3 that extends some positive distance toward 1^{23}.
  - one endpoint at 1^1 that extends some positive distance toward the origin.
  - one endpoint at 1^2 that extends some positive distance toward the origin.
- the points 1^1 and 1^2 are each Pareto minimal but not Pareto maximal, and
- the point 1^3 is Pareto maximal but not Pareto minimal.

Next, for each of the preceding two situations, we construct a cake C and measures m_1, m_2, and m_3 whose IPS is as given. We shall use the first of these examples in the proofs of Lemma 5.45 and Theorem 5.37. We shall return to these two examples in Chapter 9.

Example 5.43 Let C be the interval $[0, 3)$ on the real number line and let m_L be Lebesgue measure on this set. Suppose that there are three players, Player 1, Player 2, and Player 3, with corresponding measures m_1, m_2, and m_3, respectively, defined as follows: for any $A \subseteq C$,

$$m_1(A) = \frac{1}{2} m_L(A \cap [0, 2))$$

$$m_2(A) = \frac{1}{2} m_L(A \cap [1, 3))$$

$$m_3(A) = m_L(A \cap [1, 2))$$

Then, m_1, m_2, and m_3 are measures on C and

- m_1 and m_2 each fail to be absolutely continuous with respect to the other and with respect to m_3, and
- m_3 is absolutely continuous with respect to both m_1 and m_2.

This situation is as described earlier and illustrated in Figure 5.2.

Notice that the corresponding IPS intersects the $z = 1$ plane in a square with vertices $(0, 0, 1)$, $(\frac{1}{2}, 0, 1)$, $(\frac{1}{2}, \frac{1}{2}, 1)$, and $(0, \frac{1}{2}, 1)$. (This can be seen by considering all partitions in which Player 3 receives the interval $[1, 2)$ of C.) This fact will be used in the proof of Lemma 5.45.

Example 5.44 Let C be the interval $[0, 3)$ on the real number line and let m_L be Lebesgue measure on this set. Suppose that there are three players, Player 1, Player 2, and Player 3, with corresponding measures m_1, m_2,and m_3, respectively, defined as follows: for any $A \subseteq C$,

$$m_1(A) = \frac{2}{3}m_L(A \cap [0, 1)) + \frac{1}{3}m_L(A \cap [1, 2))$$
$$m_2(A) = \frac{1}{3}m_L(A \cap [0, 1)) + \frac{2}{3}m_L(A \cap [1, 2))$$
$$m_3(A) = \frac{1}{3}m_L(A \cap [0, 3))$$

Then, m_1, m_2, and m_3 are measures on C and

- m_1 and m_2 are each absolutely continuous with respect each other and with respect to m_3, and
- m_3 is fails to be absolutely continuous with respect to both m_1 and m_2.

This situation is as described earlier and illustrated in Figure 5.3.

Since the IPS is a subset of the unit hypercube, it is clear that, for each $i = 1, 2, \ldots, n$, the point 1^i is not an interior point of a line segment contained in the IPS. Lemma 3.29 tells us that, in the two-player context, any point that is neither Pareto maximal nor Pareto minimal, and is not equal to $(1, 0)$ or to $(0, 1)$, is an interior point of a line segment contained in the IPS. The natural generalization of this statement to the context of more than two players fails.

Lemma 5.45 *There is a cake C, corresponding measures $m_1, m_2, \ldots, m_n$ on C, and a point p in the associated IPS that is neither Pareto maximal nor Pareto minimal, is not equal to any of the points $1^1, 1^2, \ldots, 1^n$, and is not an interior point of a line segment contained in the IPS.*

Proof: We shall present an example involving three players. It will not be hard to see how to adapt this idea to more than three players.

Let C, m_1, m_2, and m_3 be as in Example 5.43. We noted previously that the corresponding IPS intersects the $z = 1$ plane in a square with vertices $(0, 0, 1)$, $(\frac{1}{2}, 0, 1)$, $(\frac{1}{2}, \frac{1}{2}, 1)$, and$(0, \frac{1}{2}, 1)$. This implies that neither $(\frac{1}{2}, 0, 1)$ nor

$(0, \frac{1}{2}, 1)$ is an interior point of a line segment contained in the IPS. Neither of these points is Pareto maximal, since the point $(\frac{1}{2}, \frac{1}{2}, 1)$ is Pareto bigger than each. Each is obviously not Pareto minimal. □

The preceding argument can be seen clearly in Figure 5.2a.

It is not hard to see that the lemma, together with Theorem 4.4, implies that the natural generalization of Theorem 3.30 to our present context fails.

Armed with Example 5.43, we are now ready to prove Theorem 5.37.

Proof of Theorem 5.37: We shall again present an example involving three players, and it will not be hard to see how to adapt this idea to more than three players.

Let C, m_1, m_2, and m_3 be as in Example 5.43. Notice that the points $(\frac{1}{2}, 0, 1)$, $(\frac{1}{2}, \frac{1}{2}, 1)$, $(1, \frac{1}{2}, 0)$, and $(1, 0, 0)$ are all in the IPS and are coplanar. Also, the rectangle determined by these points lies on the boundary of the IPS. Pick a point p that is an interior point of this rectangle, such as the center of the rectangle, $p = (\frac{3}{4}, \frac{1}{4}, \frac{1}{2})$.

This point p is on the boundary of the IPS and is an interior point of the unit hypercube. We must show that p is not on the Pareto boundary of the IPS.

Since the coordinates of p sum to more than one, we know that p is not a Pareto minimal point (since there are certainly points on the simplex such as, for example, $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$, that are Pareto smaller than p). The point p is not Pareto maximal since, for example, the point $(\frac{3}{4}, \frac{1}{2}, \frac{1}{2})$ is in the IPS (it is on the line segment between the points $(\frac{1}{2}, \frac{1}{2}, 1)$ and $(1, \frac{1}{2}, 0)$, both of which are in the IPS) and is Pareto bigger than p. Hence, p is not on the Pareto boundary of the IPS. □

As was the case for Lemma 5.45, the preceding argument can be seen clearly in Figure 5.2.

Next, we consider how the results of Section 5C, where we considered fairness and efficiency together, adjust to our present setting. As in that section, the only fairness properties that we presently consider are proportionality, strong proportionality, and the related chores notions.

The failure of absolute continuity implies that the measures are not all equal and, hence, in considering Theorem 5.23, we need only investigate how to adjust part b of the theorem. The appropriate adjustment is the following.

Theorem 5.46

a. If the measures concentrate on disjoint sets, then

i. the IPS has exactly one point that is both proportional and Pareto maximal, and that point is $(1, 1, \ldots, 1)$.

   ii. *the IPS has exactly one point that is both strongly proportional and Pareto maximal, and that point is* $(1, 1, \ldots, 1)$.

b. *If the measures do not concentrate on disjoint sets, then*
   i. *the IPS has infinitely many points that are both proportional and Pareto maximal.*
   ii. *the IPS has infinitely many points that are both strongly proportional and Pareto maximal.*

Proof: Part a follows easily from part a of Theorem 5.40, since the point $(1, 1, \ldots, 1)$ is certainly proportional and strongly proportional. Part bi follows from part bii. The proof for bii is similar to the proof of part b of Theorem 5.40, but requires some adjustments. We describe these adjustments here.

We shall need the following observation: any point that is Pareto bigger than a strongly proportional point is strongly proportional.

We assume that the measures do not concentrate on disjoint sets. Instead of beginning with any Pareto maximal point (as we did in the proof of Theorem 5.40), we instead begin with a point that is strongly proportional (which we know exists, by Theorem 5.27) and then use Theorem 5.38 to obtain a point that is Pareto maximal and is Pareto bigger than this point. Call the point we obtain in this way $p = (p_1, p_2, \ldots, p_n)$. By our preceding observation, p is also strongly proportional.

As in the proof of Theorem 5.40, we see that, since the measures do not concentrate on disjoint sets, $(1, 1, \ldots, 1) \notin$IPS, and therefore, for at least one $i = 1, 2, \ldots, n$, $p_i < 1$. As in that proof, we wish to pick a point q that is larger than p in at least one coordinate and then to pick a point r that is Pareto maximal and is equal to, or Pareto bigger than, q. However, we now need to be a bit more careful. We must pick such a point q that is strongly proportional.

Fix some $i = 1, 2, \ldots, n$ such that $p_i < 1$ and consider the line segment between p and 1^i. (Recall that 1^i is the point with ith coordinate equal to one and zeros elsewhere.) Since $p \in$ IPS and $1^i \in$ IPS, convexity implies that this line segment lies completely in the IPS. Since p is strongly proportional, points sufficiently close to p on this line segment are strongly proportional. Let $q = (q_1, q_2, \ldots, q_n)$ be such a point (where $q \neq p$). Then $q_i > p_i$.

If q is not Pareto maximal then, by Theorem 5.38, let r be a Pareto maximal point that is Pareto bigger than q. If q is Pareto maximal, let $r = q$. Then r is Pareto maximal and, by our preceding observation, r is strongly proportional. Set $r = (r_1, r_2, \ldots, r_n)$. Since $r_i \geq q_i > p_i$, we know that $r \neq p$. Hence, there are at least two points that are strongly proportional and Pareto maximal.

The remainder of the proof is almost the same as the proof of part b of Theorem 5.40. The only difference is that we assume, by way of contradiction, that there are finitely many points that are both strongly proportional and Pareto maximal. We also need to observe that the midpoint of a line segment between two strongly proportional points is strongly proportional. □

Corollary 5.47

a. If the measures $m_1, m_2, \ldots, m_n$ concentrate on the disjoint sets P_1, $P_2, \ldots, P_n$, respectively, then

i. all partitions that are both proportional and Pareto maximal are s-equivalent and p-equivalent (and are s-equivalent and p-equivalent to $\langle P_1, P_2, \ldots, P_n \rangle$).

ii. all partitions that are both strongly proportional and Pareto maximal are s-equivalent and p-equivalent (and are s-equivalent and p-equivalent to $\langle P_1, P_2, \ldots, P_n \rangle$).

b. If the measures do not concentrate on disjoint sets, then

i. there are infinitely many mutually non-s-equivalent partitions that are both proportional and Pareto maximal.

ii. there are infinitely many mutually non-p-equivalent partitions that are both proportional and Pareto maximal.

iii. there are infinitely many mutually non-s-equivalent partitions that are both strongly proportional and Pareto maximal.

iv. there are infinitely many mutually non-p-equivalent partitions that are both strongly proportional and Pareto maximal.

Proof: Part a follows from part a of the theorem, together with Theorem 4.4 and the fact that $(1, 1, \ldots, 1)$ does not lie in the interior of a line segment contained in the IPS.

Parts bi, bii, and biii follow from part biv. Part biv follows from part bii of the theorem and the fact that distinct points of the IPS are the image, under m, of non-p-equivalent partitions. □

Corollary 5.47 – Equivalence Class Version

a. If the measures $m_1, m_2, \ldots, m_n$ concentrate on the disjoint sets $P_1, P_2, \ldots, P_n$, respectively, then

i. there is exactly one s-class that is both proportional and Pareto maximal (and that class is $[\langle P_1, P_2, \ldots, P_n \rangle]_s$), and there is exactly one p-class that is both proportional and Pareto maximal (and that class is $[\langle P_1, P_2, \ldots P_n \rangle]_p$).

ii. there is exactly one s-class that is both strongly proportional and Pareto maximal (and that class is $[\langle P_1, P_2, \ldots, P_n \rangle]_s$), and there is exactly one p-class that is both strongly proportional and Pareto maximal (and that class is $[\langle P_1, P_2, \ldots, P_n \rangle]_p$).

b. If the measures do not concentrate on disjoint sets, then

i. there are infinitely many s-classes that are both proportional and Pareto maximal.

ii. there are infinitely many p-classes that are both proportional and Pareto maximal.

iii. there are infinitely many s-classes that are both strongly proportional and Pareto maximal.

iv. there are infinitely many p-classes that are both strongly proportional and Pareto maximal.

Of course, the s-classes and the p-classes in parts ai and aii are the same set.

The chores versions of Theorem 5.46 and Corollary 5.47 are as follows. The proofs are similar and we omit them.

Theorem 5.48

a. If the measures concentrate on the complements of disjoint sets, then

i. the IPS has exactly one point that is both c-proportional and Pareto minimal, and that point is $(0, 0, \ldots, 0)$.

ii. the IPS has exactly one point that is both strongly c-proportional and Pareto minimal, and that point is $(0, 0, \ldots, 0)$.

b. If the measures do not concentrate on the complements of disjoint sets, then

i. the IPS has infinitely many points that are both c-proportional and Pareto minimal.

ii. the IPS has infinitely many points that are both strongly c-proportional and Pareto minimal.

Corollary 5.49

a. If the measures $m_1, m_2, \ldots, m_n$ *concentrate on the complements of the disjoint sets* $Q_1, Q_2, \ldots, Q_n$*, respectively, then*

i. all partitions that are both c-proportional and Pareto minimal are s-equivalent and p-equivalent (and are s-equivalent and p-equivalent to $\langle Q_1, Q_2, \ldots, Q_n \rangle$*).*

ii. all partitions that are both strongly c-proportional and Pareto minimal are s-equivalent and p-equivalent (and are s-equivalent and p-equivalent to $\langle Q_1, Q_2, \ldots, Q_n \rangle$*).*

b. If the measures do not concentrate on the complements of disjoint sets, then

i. there are infinitely many mutually non-s-equivalent partitions that are both c-proportional and Pareto minimal.

ii. there are infinitely many mutually non-p-equivalent partitions that are both c-proportional and Pareto minimal.

iii. there are infinitely many mutually non-s-equivalent partitions that are both strongly c-proportional and Pareto minimal.

iv. there are infinitely many mutually non-p-equivalent partitions that are both strongly c-proportional and Pareto minimal.

Corollary 5.49 – Equivalence Class Version

a. If the measures $m_1, m_2, \ldots, m_n$ concentrate on the complements of the disjoint sets $Q_1, Q_2, \ldots, Q_n$, respectively, then

i. there is exactly one s-class that is both c-proportional and Pareto minimal (and that class is $[\langle Q_1, Q_2, \ldots, Q_n\rangle]_s$), and there is exactly one p-class that is both c-proportional and Pareto minimal (and that class is $[\langle Q_1, Q_2, \ldots, Q_n\rangle]_p$).

ii. there is exactly one s-class that is both strongly c-proportional and Pareto minimal (and that class is $[\langle Q_1, Q_2, \ldots, Q_n\rangle]_s$), and there is exactly one p-class that is both strongly c-proportional and Pareto minimal (and that class is $[\langle Q_1, Q_2, \ldots, Q_n\rangle]_p$).

b. If the measures do not concentrate on the complements of disjoint sets, then

i. there are infinitely many s-classes that are both c-proportional and Pareto minimal.

ii. there are infinitely many p-classes that are both c-proportional and Pareto minimal.

iii. there are infinitely many s-classes that are both strongly c-proportional and Pareto minimal.

iv. there are infinitely many p-classes that are both strongly c-proportional and Pareto minimal.

Again, we note that the s-classes and the p-classes in parts ai and aii are the same set.

Next, we reconsider the existence of egalitarian partitions (which we discussed at the end of Section 5C) in light of the failure of absolute continuity. We recall that a partition $P = \langle P_1, P_2, \ldots, P_n\rangle$ is *egalitarian* if and only if $m_1(P_1) = m_2(P_2) = \cdots = m_n(P_n)$. As we saw in Section 5C, it is easy to show that egalitarian partitions exist. In that section, we saw that if the measures are absolutely continuous with respect to each other, then it is also straightforward to show combine egalitarianism with fairness and efficiency notions. If absolute continuity fails, it is still straightforward to combine egalitarianism with proportionality or strong proportionality (or the corresponding chores notions), as we did in Section 5C. However, such is not the case for Pareto maximality. Suppose that there are three players, Player 1, Player 2, and Player 3, with measures

m_1, m_2, and m_3, respectively. Assume that $m_1 = m_2$ and that m_1 and m_2 on the one hand, and m_3 on the other, concentrate on disjoint sets. Then the IPS is the convex hull of the set {(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)}. The outer Pareto boundary of this IPS consists of the closed line segment between (1, 0, 1) and (0, 1, 1). Clearly, none of the points on this line segment correspond to egalitarian partitions.

We close this section as we did Sections 5A and 5B, by commenting on the convexity of the sets of points satisfying our various fairness and efficiency properties. We first note that our remarks on fairness properties in Section 5A hold with or without absolute continuity. Hence:

- The set of all proportional points is a convex subset of the IPS.
- The set of all strongly proportional points is a convex subset of the IPS.
- The set of all envy-free points is a convex subset of the FIPS.
- The set of all strongly envy-free points is a convex subset of the FIPS.
- The set of all super envy-free points is a convex subset of the FIPS.

Analogous facts hold for the chores fairness properties. However, our previous remarks on the set of all Pareto maximal points and the set of all Pareto minimal points is not quite correct if absolute continuity fails. Theorem 5.40 implies that there may be only one Pareto maximal point or only one Pareto minimal point. Since a one-point set is convex, we must adjust our previous statement from Section 5B. Curiously, the sets under consideration will be convex only if one of the two extremes of agreement or disagreement of the measures occurs. In particular:

- The set of all Pareto maximal points in the IPS is a convex subset of the IPS if and only if either the measures are all equal or the measures concentrate on disjoint sets.
- The set of all Pareto minimal points in the IPS is a convex subset of the IPS if and only if either the measures are all equal or the measures concentrate on the complements of disjoint sets.

5E. Examples and Open Questions

In this section, we present examples and a theorem to illustrate how some of our results and our geometric perspectives may be used to establish the existence of partitions with various desired properties, and we conclude with open questions. We make no general assumption in this section about absolute continuity. We first present examples that rely on the IPS, and then examples that rely on the FIPS.

Example 5.50

a. A strongly proportional partition $P = \langle P_1, P_2, P_3 \rangle$ with $m_1(P_1) = \frac{1}{3} + 4\varepsilon$, $m_2(P_2) = \frac{1}{3} + 5\varepsilon$, and $m_3(P_3) = \frac{1}{3} + 6\varepsilon$ for some $\varepsilon > 0$.
b. A strongly c-proportional partition $P = \langle P_1, P_2, P_3 \rangle$ with $m_1(P_1) = \frac{1}{3} - 4\varepsilon$, $m_2(P_2) = \frac{1}{3} - 5\varepsilon$, and $m_3(P_3) = \frac{1}{3} - 6\varepsilon$ for some $\varepsilon > 0$.

If the measures are all equal, then such partitions do not exist, since in this case $m_1(P_1) + m_2(P_2) + m_3(P_3) = m_1(P_1) + m_1(P_2) + m_1(P_3) = m_1(C) = 1$. Assume then that the measures are not all equal, and consider part a. The existence of such a partition follows easily from Theorems 5.1 or 5.27 (depending on whether or not the measures are absolutely continuous with respect to each other) with $q = (4, 5, 6)$ and $\lambda = \varepsilon$.

Assuming that the measures are not all equal, there are actually infinitely many non-p-equivalent partitions satisfying the given conditions. Given a point of the IPS corresponding to a partition that satisfies these conditions, let q be any point in the IPS that is on the open line segment between the given point and the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. If Q is any partition such that $m(Q) = q$, then Q satisfies the given conditions. Since there are infinitely many such points q, and distinct points of the IPS are the image, under m, of non-p-equivalent partitions, it follows that there are infinitely many non-p-equivalent partitions satisfying the given conditions.

Notice that in this example we cannot insist that ε have any particular size. This is so since the λ of Theorems 5.1 and 5.27 can be small. If, for example, the measures are not very different, then the IPS will not be much larger than the simplex, and we may not be able to move far from the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ without leaving the IPS. On the other hand, if the measures are very different from each other, then the IPS will be larger and a larger value of ε may be possible.

If the measures are all absolutely continuous with respect to each other, then we could have applied Theorem 5.23 instead of Theorem 5.1 and obtained a partition that satisfies the given conditions and is also Pareto maximal. However, if the measures are not all absolutely continuous with respect to each other, then such a partition may or may not exist, since the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) + \lambda(4, 5, 6)$ (as in Theorem 5.27), where λ is chosen so that this point is on the outer boundary of the IPS, may or may not be on the outer Pareto boundary of the IPS.

The arguments for part b are similar and we omit them.

Our next example illustrates that, if the measures are absolutely continuous with respect to each other, then for any desired ratios between players' evaluations of their own pieces there is a Pareto maximal partition and a Pareto minimal partition for which these ratios are satisfied. As we shall see, this need not be true if the measures are not absolutely continuous with respect to each other.

Example 5.51

a. A Pareto maximal partition $P = \langle P_1, P_2, P_3 \rangle$ with $m_2(P_2) = 3m_1(P_1)$ and $m_3(P_3) = 5m_1(P_1)$.
b. A Pareto minimal partition $P = \langle P_1, P_2, P_3 \rangle$ with $m_2(P_2) = 3m_1(P_1)$ and $m_3(P_3) = 5m_1(P_1)$.

If the measures are absolutely continuous with respect to each other, then the existence of such partitions follows easily from Theorem 5.18 by setting $p = (1, 3, 5)$. Of course, any such set of ratios can be satisfied by choosing the correct p.

Notice that there is no partition satisfying the conditions of part a which is also proportional since, if there were such a partition $P = \langle P_1, P_2, P_3 \rangle$, then we would have $m_1(P_1) \geq \frac{1}{3}$ and, hence, $m_3(P_3) = 5m_1(P_1) \geq \frac{5}{3} > 1$. Geometrically, the idea here is that if we move from the origin in the direction given by $(1, 3, 5)$, we leave the unit cube at the point $(\frac{1}{5}, \frac{3}{5}, 1)$ without ever passing through any point with first coordinate at least $\frac{1}{3}$. Then, certainly, we hit no proportional points. On the other hand, as we leave the origin, moving in the direction given by $(1, 3, 5)$, we go through many points with all coordinates at most $\frac{1}{3}$. Whether we hit a point of the IPS that is c-proportional and Pareto minimal depends on whether we hit the IPS at all before leaving this region (i.e., the region in which all coordinates are at most $\frac{1}{3}$). This depends on the shape of the IPS. If, for example, the measures are all equal, then the IPS is equal to the simplex, and we do not hit the IPS until the point $(\frac{1}{9}, \frac{3}{9}, \frac{5}{9})$. In this case, there is no point of the IPS that is c-proportional and Pareto minimal.

Suppose now that the measures are not absolutely continuous with respect to each other. For part a, suppose that the measures concentrate on disjoint sets. Then, by Theorem 5.40, $(1, 1, 1)$ is the only Pareto maximal point. A partition corresponding to this point does not satisfy the conditions of part a. On the other hand, if the measures concentrate on the complements of disjoint sets, then, by Theorem 5.40, $(0, 0, 0)$ is the only Pareto minimal point. A partition corresponding to this point does satisfy the conditions of part b.

It is not hard to see that even if the measures do not concentrate on disjoint sets, but the failure of absolute continuity is sufficiently strong so that the intersection of the IPS with the $z = 1$ plane is large enough to include the point $(\frac{1}{5}, \frac{3}{5}, 1)$ in its interior, then there is no partition that satisfies the conditions of part a.

We note that the relationship between the standard and the chores issues in this example represents a departure from our usual symmetry between these settings.

Next, we show that it may or may not be possible to specify in advance the amount of cake certain players get in a Pareto maximal or a Pareto minimal partition.

Example 5.52

a. A Pareto maximal partition $P = \langle P_1, P_2\rangle$ with $m_1(P_1) = .2$.
b. A Pareto minimal partition $P = \langle P_1, P_2\rangle$with $m_1(P_1) = .2$.
c. A Pareto maximal partition $P = \langle P_1, P_2, P_3\rangle$ with $m_1(P_1) = .2$ and $m_2(P_2) = .3$.
d. A Pareto minimal partition $P = \langle P_1, P_2, P_3\rangle$ with $m_1(P_1) = .2$ and $m_2(P_2) = .3$.

There may be no partitions satisfying any of the preceding conditions if the associated measures are not absolutely continuous with respect to each other. For example, for parts a and b, suppose that the two measures concentrate on disjoint sets. Then, by Theorem 3.27, we know that (1, 1) is the only Pareto maximal point and (0, 0) is the only Pareto minimal point. This implies that there is no Pareto maximal partition $P = \langle P_1, P_2\rangle$ with $m_1(P_1) = .2$ and no Pareto minimal partition $P = \langle P_1, P_2\rangle$ with $m_1(P_1) = .2$. A similar argument shows that there may be no partition that satisfies the conditions of parts c or d if absolute continuity fails.

Consider parts a and b, and suppose now that the two measures are absolutely continuous with respect to each other. The vertical line $x = .2$ contains at least one point of the IPS, since this line intersects the simplex. (It will contain exactly one point, the point (.2, .8), which is on the simplex, if and only if the measures are equal.) We imagine beginning at the point (.2, 0) and moving up (i.e., in the positive y direction). The first point $p = (p_1, p_2)$ of the IPS that we encounter is on the inner boundary of the IPS and, hence, by Theorem 3.9, is on the inner Pareto boundary of the IPS. Thus, p is a Pareto minimal point and if $P = \langle P_1, P_2\rangle$ is such that $m(P) = p$, then P is a Pareto minimal partition and $m_1(P_1) = .2$. Similarly, if $q = (q_1, q_2)$ is the last point of the IPS that we encounter as we move up from the point (.2, 0), then q is on the outer boundary of the IPS and, hence, by Theorem 3.9, q is on the outer Pareto boundary of the IPS. Thus, q is a Pareto maximal point and, if $Q = \langle Q_1, Q_2\rangle$ is such that $m(Q) = q$, then Q is a Pareto maximal partition and $m_1(Q_1) = .2$. (The existence of such a first point and last point of contact with the IPS as we move up from (.2, 0) follows from the fact that the IPS is closed.)

Next we consider parts c and d, and we assume that the measures are absolutely continuous with respect to each other. As in the previous example, we will see that our usual symmetry between Pareto maximality and Pareto minimality does not hold. It turns out that there is always a partition satisfying the conditions of part c, but there may or may not be a partition that satisfies the conditions of part d.

For part c, we first make an observation. If $P = \langle P_1, P_2, P_3 \rangle$ is Pareto maximal and, for some $i = 1, 2, 3$, $m_i(P_i) = 1$, then, for any $j = 1, 2, 3$ with $j \neq i$, $m_j(P_j) = 0$. This is so because if it were the case that $m_j(P_j) > 0$ for some such j then, by absolute continuity, $m_i(P_j) > 0$. But this is impossible, since $m_i(P_i) = 1$. This observation tells us that the intersection of the IPS with the $x = 1$ plane is the single point $(1, 0, 0)$, the intersection of the IPS with the $y = 1$ plane is the single point $(0, 1, 0)$, and the intersection of the IPS with the $z = 1$ plane is the single point $(0, 0, 1)$.

To show that there is a partition that satisfies the conditions of part c, we proceed as we did for part a. We begin at the point $(.2, .3, 0)$ and move up (i.e., in the positive z direction). We know that we must hit the IPS, since the point $(.2, .3, .5)$ is on the simplex and therefore is in the IPS. Let $p = (.2, .3, p_3)$ be the last point of contact with the IPS. Then p is on the outer boundary of the IPS and, by our preceding observation, $p_3 < 1$. Hence, p is an interior point of the unit hypercube. Theorem 5.16 implies that p is on the outer Pareto boundary of the IPS. Hence, p is a Pareto maximal point and any partition P with $m(P) = p$ is a Pareto maximal partition satisfying the conditions of part c.

For part d, consider Example 5.14 (which is illustrated in Figure 5.1). If m_1 and m_2 are sufficiently different, then $(.2, .3, 0)$ is an interior point of IPS_{12}, the intersection of the IPS and the xy plane. This tells us that $(.2, .3, 0)$ is on the inner boundary but not on the inner Pareto boundary of the IPS and, hence, there are no Pareto minimal points of the form $(.2, .3, p_3)$. It follows that, in this case, there is no partition P that satisfies the conditions of part d.

What if we had asked for a Pareto maximal partition $P = \langle P_1, P_2, P_3 \rangle$ with $m_1(P_1) = .7$ and $m_2(P_2) = .6$. Is this possible? The answer is maybe and maybe not. Certainly our preceding description would not be correct since, beginning with the point $(.7, .6, 0)$ in the xy plane and moving up, we do not hit the simplex. Whether or not we hit the IPS, and can therefore continue until we reach the outer boundary of the IPS, depends on the shape of the IPS, which depends on the particular measures. It is not hard to see that such a partition exists if and only if $(.7, .6, 0) \in \text{IPS}_{12}$.

Next, we move on to examples that involve the FIPS. These examples will use Theorem 4.18.

Example 5.53 – The Examples from Section 4B In Section 4B, we listed four situations meant to illustrate the differences between the various fairness properties. However, in that section, we did not show that the given situations were possible. We do so here. For convenience, we first restate these four situations. In each case, we assume that there are three players and ask for a partition $P = \langle P_1, P_2, P_3 \rangle$ so that the given conditions are satisfied for some $\varepsilon > 0$. The measures may or may not be absolutely continuous with respect to each other.

a.
$$\begin{aligned} m_1(P_1) &= \tfrac{1}{3} + 2\varepsilon & m_1(P_2) &= \tfrac{1}{3} + 5\varepsilon & m_1(P_3) &= \tfrac{1}{3} - 7\varepsilon \\ m_2(P_1) &= \tfrac{1}{3} + 20\varepsilon & m_2(P_2) &= \tfrac{1}{3} + 5\varepsilon & m_2(P_3) &= \tfrac{1}{3} - 25\varepsilon \\ m_3(P_1) &= \tfrac{1}{3} - 7\varepsilon & m_3(P_2) &= \tfrac{1}{3} + 5\varepsilon & m_3(P_3) &= \tfrac{1}{3} + 2\varepsilon \end{aligned}$$

b.
$$\begin{aligned} m_1(P_1) &= \tfrac{1}{3} + 2\varepsilon & m_1(P_2) &= \tfrac{1}{3} - 4\varepsilon & m_1(P_3) &= \tfrac{1}{3} + 2\varepsilon \\ m_2(P_1) &= \tfrac{1}{3} - 3\varepsilon & m_2(P_2) &= \tfrac{1}{3} + 5\varepsilon & m_2(P_3) &= \tfrac{1}{3} - 2\varepsilon \\ m_3(P_1) &= \tfrac{1}{3} + 2\varepsilon & m_3(P_2) &= \tfrac{1}{3} - 4\varepsilon & m_3(P_3) &= \tfrac{1}{3} + 2\varepsilon \end{aligned}$$

c.
$$\begin{aligned} m_1(P_1) &= \tfrac{1}{3} + 2\varepsilon & m_1(P_2) &= \tfrac{1}{3} + \varepsilon & m_1(P_3) &= \tfrac{1}{3} - 3\varepsilon \\ m_2(P_1) &= \tfrac{1}{3} + 3\varepsilon & m_2(P_2) &= \tfrac{1}{3} + 5\varepsilon & m_2(P_3) &= \tfrac{1}{3} - 8\varepsilon \\ m_3(P_1) &= \tfrac{1}{3} + \varepsilon & m_3(P_2) &= \tfrac{1}{3} - 3\varepsilon & m_3(P_3) &= \tfrac{1}{3} + 2\varepsilon \end{aligned}$$

d.
$$\begin{aligned} m_1(P_1) &= \tfrac{1}{3} + 2\varepsilon & m_1(P_2) &= \tfrac{1}{3} - \varepsilon & m_1(P_3) &= \tfrac{1}{3} - \varepsilon \\ m_2(P_1) &= \tfrac{1}{3} - 3\varepsilon & m_2(P_2) &= \tfrac{1}{3} + 5\varepsilon & m_2(P_3) &= \tfrac{1}{3} - 2\varepsilon \\ m_3(P_1) &= \tfrac{1}{3} - \varepsilon & m_3(P_2) &= \tfrac{1}{3} - \varepsilon & m_3(P_3) &= \tfrac{1}{3} + 2\varepsilon \end{aligned}$$

As noted in Section 4B, in all four of the preceding situations, P is strongly proportional and,

- in situation a, P is not envy-free.
- in situation b, P is envy-free but not strongly envy-free.
- in situation c, P is strongly envy-free but not super envy-free.
- in situation d, P is super envy-free.

We now consider each of these situations. In situation a, P is strongly proportional but satisfies no additional fairness properties. By Corollaries 5.2

and 5.28, we know that a strongly proportional partition exists if and only if the measures are not all equal. Of course, the existence of dependence relationships between the measures may imply that there is no partition P satisfying the properties given in situation a. For example, if $m_1 = m_2$, or if $m_1 = \frac{1}{2}m_2 + \frac{1}{2}m_3$, then it is clear that situation a is impossible since, for example, the players' evaluations of piece P_1 (i.e., $m_1(P_1) = \frac{1}{3} + 2\varepsilon$, $m_2(P_1) = \frac{1}{3} + 20\varepsilon$, $m_3(P_1) = \frac{1}{3} - 7\varepsilon$) directly contradict these dependence relationships. However, note that the conditions of situation a are consistent with the dependence relationship $m_1 = \frac{1}{3}m_2 + \frac{2}{3}m_3$.. Let us assume that this is the only dependence relationship that holds between the three measures and hence that DEP (see Definition 4.16) contains just this one equation. We can then use Theorem 4.18 to show that there is a partition P satisfying the conditions of situation a.

Let

$$q = \begin{bmatrix} 2 & 5 & -7 \\ 20 & 5 & -25 \\ -7 & 5 & 2 \end{bmatrix}.$$

Note that each row of q sums to one and each column is consistent with the one assumed dependence relationship, $m_1 = \frac{1}{3}m_2 + \frac{2}{3}m_3$. Hence, q is a proper matrix. Theorem 4.18 (with ε in place of λ) implies that, for some $\varepsilon > 0$,

$$\begin{bmatrix} \frac{1}{3} + 2\varepsilon & \frac{1}{3} + 5\varepsilon & \frac{1}{3} - 7\varepsilon \\ \frac{1}{3} + 20\varepsilon & \frac{1}{3} + 5\varepsilon & \frac{1}{3} - 25\varepsilon \\ \frac{1}{3} - 7\varepsilon & \frac{1}{3} + 5\varepsilon & \frac{1}{3} + 2\varepsilon \end{bmatrix} = \left[\frac{1}{3} + \varepsilon q_{ij}\right]_{i,j \le n} \in \text{FIPS}.$$

Hence, for some partition $P = \langle P_1, P_2, P_3 \rangle$,

$$\begin{array}{lll} m_1(P_1) = \dfrac{1}{3} + 2\varepsilon & m_1(P_2) = \dfrac{1}{3} + 5\varepsilon & m_1(P_3) = \dfrac{1}{3} - 7\varepsilon \\ m_2(P_1) = \dfrac{1}{3} + 20\varepsilon & m_2(P_2) = \dfrac{1}{3} + 5\varepsilon & m_2(P_3) = \dfrac{1}{3} - 25\varepsilon \\ m_3(P_1) = \dfrac{1}{3} - 7\varepsilon & m_3(P_2) = \dfrac{1}{3} + 5\varepsilon & m_3(P_3) = \dfrac{1}{3} + 2\varepsilon \end{array}$$

and so P satisfies the conditions of situation a. Of course, this construction would also work if the measures were linearly independent (i.e., if DEP were empty).

We next consider situation b. If P is a partition that satisfies the conditions of situation b, then P is strongly proportional and envy-free but is not strongly envy-free. We note that, as in situation a, it is easy see that certain dependence relationships among the measures would make the existence of such a partition impossible. Let us assume that the only dependence relationship is $m_1 = m_3$ and, hence, DEP contains just this one equality. By Corollaries 5.7 and 5.32, the assumption that two of the measures are equal implies that there does not exist a strongly envy-free partition P. We shall show that there is a partition that is strongly proportional, envy-free, and, in particular, satisfies the conditions of situation b.

Let

$$q = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 5 & -2 \\ 2 & -4 & 2 \end{bmatrix}.$$

Since each row of q sums to one and in each column the first and third entries are equal (and so each column is consistent with the one equation in DEP), we know that q is a proper matrix. It follows from Theorem 4.18 that, for some $\varepsilon > 0$,

$$\begin{bmatrix} \frac{1}{3}+2\varepsilon & \frac{1}{3}-4\varepsilon & \frac{1}{3}+2\varepsilon \\ \frac{1}{3}-3\varepsilon & \frac{1}{3}+5\varepsilon & \frac{1}{3}-2\varepsilon \\ \frac{1}{3}+2\varepsilon & \frac{1}{3}-4\varepsilon & \frac{1}{3}+2\varepsilon \end{bmatrix} = \left[\frac{1}{3}+\varepsilon q_{ij}\right]_{i,j\le n} \in \text{FIPS}.$$

Hence, for some partition $P = \langle P_1, P_2, P_3 \rangle$,

$$\begin{array}{lll} m_1(P_1) = \frac{1}{3}+2\varepsilon & m_1(P_2) = \frac{1}{3}-4\varepsilon & m_1(P_3) = \frac{1}{3}+2\varepsilon \\ m_2(P_1) = \frac{1}{3}-3\varepsilon & m_2(P_2) = \frac{1}{3}+5\varepsilon & m_2(P_3) = \frac{1}{3}-2\varepsilon \\ m_3(P_1) = \frac{1}{3}+2\varepsilon & m_3(P_2) = \frac{1}{3}-4\varepsilon & m_3(P_3) = \frac{1}{3}+2\varepsilon \end{array}$$

and so P satisfies the conditions of situation b. As in situation a, we note that this construction would also work if the measures were linearly independent.

Next, we consider situation c. A partition that satisfies situation c is strongly proportional and strongly envy-free, but not super envy-free. Corollaries 5.2 and 5.28 tell us that, in order for there to exist a strongly proportional partition, the measures must not all be equal. However, Corollaries 5.7 and 5.32 tell us that, in order for there to exist a strongly envy-free partition, we must have the stronger condition that no two of the measures are equal. Dependence relationships are not ruled out by these results. Of course, as in situations a and b, it is easy to find dependence relationships involving the three measures that are inconsistent with the given conditions.

Let us assume that there is one dependence relationship between the measures, and this relationship is $m_1 = \frac{1}{2}m_2 + \frac{1}{2}m_3$. It is clear that the conditions of situation c are consistent with this relationship. We note that by Corollaries 5.7 and 5.32 such a dependence relationship implies that there does not exist a super envy-free partition. We shall show that there is a partition that is strongly proportional, strongly envy-free, and, in particular, satisfies the conditions of situation c.

Let

$$q = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 5 & -8 \\ 1 & -3 & 2 \end{bmatrix}.$$

Note that each row of q sums to one. Also, since the only dependence relationship between the measures is $m_1 = \frac{1}{2}m_2 + \frac{1}{2}m_3$, we know that this is the only equation in DEP. Since the columns of q are consistent with this equation, it follows that q is a proper matrix. Theorem 4.18 implies that, for some $\varepsilon > 0$,

$$\begin{bmatrix} \frac{1}{3}+2\varepsilon & \frac{1}{3}+\varepsilon & \frac{1}{3}-3\varepsilon \\ \frac{1}{3}+3\varepsilon & \frac{1}{3}+5\varepsilon & \frac{1}{3}-8\varepsilon \\ \frac{1}{3}+\varepsilon & \frac{1}{3}-3\varepsilon & \frac{1}{3}+2\varepsilon \end{bmatrix} = \left[\frac{1}{3}+\varepsilon q_{ij}\right]_{i,j\le n} \in \text{FIPS}.$$

Hence, for some partition $P = \langle P_1, P_2, P_3 \rangle$,

$$\begin{array}{lll} m_1(P_1) = \dfrac{1}{3}+2\varepsilon & m_1(P_2) = \dfrac{1}{3}+\varepsilon & m_1(P_3) = \dfrac{1}{3}-3\varepsilon \\ m_2(P_1) = \dfrac{1}{3}+3\varepsilon & m_2(P_2) = \dfrac{1}{3}+5\varepsilon & m_2(P_3) = \dfrac{1}{3}-8\varepsilon \\ m_3(P_1) = \dfrac{1}{3}+\varepsilon & m_3(P_2) = \dfrac{1}{3}-3\varepsilon & m_3(P_3) = \dfrac{1}{3}+2\varepsilon. \end{array}$$

Therefore, P satisfies the conditions of situation c. As in situations a and b, we observe that the construction would also work if the measures were linearly independent.

Finally, we consider situation d. If P is a partition satisfying the conditions of situation d, then P is super envy-free. By Corollaries 5.7 and 5.32, we know that such a partition exists if and only if the measures are linearly independent. We make that assumption now, and thus DEP is empty.

Let

$$q = \begin{bmatrix} 2 & -1 & -1 \\ -3 & 5 & -2 \\ -1 & -1 & 2 \end{bmatrix}.$$

Since each row of q sums to one and DEP is empty, it follows that q is a proper matrix. Then, by Theorem 4.18, we know that, for some $\varepsilon > 0$,

$$\begin{bmatrix} \frac{1}{3}+2\varepsilon & \frac{1}{3}-\varepsilon & \frac{1}{3}-\varepsilon \\ \frac{1}{3}-3\varepsilon & \frac{1}{3}+5\varepsilon & \frac{1}{3}-2\varepsilon \\ \frac{1}{3}-\varepsilon & \frac{1}{3}-\varepsilon & \frac{1}{3}+2\varepsilon \end{bmatrix} = \left[\frac{1}{3}+\varepsilon q_{ij}\right]_{i,j\le n} \in \text{FIPS}.$$

Hence, for some partition $P = \langle P_1, P_2, P_3\rangle$,

$$\begin{array}{lll} m_1(P_1) = \dfrac{1}{3}+2\varepsilon & m_1(P_2) = \dfrac{1}{3}-\varepsilon & m_1(P_3) = \dfrac{1}{3}-\varepsilon \\ m_2(P_1) = \dfrac{1}{3}-3\varepsilon & m_2(P_2) = \dfrac{1}{3}+5\varepsilon & m_2(P_3) = \dfrac{1}{3}-2\varepsilon \\ m_3(P_1) = \dfrac{1}{3}-\varepsilon & m_3(P_2) = \dfrac{1}{3}-\varepsilon & m_3(P_3) = \dfrac{1}{3}+2\varepsilon \end{array}$$

and so P satisfies the conditions of situation d.

Fix some point $p = (p_1, p_2, \ldots, p_n)$ in the simplex that has all positive coordinates (i.e., p is an interior point of the simplex). Corollary 1.5 implies that there is a partition $P = \langle P_1, P_2, \ldots, P_n\rangle$ such that $m_i(P_i) = p_i$ for each $i = 1, 2, \ldots, n$. Can we change the equalities in some or all of these equations to inequalities and find a partition P satisfying these n relationships? Clearly, this is not always possible. If the measures are all equal, then it is certainly not possible to find a partition $P = \langle P_1, P_2, \ldots, P_n\rangle$ such that $m_i(P_i) > p_i$ for each i, since this would imply that each player's measure of the entire cake C is greater than one. Our next result shows that having the measures not all equal is the only necessary restriction. After proving this result, we will present an example.

Theorem 5.54 *Suppose that the measures are not all equal and* $p = (p_1, p_2, \ldots, p_n)$ *is a point in the simplex with all positive coordinates. In addition, assume that, for each* $i = 1, 2, \ldots, n$, σ_i *is one of the relations "<," "=," or ">." Then there exists a partition* $P = \langle P_1, P_2, \ldots, P_n\rangle$ *such that* $m_i(P_i)\sigma_i p_i$ *for each such i.*

Proof: Since the measures are not all equal, we may assume, by renumbering if necessary, that $m_{n-1} \neq m_n$. We wish to define an appropriate proper matrix q and to then apply Theorem 4.18.

For each $i = 1, 2, \ldots, n$, define q_{ii} as follows:

$$\begin{aligned} q_{ii} &= -1 \quad \text{if } \sigma_i \text{ is the "<" relation} \\ q_{ii} &= 0 \quad \text{if } \sigma_i \text{ is the "=" relation} \\ q_{ii} &= 1 \quad \text{if } \sigma_i \text{ is the ">" relation} \end{aligned}$$

We need to define q_{ij} for distinct $i, j = 1, 2, \ldots, n$ in such a way that the resulting matrix q is proper. We begin by assigning values for all such q_{ij} that are not in either of the last two columns of q as follows: for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n-2$ with $i \neq j$, define q_{ij} arbitrarily, subject to the constraint that each of these $n-2$ columns is consistent with each of the equations in DEP.

Next, we note that there is only one as yet undefined entry in each of the last two rows of q ($q_{n-1,n}$ in row $n-1$, and $q_{n,n-1}$ in row n). Define these two values in the unique way that makes each row sum to zero. Observe that since "$m_{n-1} = m_n$" is not one of the equations in DEP, the assignment that we have just made for $q_{n-1,n}$ and $q_{n,n-1}$ is not inconsistent with any equation in DEP.

It remains for us to define q_{ij} for $i = 1, 2, \ldots, n-2$ and $j = n-1, n$. For each such i, define $q_{i,n-1}$ arbitrarily, subject to the constraint that the $(n-1)$th column of q is consistent with each of the equations in DEP.

Finally, we must define q_{in} for $i = 1, 2, \ldots, n-2$. For each such i, we define q_{in} to be the unique number that makes each row sum to zero. This completes the definition of the matrix q. We must show that q is proper.

It is immediate from our construction that each row of q sums to zero. It is also immediate that each of the first $n-1$ columns is consistent with each equation in DEP. These two facts together imply that the nth column is consistent with each equation in DEP. This establishes that q is a proper matrix.

By Theorem 4.18, we know that, for some $\varepsilon > 0$, $[p_j + \varepsilon q_{ij}]_{i,j \leq n} \in$ FIPS. Hence, for some partition $P = \langle P_1, P_2, \ldots, P_n \rangle$, $m_1(P_1) = p_1 + \varepsilon q_{11}$, $m_2(P_2) = p_2 + \varepsilon q_{22}, \ldots, m_n(P_n) = p_n + \varepsilon q_{nn}$. It then follows from the definition of the q_{ii} that, for each $i = 1, 2, \ldots, n$, $m_i(P_i)\sigma_i q_i$, as desired. This completes the proof of the theorem. □

The following is an easy application of the theorem.

Example 5.55 A Pareto maximal partition $P = \langle P_1, P_2, P_3, P_4 \rangle$ with $m_1(P_1) > .1$, $m_2(P_2) > .2$, $m_3(P_3) = .3$, and $m_4(P_4) = .4$.

If the measures are all equal, then it is easy to see that there is no such partition P since, if there were, then each measure would assign a value greater than one to the whole cake.

Assume then that the measures are not all equal, set $p = (.1, .2, .3, .4)$, let σ_1 and σ_2 be the "$>$" relation, and let σ_3 and σ_4 be the "$=$" relation. Theorem 5.54 implies that there is a partition $P = \langle P_1, P_2, P_3, P_4 \rangle$ such that $m_i(P_i)\sigma_i p_i$ for each $i = 1, 2, 3, 4$. This P satisfies the desired equalities and inequalities.

Example 5.56 A partition on which each player's measure of the relative sizes of the n pieces of cake can satisfy any given order.

Assume that the measures are linearly independent. Suppose there are four players and we wish to obtain a partition $P = \langle P_1, P_2, P_3, P_4 \rangle$ satisfying the following:

$$\begin{aligned} m_1(P_1) = m_1(P_2) < m_1(P_3) = m_1(P_4) \\ m_2(P_2) < m_2(P_4) < m_2(P_3) < m_2(P_1) \\ m_3(P_3) < m_3(P_1) = m_3(P_4) < m_3(P_2) \\ m_4(P_1) = m_4(P_2) = m_4(P_3) = m_4(P_4) \end{aligned}$$

Since the measures are linearly independent, a matrix is proper if and only if each of its rows sums to one. Thus, finding a proper matrix that correctly represents the preceding situation is quite easy. For example, we can let

$$q = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 2 & -2 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows from Theorem 4.18 that, for some $\varepsilon > 0$,

$$\begin{bmatrix} \frac{1}{4} - \varepsilon & \frac{1}{4} - \varepsilon & \frac{1}{4} + \varepsilon & \frac{1}{4} + \varepsilon \\ \frac{1}{4} + 2\varepsilon & \frac{1}{4} - 2\varepsilon & \frac{1}{4} + \varepsilon & \frac{1}{4} - \varepsilon \\ \frac{1}{4} & \frac{1}{4} + \varepsilon & \frac{1}{4} - \varepsilon & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \left[\frac{1}{4} + \varepsilon q_{ij}\right]_{i,j \le n} \in \text{FIPS}.$$

Hence, for some partition $P = \langle P_1, P_2, P_3, P_4 \rangle$,

$$\begin{array}{llll} m_1(P_1) = \frac{1}{4} - \varepsilon & m_1(P_2) = \frac{1}{4} - \varepsilon & m_1(P_3) = \frac{1}{4} + \varepsilon & m_1(P_4) = \frac{1}{4} + \varepsilon \\ m_2(P_1) = \frac{1}{4} + 2\varepsilon & m_2(P_2) = \frac{1}{4} - 2\varepsilon & m_2(P_3) = \frac{1}{4} + \varepsilon & m_2(P_4) = \frac{1}{4} - \varepsilon \\ m_3(P_1) = \frac{1}{4} & m_3(P_2) = \frac{1}{4} + \varepsilon & m_3(P_3) = \frac{1}{4} - \varepsilon & m_3(P_4) = \frac{1}{4} \\ m_4(P_1) = \frac{1}{4} & m_4(P_2) = \frac{1}{4} & m_4(P_3) = \frac{1}{4} & m_4(P_4) = \frac{1}{4} \end{array}$$

and therefore,

$$\begin{array}{l} m_1(P_1) = m_1(P_2) < m_1(P_3) = m_1(P_4) \\ m_2(P_2) < m_2(P_4) < m_2(P_3) < m_2(P_1) \\ m_3(P_3) < m_3(P_1) = m_3(P_4) < m_3(P_2) \\ m_4(P_1) = m_4(P_2) = m_4(P_3) = m_4(P_4) \end{array}$$

as desired.

It is not hard to see that certain dependence relationships between the measures are inconsistent with the desired equalities and inequalities. To illustrate, note that the given equalities and inequalities imply that $m_1(P_2) < \frac{1}{4}$, $m_2(P_2) < \frac{1}{4}$, and $m_4(P_2) = \frac{1}{4}$. Hence, for example, no such partition could exist if it $m_4 = \frac{1}{2}m_1 + \frac{1}{2}m_2$.

The conditions of our final two examples have a similar appearance, but one turns out to be possible and one impossible. This contrast will lead us to two open questions.

Example 5.57 A partition $P = \langle P_1, P_2, P_3, P_4 \rangle$ satisfying the following relationships:

$$\begin{array}{llll} m_1(P_1) > .1 & m_1(P_2) > .2 & m_1(P_3) < .3 & m_1(P_4) = .4 \\ m_2(P_1) < .1 & m_2(P_2) = .2 & m_2(P_3) < .3 & m_2(P_4) > .4 \\ m_3(P_1) = .1 & m_3(P_2) > .2 & m_3(P_3) < .3 & m_3(P_4) > .4 \\ m_4(P_1) > .1 & m_4(P_2) > .2 & m_4(P_3) < .3 & m_4(P_4) > .4 \end{array}$$

Certain dependence relationships imply the impossibility of obtaining such a partition. For example, the dependence relationship $m_1 = \frac{1}{2}m_2 + \frac{1}{2}m_3$ is inconsistent with the relationships $m_1(P_1) > .1$, $m_2(P_1) < .1$, and $m_3(P_1) = .1$.

Let us assume that the following dependence relationships hold:

$$m_3 = \frac{1}{2}m_1 + \frac{1}{2}m_2 \qquad \text{and} \qquad m_4 = \frac{3}{4}m_1 + \frac{1}{4}m_2$$

We assume that this list is complete, except for the dependence relationships that are implied by these two.

It is not hard to see that the given relationships involving P are consistent with the given dependence relationships. We shall define a proper matrix and then use Theorem 4.18 to obtain a partition P satisfying the desired conditions.

Let

$$q = \begin{bmatrix} 4 & 4 & -8 & 0 \\ -4 & 0 & -4 & 8 \\ 0 & 2 & -6 & 4 \\ 2 & 3 & -7 & 2 \end{bmatrix}.$$

Each row of q sums to one and each column is consistent with each of the given dependence relationships. Hence, q is a proper matrix. Theorem 4.18 tells us that, for some $\varepsilon > 0$,

$$\begin{bmatrix} .1+4\varepsilon & .2+4\varepsilon & .3-8\varepsilon & .4 \\ .1-4\varepsilon & .2 & .3-4\varepsilon & .4+8\varepsilon \\ .1 & .2+2\varepsilon & .3-6\varepsilon & .4+4\varepsilon \\ .1+2\varepsilon & .2+3\varepsilon & .3-7\varepsilon & .4+2\varepsilon \end{bmatrix} = [r_j + \varepsilon q_{ij}]_{i,j \le n} \in \text{FIPS}$$

where we have set $r = (.1, .2, .3, .4)$. Hence, for some partition $P = \langle P_1, P_2, P_3, P_4 \rangle$,

$$\begin{array}{llll} m_1(P_1) = .1+4\varepsilon & m_1(P_2) = .2+4\varepsilon & m_1(P_3) = .3-8\varepsilon & m_1(P_4) = .4 \\ m_2(P_1) = .1-4\varepsilon & m_2(P_2) = .2 & m_2(P_3) = .3-4\varepsilon & m_2(P_4) = .4+8\varepsilon \\ m_3(P_1) = .1 & m_3(P_2) = .2+2\varepsilon & m_3(P_3) = .3-6\varepsilon & m_3(P_4) = .4+4\varepsilon \\ m_4(P_1) = .1+2\varepsilon & m_4(P_2) = .2+3\varepsilon & m_4(P_3) = .3-7\varepsilon & m_4(P_4) = .4+2\varepsilon \end{array}$$

and therefore,

$$\begin{array}{llll} m_1(P_1) > .1 & m_1(P_2) > .2 & m_1(P_3) < .3 & m_1(P_4) = .4 \\ m_2(P_1) < .1 & m_2(P_2) = .2 & m_2(P_3) < .3 & m_2(P_4) > .4 \\ m_3(P_1) = .1 & m_3(P_2) > .2 & m_3(P_3) < .3 & m_3(P_4) > .4 \\ m_4(P_1) > .1 & m_4(P_2) > .2 & m_4(P_3) < .3 & m_4(P_4) > .4, \end{array}$$

as desired.

Example 5.58 A partition $P = \langle P_1, P_2, P_3, P_4\rangle$ satisfying the following relationships:

$m_1(P_1) > .25$	$m_1(P_2) = .25$	$m_1(P_3) < .25$	$m_1(P_4) > .25$
$m_2(P_1) = .25$	$m_2(P_2) < .25$	$m_2(P_3) < .25$	$m_2(P_4) > .25$
$m_3(P_1) > .25$	$m_3(P_2) < .25$	$m_3(P_3) < .25$	$m_3(P_4) < .25$
$m_4(P_1) > .25$	$m_4(P_2) < .25$	$m_4(P_3) > .25$	$m_4(P_4) > .25$

Let us assume that the following dependence relationship holds:

$$m_1 + m_2 = m_3 + m_4$$

We first observe that there is nothing obvious preventing us from using Theorem 4.18, as in the previous example, to obtain a partition satisfying the given conditions. In particular, the four conditions involving each measure are consistent with the fact that each measure assigns value one to C (in contrast with, for example, the conditions $m_1(P_1) > .25$, $m_1(P_2) > .25$, $m_1(P_3) > .25$, and $m_1(P_4) > .25$), and the four conditions involving each piece of the partition are consistent with the given dependence relationship (in contrast with, for example, the conditions $m_1(P_1) > .25$, $m_2(P_1) > .25$, $m_3(P_1) < .25$, and $m_4(P_1) < .25$). However, there is a subtler problem here that arises from the interaction between these two types of constraints.

We claim that, the given conditions on the partition, together with the one dependence relationship, imply that m_1 gives a larger value to each of the four pieces of cake than does m_3.

Consider piece P_1. We are given that $m_2(P_1) = .25$ and $m_4(P_1) > .25$. Thus, $m_2(P_1) < m_4(P_1)$ and so, since $m_1(P_1) + m_2(P_1) = m_3(P_1) + m_4(P_1)$, it follows that $m_1(P_1) > m_3(P_1)$.

Since $m_1(P_2) = .25$ and $m_3(P_2) < .25$, it is obvious that $m_1(P_2) > m_3(P_2)$.

Next, consider piece P_3. We are given that $m_2(P_3) < .25$ and $m_4(P_3) > .25$. Thus, $m_2(P_3) < m_4(P_3)$ and so, since $m_1(P_3) + m_2(P_3) = m_3(P_3) + m_4(P_3)$, it follows that $m_1(P_3) > m_3(P_3)$.

Finally, since $m_1(P_4) > .25$ and $m_3(P_4) < .25$, it is obvious that $m_1(P_4) > m_3(P_4)$.

We have shown that, for each $i = 1, 2, 3, 4$, $m_1(P_i) > m_3(P_i)$. But then

$$\begin{aligned} 1 = m_1(C) &= m_1(P_1) + m_1(P_2) + m_1(P_3) + m_1(P_4) \\ &> m_3(P_1) + m_3(P_2) + m_3(P_3) + m_3(P_4) = m_3(C) = 1 \end{aligned}$$

which is a contradiction. Hence, there is no partition satisfying the given conditions and dependence relationship.

A comparison between the last two examples presents us with a question. In general, given certain dependence relationships, equalities, and inequalities, as in the last two examples, how can we determine whether there exists a partition satisfying these dependence relationships, equalities, and inequalities? We do not know the answer to this question, but we wish to develop some terminology in order to ask this question more precisely.

Definition 5.59 A *relation matrix* is an $n \times n$ matrix, each of whose entries is one of the relations “$<$,” “$=$,” or “$>$.” If $\sigma = [\sigma_{ij}]_{i,j \le n}$ is a relation matrix, $p = (p_1, p_2, \ldots, p_n)$ is a point in the simplex with all positive coordinates, and $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition, we shall say that P *satisfies* σ *with respect to* p if and only if, for each $i, j = 1, 2, \ldots, n$, $m_i(P_j)\sigma_{ij} p_j$. Also, a proper matrix $q = [q_{ij}]_{i,j \le n}$ *satisfies* σ if and only if, for each $i, j = 1, 2, \ldots, n$, $q_{ij}\sigma_{ij}0$.

Suppose that σ and p are as in the definition. If there exists a proper matrix q that satisfies σ, then it follows from Theorem 4.18 that there is a partition P that satisfies σ with respect to p. Conversely, if $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition that satisfies σ with respect to p, then $q = [m_i(P_j) - p_j]_{i,j \le n}$ is a proper matrix that satisfies σ. Hence, the question of whether there exists a partition that satisfies σ with respect to p is equivalent to the question of whether there exists a proper matrix that satisfies σ.

To illustrate this idea, we note that the existence of a partition satisfying the conditions of Example 5.57 is equivalent to the existence of a proper matrix that satisfies the relation matrix

$$\begin{bmatrix} > & > & < & = \\ < & = & < & > \\ = & > & < & > \\ > & > & < & > \end{bmatrix}$$

and the existence of a partition satisfying the conditions of Example 5.58 is equivalent to the existence of a proper matrix that satisfies the relation matrix

$$\begin{bmatrix} > & = & < & > \\ = & < & < & > \\ > & < & < & < \\ > & < & > & > \end{bmatrix}.$$

We also observe that the existence of a super envy-free partition is equivalent to the existence of a proper matrix that satisfies the relation matrix

$$\begin{bmatrix} > & < & < & < \\ < & > & < & < \\ < & < & > & < \\ < & < & < & > \end{bmatrix}.$$

We are now ready to state our question more precisely.

Open Question 5.60 Suppose that $p = (p_1, p_2, \ldots, p_n)$ is a point in the simplex with all positive coordinates and $\sigma = [\sigma_{ij}]_{i,j \le n}$ a relation matrix. Is there a procedure for determining whether there exists a partition that satisfies σ with respect to p? Or, equivalently (by Theorem 4.18), is there a procedure for determining whether there exists a proper matrix that satisfies σ?

An affirmative answer to this question would still leave the following.

Open Question 5.61 Suppose that we know that there exists a partition that satisfies σ with respect to p or, equivalently, that there is a proper matrix that satisfies σ. How do we actually find such a matrix?

For example, given the relation matrix

$$\sigma = \begin{bmatrix} > & > & < & = \\ < & = & < & > \\ = & > & < & > \\ > & > & < & > \end{bmatrix}$$

of Example 5.57, how can we find a proper matrix satisfying σ? In this example, we presented the matrix

$$q = \begin{bmatrix} 4 & 4 & -8 & 0 \\ -4 & 0 & -4 & 8 \\ 0 & 2 & -6 & 4 \\ 2 & 3 & -7 & 2 \end{bmatrix}$$

and observed that this matrix is proper and satisfies σ. We obtained this matrix by “fiddling around.” We do not know a procedure for finding such a matrix in general.

6

Characterizing Pareto Optimality

Introduction and Preliminary Ideas

We now turn our attention to characterizations of Pareto optimality. Suppose that P is a partition. How can we determine whether or not P is Pareto maximal or Pareto minimal? We have seen that P is Pareto maximal if and only if $m(P)$ is on the outer Pareto boundary of the IPS and that P is Pareto minimal if and only if $m(P)$ is on the inner Pareto boundary of the IPS. However, the relevant IPS is not always available to us. Our goal is to find other ways to make this determination. We begin by considering Pareto maximality.

Of course, if P is not Pareto maximal, the presentation of a Pareto bigger partition establishes this. But it may not, in general, be clear how to find such a Pareto bigger partition and, of course, our inability to find such a Pareto bigger partition is not a proof that there is none. The methods presented in Examples 6.3 (where we show that a certain partition is not Pareto maximal) and 6.6 (where we show that a certain partition is Pareto maximal) are rather ad hoc. We wish to develop general characterizations for Pareto maximality (and, of course, Pareto minimality too).

Chapters 7, 8, and 10 each focus on a different approach. (Chapter 9 is devoted to the dvelopment of the framework needed in Chapter 10.) Our first characterization, presented in Chapter 7, involves the maximization (for Pareto maximality) or minimization (for Pareto minimality) of certain linear combinations of the measures. In Chapter 8, we present our second characterization. This characterization involves certain numbers called partition ratios that we shall associate with a partition. Partition ratios provide us with a comparison between how much a given player values his or her piece of cake compared to how other players value that piece. Our third characterization, which we present in Chapter 10, involves a geometric construction on the simplex, and is attributable to D. Weller ([43]). In the present chapter, we state some definitions and prove two theorems (Theorems 6.2 and 6.4) that will be needed for these characterizations. We make no assumptions in this chapter about the absolute continuity of the measures.

Suppose $P = \langle P_1, P_2, \ldots, P_n\rangle$ is a partition of C. For any non-empty $\delta \subseteq \{1, 2, \ldots, n\}$, $\langle P_i : i \in \delta\rangle$ is a partition of $\bigcup_{i\in\delta} P_i$ among the players named by δ. What is the relationship between the Pareto maximality of such a partition of $\bigcup_{i\in\delta} P_i$ and the Pareto maximality of the partition P of C? We introduce some terminology to simplify our discussion.

Definition 6.1 A partition $P = \langle P_1, P_2, \ldots, P_n\rangle$ of C is *proper subpartition Pareto maximal* if and only if, for every proper and non-empty $\delta \subseteq \{1, 2, \ldots, n\}$, $\langle P_i : i \in \delta\rangle$ is a Pareto maximal partition of $\bigcup_{i\in\delta} P_i$ among the players named by δ.

We wish study the relationship between Pareto maximality and proper subpartition Pareto maximality. The relationship in one direction is easy.

Theorem 6.2 *If P is a Pareto maximal partition of C, then P is proper subpartition Pareto maximal.*

Proof: Fix a partition $P = \langle P_1, P_2, \ldots, P_n\rangle$ of C and suppose that P is not proper subpartition Pareto maximal. Then for some proper and non-empty $\delta \subseteq \{1, 2, \ldots, n\}$ there is a partition $\langle Q_i : i \in \delta\rangle$ of $\bigcup_{i\in\delta} P_i$ that is Pareto bigger than $\langle P_i : i \in \delta\rangle$. Define a partition $R = \langle R_1, R_2, \ldots, R_n\rangle$ of C as follows: for each $i = 1, 2, \ldots, n$,

$$R_i = \begin{cases} Q_i & \text{if } i \in \delta \\ P_i & \text{if } i \notin \delta \end{cases}$$

Then R is a partition of C that is Pareto bigger than P, and so P is not Pareto maximal. □

The converse of the theorem is false. In other words, there exist partitions $P = \langle P_1, P_2, \ldots, P_n\rangle$ that are not Pareto maximal but are such that, for any proper and non-empty $\delta \subseteq \{1, 2, \ldots, n\}$, $\langle P_i : i \in \delta\rangle$ is a Pareto maximal partition of $\bigcup_{i\in\delta} P_i$ among the players named by δ. As a trivial example, let P be any non-Pareto maximal partition of C among two players. Then certainly for any proper and non-empty $\delta \subseteq \{1, 2\}$, $\langle P_i : i \in \delta\rangle$ is a Pareto maximal partition of $\bigcup_{i\in\delta} P_i$ among the players named by δ since, in this case, δ is a singleton. (For any piece of cake A and any one player, the trivial partition of A into one piece that is given to that one player is a Pareto maximal partition of A to the one player.) The following is a less trivial example.

Example 6.3 Let C be the interval $[0, 3)$ on the real number line and let m_L be Lebesgue measure on this set. Suppose that there are three players, Player 1, Player 2, and Player 3, with corresponding measures m_1, m_2, and m_3, respectively, defined as follows: for any $A \subseteq C$,

$$m_1(A) = .3m_L(A \cap [0, 1)) + .1m_L(A \cap [1, 2)) + .6m_L(A \cap [2, 3))$$
$$m_2(A) = .6m_L(A \cap [0, 1)) + .3m_L(A \cap [1, 2)) + .1m_L(A \cap [2, 3))$$
$$m_3(A) = .1m_L(A \cap [0, 1)) + .6m_L(A \cap [1, 2)) + .3m_L(A \cap [2, 3))$$

We note that

$$\begin{aligned} m_1(C) &= .3m_L(C \cap [0, 1)) + .1m_L(C \cap [1, 2)) + .6m_L(C \cap [2, 3)) \\ &= .3 + .1 + .6 = 1. \end{aligned}$$

Similarly, $m_2(C) = m_3(C) = 1$. Thus, m_1, m_2, and m_3 are measures on C, and it is easy to see that these measures are absolutely continuous with respect to each other.

Let $P = \langle [0, 1), [1, 2), [2, 3) \rangle$. We claim that P is not Pareto maximal but is proper subpartition Pareto maximal. We begin by computing each player's measure of his or her own piece of cake:

$$m_1([0, 1)) = .3m_L([0, 1)) + .1m_L(\emptyset) + .6m_L(\emptyset) = .3$$
$$m_2([1, 2)) = .6m_L(\emptyset) + .3m_L([1, 2)) + .1m_L(\emptyset) = .3$$
$$m_3([2, 3)) = .1m_L(\emptyset) + .6m_L(\emptyset) + .3m_L([2, 3)) = .3$$

Next, we consider the partition $Q = \langle [2, 3), [0, 1), [1, 2) \rangle$. Let us compute each player's measure of his or her own piece of cake according to this partition:

$$m_1([2, 3)) = .3m_L(\emptyset) + .1m_L(\emptyset) + .6m_L([2, 3)) = .6$$
$$m_2([0, 1)) = .6m_L([0, 1)) + .3m_L(\emptyset) + .1m_L(\emptyset) = .6$$
$$m_3([1, 2)) = .1m_L(\emptyset) + .6m_L([1, 2)) + .3m_L(\emptyset) = .6$$

Thus, Q is Pareto bigger than P and so P is not Pareto maximal. It remains for us to show that P is proper subpartition Pareto maximal.

Consider the partition $R = \langle [0, 1), [1, 2) \rangle$ of $[0, 2)$ between Players 1 and 2. We must show that this partition is a Pareto maximal partition of $[0, 2)$ between Player 1 and Player 2. Suppose, by way of contradiction, that it is not. Then some transfer of cake from Player 1 to Player 2, call it piece A, and from Player 2 to Player 1, call it piece B, must result in a Pareto bigger partition. Player 1's change due to this trade is $m_1(B) - m_1(A)$ and Player 2's change due to this trade is $m_2(A) - m_2(B)$. Since $A \subseteq [0, 1)$ and $B \subseteq [1, 2)$, the changes for Players 1 and 2 will be $.1m_L(B) - .3m_L(A)$ and $.6m_L(A) - .3m_L(B)$, respectively. Since we are assuming that the change produced by this trade results in a partition that is Pareto bigger than the original partition, it must be that

$$.1m_L(B) - .3m_L(A) \geq 0 \qquad \text{and} \qquad .6m_L(A) - .3m_L(B) \geq 0$$

with at least one of these inequalities being strict. Adding twice the first inequality to the second, and using the fact that at least one of the inequalities is strict, we obtain $-.1m_L(B) > 0$, and thus $m_L(B) < 0$, a contradiction. Thus, R is a Pareto maximal partition of $[0, 2)$ between Players 1 and 2. The proofs that $\langle [0, 1), [2, 3) \rangle$ is a Pareto maximal partition of $[0, 1) \cup [2, 3)$ between Players 1 and 3, and that $\langle [1, 2), [2, 3) \rangle$ is a Pareto maximal partition of $[1, 3)$ between Players 2 and 3, are similar. This establishes that P is proper subpartition Pareto maximal.

This example shows that the converse of Theorem 6.2 is false. We shall gain some additional insight into the relationship between Pareto maximality and proper subpartition Pareto maximality when we revisit this example in Chapters 8, 10, and 13 (see Examples 8.10, 10.10, and 13.23).

We can establish something resembling a converse to Theorem 6.2 if we restrict the subpartitions that we consider and add some assumptions.

Theorem 6.4 *Suppose that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of C and the following conditions hold:*

a. For any $i = 1, 2, \ldots, n$, if $A \subseteq P_i$ and $m_i(A) = 0$, then for every $j = 1, 2, \ldots, n$, $m_j(A) = 0$.
b. There exists a partition $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_m \rangle$ of $\{1, 2, \ldots, n\}$ such that
  i. for every $k = 1, 2, \ldots, m$, $\langle P_i : i \in \gamma_k \rangle$ is a Pareto maximal partition of $\bigcup_{i \in \gamma_k} P_i$ among the players named by γ_k, and either
  ii. for every $k, k' = 1, 2, \ldots, m$ with $k < k'$, if $j \in \gamma_k$ and $j' \in \gamma_{k'}$, then $m_{j'}(P_j) = 0$ or
  iii. for every $k, k' = 1, 2, \ldots, m$ with $k < k'$, if $j \in \gamma_k$ and $j' \in \gamma_{k'}$, then $m_j(P_{j'}) = 0$.

Then P is Pareto maximal.

It is easy to see that one can easily change a partition $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_m \rangle$ that satisfies condition bii to one that satisfies condition biii, or vice versa, simply by reversing the order of the γ_k.

Before proving the theorem, we first comment on the sense in which it resembles a converse to Theorem 6.2 and then develop some informal perspective on the theorem.

The converse to Theorem 6.2 would have as its premise that P is proper subpartition Pareto maximal; i.e., for any proper and non-empty $\delta \subseteq \{1, 2, \ldots, n\}$, $\langle P_i : i \in \delta \rangle$ is a Pareto maximal partition of $\bigcup_{i \in \delta} P_i$ among the players named by δ. Condition bi of Theorem 6.4 says that we need not check to see whether this condition holds for every proper and non-empty $\delta \subseteq \{1, 2, \ldots, n\}$. Instead, we need only consider the collection of m disjoint subsets of $\{1, 2, \ldots, n\}$ given by γ. However, condition bii or biii gives an additional requirement. We wish to develop some informal perspective about this requirement.

We think of the conditions of the theorem as describing an iterative process, consisting of m stages. At each stage certain pieces of cake are allotted to certain players. With $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_m \rangle$ as in the theorem, we shall think of the first allotment as going to the players named by γ_1, the second allotment as going to the players named by γ_2, and so on. Condition bii says that a piece of cake allotted to some player at some stage k has value zero to any player that receives his or her piece at a later stage k'. Similarly, condition biii says that a piece of cake allotted to some player at some stage k' has value zero to any player that receives his or her piece at an earlier stage k. This iterative perspective will be explored in considerably more detail in Chapters 7 and 10.

We shall discuss condition a of Theorem 6.4 following the proof of the theorem.

Proof of Theorem 6.4: We assume that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of C, $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_m \rangle$ is a partition of $\{1, 2, \ldots, n\}$, and that the given conditions hold. In particular, we shall assume that condition biii holds. If, instead, condition bii holds, then the proof is similar. We must show that P is Pareto maximal.

Suppose, by way of contradiction, that P is not Pareto maximal. Then, by Theorem 5.38, we know that there is a Pareto maximal partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ that is Pareto bigger than P.

We may view Q as the result of beginning with P and making some collection of transfers of cake between players. Suppose that one of these is a transfer from Player j to Player j' and this transfer consists of piece A. We may assume, without loss of generality, that $m_{j'}(A) > 0$, since otherwise no player would be worse off if we simply eliminate this transfer. By condition a of the theorem,

$m_j(A) > 0$. Hence, we may assume that all transfers involve a piece of cake that has positive value to both the giver and the receiver.

Claim The transition from partition P to partition Q involves no transfer of cake between players named by different pieces of γ. In other words, for any distinct $k, k' = 1, 2, \ldots, m$, and any $j \in \gamma_k$ and $j' \in \gamma_{k'}$, the transition from partition P to partition Q involves no transfer from Player j to Player j', or from Player j' to Player j.

Proof of Claim: Fix $k, k' = 1, 2, \ldots, m$ with $k < k'$. We first show that the transition from partition P to partition Q involves no transfer from a player named by $\gamma_{k'}$ to a player named by γ_k, and then we show that this transition involves no transfer from a player named by γ_k to a player named by $\gamma_{k'}$.

Fix $j \in \gamma_k$ and $j' \in \gamma_{k'}$. We must show that the transition from partition P to partition Q involves no transfer of cake from Player j' to Player j. By condition biii of the theorem, $m_j(P_{j'}) = 0$. Then, since we have assumed that all transfers involve a piece of cake that has positive measure to both the giver and the receiver, it follows that the transition from partition P to partition Q does not involve any transfer from Player j' to Player j.

To show that the transition from partition P to partition Q involves no transfer of cake from a player named by γ_k to a player named by $\gamma_{k'}$, we assume by way of contradiction that this is not the case and that $k = 1, 2, \ldots, m-1$ is minimal such that, for some $k' = k+1, k+2, \ldots, m$, there is a $j \in \gamma_k$ and $j' \in \gamma_{k'}$ so that the transition from partition P to partition Q involves a transfer from Player j to Player j'. The previous paragraph and the minimality of k imply that this transition involves no transfer of cake from any player not named by γ_k to any player named by γ_k. Then, since $\langle P_i : i \in \gamma_k \rangle$ is a Pareto maximal partition of $\bigcup_{i \in \gamma_k} P_i$ among the players named by γ_k, it is clear that since the transition from partition P to partition Q involves a transfer from a player named by γ_k (i.e., Player j) to a player not named by γ_k (i.e., Player j'), $m_i(Q_i) < m_i(P_i)$ for some $i \in \gamma_k$. This contradicts the fact that Q is Pareto bigger than P and, hence, establishes the claim.

We return to the proof of the theorem. The claim tells us that each transfer in the transition from partition P to partition Q takes place within some γ_k. (Different transfers can take place within different γ_k.) Then, for each $k = 1, 2, \ldots, m$, $\langle Q_i : i \in \gamma_k \rangle$ is a partition of $\bigcup_{i \in \gamma_k} P_i$. Since Q is Pareto bigger than P, it follows that, for some such k, $\langle Q_i : i \in \gamma_k \rangle$ is Pareto bigger than $\langle P_i : i \in \gamma_k \rangle$. This contradicts condition bi and, hence, completes the proof of the theorem. □

Condition a of Theorem 6.4 was necessary because it clearly must hold for any Pareto maximal partition and it is not implied by conditions bi, bii, or biii. This condition will reappear in subsequent chapters, and so it will be convenient to give it a name. While any partition that is not Pareto maximal can be thought of as wasteful, a partition that violates condition a of the theorem can be thought of as an extreme case of wastefulness. This perspective motivates our choice of a name for this condition.

Definition 6.5 A partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is *wasteful* if, for some $i, j = 1, 2, \ldots, n$, there exists $A \subseteq P_i$ such that $m_i(A) = 0$ and $m_j(A) > 0$. If a partition is not wasteful, then it is *non-wasteful*.

Condition a of Theorem 6.4 says that P is non-wasteful. Notice that if a partition is wasteful then a single transfer (of the set A in the definition, from Player i to Player j) results in a Pareto better partition.

The following example illustrates the use of Theorem 6.4 in showing that a partition is Pareto maximal.

Example 6.6 Let C be the interval $[0, 4)$ on the real number line and let m_L be Lebesgue measure on this set. Suppose that there are four players, Player 1, Player 2, Player 3, and Player 4, with corresponding measures m_1, m_2, m_3, and m_4, respectively, defined as follows: for any $A \subseteq C$,

$$\begin{aligned}
m_1(A) &= \frac{2}{3} m_L(A \cap [0, 1)) + \frac{1}{3} m_L(A \cap [1, 2)) \\
m_2(A) &= \frac{1}{3} m_L(A \cap [0, 1)) + \frac{2}{3} m_L(A \cap [1, 2)) \\
m_3(A) &= \frac{1}{3} m_L(A \cap [0, 1)) + \frac{1}{6} m_L(A \cap [1, 2)) + \frac{1}{3} m_L(A \cap [2, 3)) \\
&\quad + \frac{1}{6} m_L(A \cap [3, 4)) \\
m_4(A) &= \frac{1}{6} m_L(A \cap [0, 1)) + \frac{1}{3} m_L(A \cap [1, 2)) + \frac{1}{6} m_L(A \cap [2, 3)) \\
&\quad + \frac{1}{3} m_L(A \cap [3, 4))
\end{aligned}$$

It is straightforward to check that $m_1(C) = m_2(C) = m_3(C) = m_4(C) = 1$, and so m_1, m_2, m_3, and m_4 are each measures on C. We also note that these measures are not all absolutely continuous with respect to each other since, for example, $m_1([2, 3)) = 0$ but $m_3([2, 3)) = \frac{1}{3}$. Consider the partition $P = \langle [0, 1), [1, 2), [2, 3), [3, 4) \rangle$. We claim that P is Pareto maximal. We shall establish this using Theorem 6.4.

We must show that the conditions of Theorem 6.4 are satisfied. To verify condition a, non-wastefulness, we must show that

- if $A \subseteq [0, 1)$ and $m_1(A) = 0$, then, for every $j = 1, 2, 3, 4$, $m_j(A) = 0$.
- if $A \subseteq [1, 2)$ and $m_2(A) = 0$, then, for every $j = 1, 2, 3, 4$, $m_j(A) = 0$.
- if $A \subseteq [2, 3)$ and $m_3(A) = 0$, then, for every $j = 1, 2, 3, 4$, $m_j(A) = 0$.
- if $A \subseteq [3, 4)$ and $m_4(A) = 0$, then, for every $j = 1, 2, 3, 4$, $m_j(A) = 0$.

The truth of each of these statements follows easily from the definitions of the measures. Thus condition a of the Theorem 6.4 holds.

For condition b, we must define a partition γ of $\{1, 2, 3, 4\}$. We do so with an eye toward satisfying condition bii.

It is clear from the definitions of the measures that if we set $\gamma_1 = \{3, 4\}$ then, for $j \in \gamma_1$ and $j' \in \{1, 2, 3, 4\} \backslash \gamma_1 = \{1, 2\}$, we have $m_{j'}(P_j) = 0$. Note that no singleton set has this property and hence $\{3, 4\}$ is the smallest such set with this property. Next, we observe that for no singleton set $\gamma_2 \subseteq \{1, 2\}$, $j \in \gamma_2$, and $j' \in \{1, 2, 3, 4\} \backslash (\gamma_1 \cup \gamma_2)$ do we have $m_{j'}(P_j) = 0$. Hence, we set $\gamma_2 = \{1, 2\}$ and, thus, $\gamma = \langle \{3, 4\}, \{1, 2\} \rangle$. We must show that condition b of the theorem holds with this γ.

For condition bi, we must show that

- $\langle [2, 3), [3, 4) \rangle$ is a Pareto maximal partition of $[2, 4)$ between Players 3 and 4 and
- $\langle [0, 1), [1, 2) \rangle$ is a Pareto maximal partition of $[0, 2)$ between Players 1 and 2.

This can be established using the same approach as was used in Example 6.3. We omit the details.

To verify condition bii, we must show that $m_1([2, 3)) = m_1([3, 4)) = m_2([2, 3)) = m_2([3, 4)) = 0$. This follows easily from the definitions of m_1 and m_2. Thus, Theorem 6.4 implies that P is Pareto maximal.

We close this chapter by stating the chores versions of Theorems 6.2 and 6.4. The proofs are similar and we omit them. Theorem 6.7 requires the notion of "proper subpartition Pareto minimal." The definition of this notion is the obvious adjustment of Definition 6.1 and we omit it.

Theorem 6.7 *If P is a Pareto minimal partition of C, then P is proper subpartition Pareto minimal.*

Theorem 6.8 *Suppose that* $P = \langle P_1, P_2, \ldots, P_n \rangle$ *is a partition of* C *and the following conditions hold:*

a. For any $i = 1, 2, \ldots, n$, *if* $A \subseteq P_i$ *and* $m_i(A) > 0$, *then, for every* $j = 1, 2, \ldots, n, m_j(A) > 0$.

b. There exists a partition $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_m \rangle$ *of* $\{1, 2, \ldots, n\}$ *such that,*

  i. for every $k = 1, 2, \ldots, m$, $\langle P_i : i \in \gamma_k \rangle$ *is a Pareto minimal partition of* $\bigcup_{i \in \gamma_k} P_i$ *among the players named by* γ_k, *and either*

  ii. for every $k, k' = 1, 2, \ldots, m$ *with* $k < k'$, *if* $j \in \gamma_k$ *and* $j' \in \gamma_{k'}$, *then* $m_{j'}(P_j) = 0$ *or*

  iii. for every $k, k' = 1, 2, \ldots, m$ *with* $k < k'$, *if* $j \in \gamma_k$ *and* $j' \in \gamma_{k'}$, *then* $m_j(P_{j'}) = 0$.

Then P *is Pareto minimal.*

7

Characterizing Pareto Optimality I

The IPS and Optimization of Convex Combinations of Measures

In this chapter, we shall use the IPS to provide characterizations of Pareto maximality and Pareto minimality using the notion of convex combinations of measures. In Section 7A, we begin with a purely geometric description of this notion in the two-player context, and in Section 7B, we establish our characterization in the general n-player context. In these sections, we assume that the measures are absolutely continuous with respect to each other. In Section 7C, we consider the situation without absolute continuity.

7A. Introduction: The Two-Player Context

We shall focus only on Pareto maximality in this section. All of the ideas in this section have analogous chores versions, but we will not state these here. We shall do so in the general n-player context in the next section.

Consider Figure 7.1. In each of the figures, we see the IPS for some cake C and measures m_1 and m_2, with the outer boundary darkened. (The IPS in Figures 7.1b and 7.1c is the same.) As illustrated in each figure, we imagine a line with negative slope, beginning to the upper right of the IPS and moving in a parallel manner until it makes contact with the IPS. Since the IPS is a closed subset of the plane, we know that there is a line in this family of parallel lines that makes first contact with the IPS.

In the situation depicted in Figure 7.1a, the family of parallel lines makes first contact with the IPS at point p, and at no other points. This is in contrast with the situation depicted in Figure 7.1b, where the family of parallel lines makes simultaneous first contact with the IPS at all points along the line segment determined by points p and q. Notice that in Figure 7.1a no other family of parallel lines makes first contact with the IPS at point p, and in Figure 7.1b no other family of parallel lines makes first contact with the IPS at all points

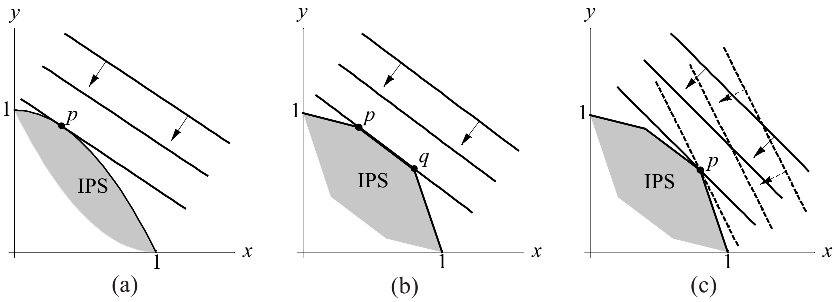


Figure 7.1

along the line segment determined by points p and q. This is in contrast with the situation depicted in Figure 7.1c, where there are an infinite number of families of parallel lines which each make first contact with the IPS at point p and at no other points. The figure shows two such families. (One family is illustrated with solid lines and one with dashed lines.) In Chapter 11, we shall see that for each of these figures there is a cake and corresponding measures that yield the given IPS, and in Sections 12B and 12C we will explore what underlies the differences among the situations depicted in Figures 7.1a, 7.1b, and 7.1c.

We shall see that every Pareto maximal point in the IPS is as above. That is, every Pareto maximal point is the point of first contact of some family of parallel lines with the IPS. Conversely, we shall also see that for any such family of parallel lines the point or points of first contact with the IPS are Pareto maximal points. Then, we shall connect this notion with the notion of maximizing convex combinations of measures. We first place some restrictions on the families of parallel lines that we consider.

Definition 7.1 We shall say that a line is *non-negative* if and only if it does not intersect the negative x axis, the negative y axis, or the origin.

Thus a line is non-negative if and only if it has an equation of the form $\alpha x + \beta y = c$, where $\alpha \geq 0$, $\beta \geq 0$, $c > 0$, and α and β are not both zero. Put another way, a line is non-negative if and only if it is either a vertical line intersecting the positive x axis, a horizontal line intersecting the positive y axis, or a line that intersects both the positive x axis and the positive y axis. We shall be interested in families of parallel non-negative lines. Notice that if $\alpha x + \beta y = c$ is a non-negative line, then all non-negative lines parallel to this line are obtained by varying the value of $c > 0$, with larger values of c yielding lines that are farther from the origin. In Observation 7.2, we imagine starting with a large enough value of c so that the line is "beyond" (i.e., farther away from the origin than) the IPS.

Observation 7.2

a. If F is a family of parallel non-negative lines, then there is at least one point of the IPS at which F makes first contact with the IPS, and any such first-contact point is Pareto maximal.
b. If p is a Pareto maximal point of the IPS, then there is at least one family of parallel non-negative lines that makes first contact with the IPS at p.

As illustrated in Figures 7.1a and 7.1b, there may be one or infinitely many points as in Observation 7.2a and, as illustrated in Figures 7.1a and 7.1c, there may be one or infinitely many families as in Observation 7.2b.

We note two special cases of families of parallel non-negative lines and their connection to Observation 7.2. If F is the family of vertical lines that intersect the positive x axis, then F is a family of parallel non-negative lines. This family makes first contact with the IPS at the Pareto maximal point $(1, 0)$ and at no other points. Similarly, the family of horizontal lines that intersect the positive y axis is a family of parallel non-negative lines that makes first contact with the IPS at the Pareto maximal point $(0, 1)$ and at no other points. We shall see in Section 7C that the situation is very different without absolute continuity.

We wish to connect the ideas presented in Observation 7.2 with the notion of maximization of expressions of the form $\alpha m_1 + \beta m_2$. Suppose that F is a family of parallel non-negative lines. Then there exist $\alpha \geq 0$ and $\beta \geq 0$, with at least one of these inequalities being strict, so that each line in F is of the form $\alpha x + \beta y = c$ for some $c > 0$. Fix any partition $P = \langle P_1, P_2 \rangle$. We shall say that P *maximizes the expression* $\alpha m_1 + \beta m_2$ if and only if, for any partition $Q = \langle Q_1, Q_2 \rangle$, $\alpha m_1(Q_1) + \beta m_2(Q_2) \leq \alpha m_1(P_1) + \beta m_2(P_2)$. Let $m(P) = p = (p_1, p_2)$.

Claim F makes first contact with the IPS at p (and perhaps at other points as well) if and only if P maximizes the expression $\alpha m_1 + \beta m_2$.

Proof of Claim: For the forward direction, assume that F makes first contact with the IPS at p, and let $k = \alpha m_1(P_1) + \beta m_2(P_2)$. Then $\alpha x + \beta y = k$ is the line in F that makes first contact with the IPS.

Next, fix any partition $Q = \langle Q_1, Q_2 \rangle$ and let $m(Q) = q$. We must show that $\alpha m_1(Q_1) + \beta m_2(Q_2) \leq \alpha m_1(P_1) + \beta m_2(P_2)$. Let $\alpha m_1(Q_1) + \beta m_2(Q_2) = k'$. Then, we wish to show that $k' \leq k$.

The line $\alpha x + \beta y = k'$ is one of the lines in F and q is on this line. Since $q \in$ IPS, F makes first contact with the IPS at or before q. But we know that the line in F that makes first contact with the IPS is the line $\alpha x + \beta y = k$. This implies that $k' \le k$, as desired.

For the reverse direction, we assume that F does not make first contact with the IPS at p. Let q be a point of first contact of F with the IPS and let Q be a partition such that $m(Q) = q$.

As before, we let $k = \alpha m_1(P_1) + \beta m_2(P_2)$ and $k' = \alpha m_1(Q_1) + \beta m_2(Q_2)$. Since $m(Q)$ is a point of first contact of F with the IPS and $m(P)$ is not, it follows that $k' > k$. Thus, $\alpha m_1(Q_1) + \beta m_2(Q_2) > \alpha m_1(P_1) + \beta m_2(P_2)$, and so P does not maximize the expression $\alpha m_1 + \beta m_2$.

This completes the proof of the claim.

Combining this claim with Observation 7.2, we see that a partition $P = \langle P_1, P_2 \rangle$ is Pareto maximal if and only if it maximizes some expression of the form $\alpha m_1 + \beta m_2$, where $\alpha \ge 0$, $\beta \ge 0$, and at least one of these inequalities is strict.

7B. The Characterization

In this section, we generalize the ideas of the previous section to the general context of n players, where we shall more precisely state and prove our results. In the present context, the IPS is a subset of $\mathbf{R}^n$ and, instead of considering families of parallel lines, we consider families of parallel hyperplanes.

We first considered hyperplanes in Chapter 4. We recall that a hyperplane in $\mathbf{R}^n$ is given by an equation of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ for some constants $\alpha_1, \alpha_2, \ldots, \alpha_n, c$, where not all of the α_i are equal to zero. As in the two-player context, the relevant hyperplanes will have $\alpha_1, \alpha_2, \ldots, \alpha_n \ge 0$, with at least one of these inequalities strict, and $c > 0$.

Definition 7.3

a. For any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$, we shall refer to the expression $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$ as a *convex combination of the measures*, or as the *convex combination of the measures associated with* α.

b. For any partition $P = \langle P_1, P_2, \cdots, P_n \rangle$, the *value of the convex combination* $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$ *applied to* P is given by $\alpha_1 m_1(P_1) + \alpha_2 m_2(P_2) + \cdots + \alpha_n m_n(P_n)$.

c. A partition P *maximizes the convex combination of measures* $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$ if and only if, for any partition Q, $\alpha_1 m_1 + \alpha_2 m_2 + \cdots +$

$\alpha_n m_n$ applied to P is greater than or equal to $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$ applied to Q.

Our characterization of Pareto maximality is the following.

Theorem 7.4 *A partition P is Pareto maximal if and only if it maximizes some convex combination of measures.*

The proof of Lemma 4.20 required a theorem from the field of convexity theory. The proof of Theorem 7.4 requires the following result, which is similar to the result used in Lemma 4.20. It is also a basic and well-known theorem in the field of convexity theory (see, for example, [25]):

> Given convex sets $G_1, G_2 \subseteq \mathbf{R}^n$ with disjoint interiors, there is a hyperplane H such that G_1 is contained in one of the closed half-spaces of $\mathbf{R}^n$ determined by H, and G_2 is contained in the other closed half-space of $\mathbf{R}^n$ determined by H.

Proof of Theorem 7.4: For the forward direction, suppose that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is Pareto maximal. We must find $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$ so that P maximizes the convex combination of measures associated with α.

Let $m(P) = p = (p_1, p_2, \ldots, p_n)$. Then $p \in$ IPS. We recall that $B^+(p) = \{(q_1, q_2, \ldots, q_n) \in \mathbf{R}^n$: for each $i = 1, 2, \ldots, n, q_i \geq p_i\}$ and that, since P is a Pareto maximal partition and hence p is a Pareto maximal point, $B^+(p) \cap$ IPS $= \{p\}$. Clearly, $B^+(p)$ is a convex set and we know that the IPS is convex. Since these two convex sets intersect at the single point p, their interiors are disjoint. It follows from the aforestated result that there is a hyperplane that separates these two sets. In other words, for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and some constant k, if H is the hyperplane given by $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = k$, then $B^+(p)$ is contained in one of the closed half-spaces determined by H and the IPS is contained in the other. By multiplying both sides of this equation by -1, if necessary, we may assume that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \leq k \text{ for every } (x_1, x_2, \ldots, x_n) \in \text{ IPS},$$
$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \geq k \text{ for every } (x_1, x_2, \ldots, x_n) \in B^+(p), \text{ and}$$
$$\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n = k.$$

Also, by multiplying both sides of each of the preceding inequalities and both sides of the equation by the positive number $\frac{1}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}$, we may assume that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$.

We claim that, for each $i = 1, 2, \ldots, n, \alpha_i \geq 0$. Suppose by way of contradiction that, for some such $i, \alpha_i < 0$. We first note that $(p_1, p_2, \ldots, p_{i-1}, p_i + 1, p_{i+1}, \ldots, p_n) \in B^+(p)$ and hence $\alpha_1 p_1 + \alpha_2 p_2 + \cdots +$

$\alpha_{i-1}p_{i-1} + \alpha_i(p_i + 1) + \alpha_{i+1}p_{i+1} + \cdots + \alpha_n p_n \geq k$. On the other hand,

$$\begin{aligned}\alpha_1 p_1 &+ \alpha_2 p_2 + \cdots + \alpha_{i-1}p_{i-1} + \alpha_i(p_i + 1) + \alpha_{i+1}p_{i+1} + \cdots + \alpha_n p_n \\ &= (\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_{i-1}p_{i-1} + \alpha_i p_i + \alpha_{i+1}p_{i+1} + \cdots + \alpha_n p_n) + \alpha_i \\ &< (\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_{i-1}p_{i-1} + \alpha_i p_i + \alpha_{i+1}p_{i+1} + \cdots + \alpha_n p_n) = k.\end{aligned}$$

This is a contradiction. Hence, for each $i = 1, 2, \ldots, n, \alpha_i \geq 0$. Since $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, this tells us that $\alpha \in S$.

We claim that P maximizes the convex combination of measures associated with α. We must show that, for any partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$,

$$\begin{aligned}\alpha_1 m_1(P_1) + \alpha_2 m_2(P_2) + \cdots + \alpha_n m_n(P_n) &\geq \alpha_1 m_1(Q_1) \\ &+ \alpha_2 m_2(Q_2) + \cdots + \alpha_n m_n(Q_n).\end{aligned}$$

We establish this as follows:

$$\begin{aligned}\alpha_1 m_1(P_1) &+ \alpha_2 m_2(P_2) + \cdots + \alpha_n m_n(P_n) \\ &= \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n = k \\ &\geq \alpha_1 m_1(Q_1) + \alpha_2 m_2(Q_2) + \cdots + \alpha_n m_n(Q_n)\end{aligned}$$

The preceding inequality holds because $(m_1(Q_1), m_2(Q_2), \ldots, m_n(Q_n)) \in$ IPS. This establishes the forward direction of the theorem.

The reverse direction is quite straightforward. We assume that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is not Pareto maximal and we must show that P does not maximize any convex combination of the measures.

Since P is not Pareto maximal, Lemma 1.13 implies that there is a partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ such that, for each $i = 1, 2, \ldots, n, m_i(Q_i) > m_i(P_i)$. Then, for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$,

$$\begin{aligned}\alpha_1 m_1(Q_1) &+ \alpha_2 m_2(Q_2) + \cdots + \alpha_n m_n(Q_n) \\ &> \alpha_1 m_1(P_1) + \alpha_2 m_2(P_2) + \cdots + \alpha_n m_n(P_n)\end{aligned}$$

and thus P does not maximize any convex combination of the measures. This completes the proof of the theorem. □

In the proof, we did not need to assume that the α_is sum to one. However, there was no harm in scaling to make these coefficients sum to one, and we chose to do so for reasons of uniformity and so that we could simply say that $\alpha \in S$.

There is a perspective on maximization of convex combinations of measures that involves maximizing "total utility." A standard example of a Pareto maximal partition is one that maximizes the quantity $m_1(P_1) + m_2(P_2) + \cdots + m_n(P_n)$. However, we may think of this sum as measuring total utility only if we assume that units of utility are the same for all players. If such is not the case, then total utility may be properly measured by some weighted sum $\alpha_1 m_1(P_1) + \alpha_2 m_2(P_2) + \cdots + \alpha_n m_n(P_n)$, where $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$. Theorem 7.4 says that a partition is Pareto maximal if and only if maximizes some such generalized total utility. We shall study standard total utility (i.e., $m_1(P_1) + m_2(P_2) + \cdots + m_n(P_n)$) further in Chapter 13.

We wish to more directly connect the proof of Theorem 7.4 with our geometric discussion from the two-player context. For any $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$, it is not hard to see that

> P maximizes the convex combination of measures $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$
> if and only if
> the family of parallel hyperplanes of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ makes first contact with the IPS at the point $m(P)$ (and possibly at other points too).

This, together with the theorem, tells us that, as in the two-player context,

> the point p is Pareto maximal
> if and only if
> for some $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$, the family of parallel hyperplanes of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ makes first contact with the IPS at the point p.

Or, equivalently,

> the partition P is Pareto maximal
> if and only if
> for some $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$, the family of parallel hyperplanes of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ makes first contact with the IPS at the point $m(P)$.

In particular, if P is Pareto maximal and $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and k are as in the proof of the theorem, then the hyperplane $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = k$ is the member of this family that makes first contact with the IPS, and the point $m(P)$ is on this hyperplane.

Since the IPS is closed, it follows that for any $\alpha \in S$ the family of parallel hyperplanes with coefficients given by α makes first contact with the IPS at one or more points. This immediately yields the following result.

Theorem 7.5 *Given any $\alpha \in S$, there is a partition P that maximizes the convex combination of measures corresponding to α.*

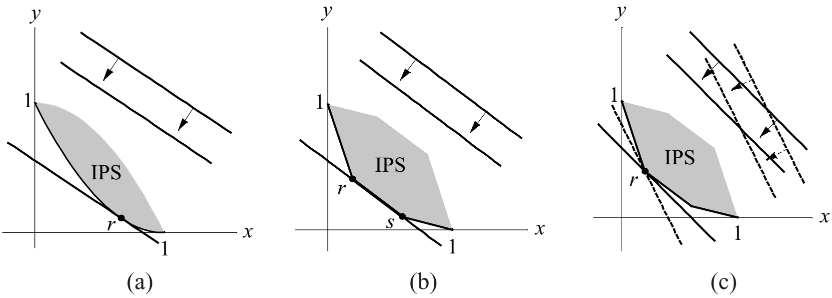


Figure 7.2

Notice that if a partition P maximizes some convex combination of measures, then every partition that is p-equivalent to P maximizes this same convex combination of measures. Hence, we may refer to the p-class of partitions that maximizes a convex combination of measures. In Section 12C, we shall consider the question of how many p-classes of partitions correspond in this way to a given α, and how many αs correspond in this way to a given p-class of partitions. In terms of our geometric perspective, this is the question of how many families of parallel hyperplanes make first contact with how many points of the IPS. (For two players, this is the issue illustrated by the contrast between Figures 7.1a, 7.1b, and 7.1c, which we discussed previously.)

Next, we focus briefly on a special case, as we did in our discussion of the two-player context. Consider the point $(1, 0, \ldots, 0) \in S$. The convex combination of measures corresponding to this point is simply m_1. It is easy to see that this convex combination is maximized by the partition $\langle C, \emptyset, \ldots, \emptyset \rangle$ and by no other partition. Of course, analogous facts are true for components other than the first. As we shall see in Section 7C, the situation is quite different without absolute continuity.

We close this section by considering the chores versions of these results. We first illustrate these ideas for two players. Consider Figure 7.2. In these figures, we have drawn the same IPSs as in Figure 7.1 but have darkened the inner boundary instead of the outer boundary. Whereas a family of parallel non-negative lines makes *first* contact with the IPS at one or more points on the outer boundary of the IPS, such a family makes *last* contact with the IPS at one or more points on the inner boundary of the IPS. Thus, any such family makes last contact with the IPS at a Pareto minimal point. As we did for the outer boundary and Pareto maximality in Figure 7.1, we illustrate various possibilities in Figure 7.2. In Figure 7.2a, the family of parallel non-negative lines makes last contact with the IPS at the point r, and at no other points, and no other family of parallel non-negative lines makes last contact with the IPS at point r. In Figure 7.2b, the family of parallel non-negative lines makes simultaneous last contact with the IPS at all points on the closed line segment determined by the points r and s (which we have made thicker), and no other family of parallel non-negative lines makes last contact with the IPS at all points of this line segment. Finally, in Figure 7.2c, there are an infinite number of families of parallel lines, each of which makes last contact with the IPS at point r and at no other points. The figure shows two such families.

In the general n-player context, we refer to the point of last contact of some family of parallel hyperplanes (rather than lines) with the IPS. The coordinates of such a point of last contact correspond to the sequence of coefficients of some convex combination of measures that is minimized. The definition and theorems corresponding to Definition 7.3c and Theorems 7.4 and 7.5 are as follows. The proofs of Theorems 7.7 and 7.8 are similar to the proofs of Theorems 7.4 and 7.5, respectively, and we omit them.

Definition 7.6 A partition P *minimizes the convex combination of measures* $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$ if and only if, for any partition Q, $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$ applied to P is less than or equal to $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$ applied to Q.

Theorem 7.7 *A partition P is Pareto minimal if and only if it minimizes some convex combination of measures.*

Theorem 7.8 *Given any $\alpha \in S$, there is a partition P that minimizes the convex combination of measures corresponding to α.*

7C. The Situation Without Absolute Continuity

In this section we make no general assumptions about absolute continuity. We consider how to adjust Theorems 7.4 and 7.7 to this situation. We first examine the two-player case and then present two characterizations of Pareto maximality (and corresponding characterizations of Pareto minimality) for the general n-player context. Both use the notion of maximization of convex combinations of measures. We shall also discuss our geometric perspective involving points of first contact with the IPS of families of parallel hyperplanes.

Consider Figure 7.3. Suppose (in either part of the figure) that $A, B \subseteq C$ are such that $m(\langle C \backslash A, A\rangle) = p$ and $m(\langle B, C \backslash B\rangle) = s$. Then $m_1(A) = 0$ and $m_2(A) > 0$, and so m_2 is not absolutely continuous with respect to m_1, and

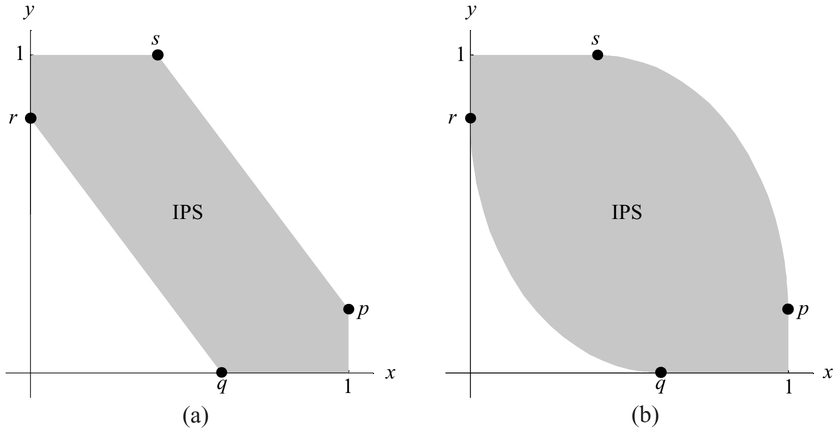


Figure 7.3

$m_1(B) > 0$ and $m_2(B) = 0$, and so m_1 is not absolutely continuous with respect to m_2. This is as in Chapter 2, in our discussion of Figure 2.1, where we observed that a vertical line segment going up from (1, 0) or down from (0, 1) corresponds to a situation where m_2 is not absolutely continuous with respect to m_1, and a horizontal line segment going to the left from (1, 0) or to the right from (0, 1) corresponds to a situation where m_1 is not absolutely continuous with respect to m_2. Given the sets A and B, we can be more specific:

a. Points on the vertical line segment between (1, 0) and p are associated with partitions that give all of $C \backslash A$ to Player 1 and distribute A between the two players.
b. Points on the horizontal line segment between (0, 1) and s are associated with partitions that give all of $C \backslash B$ to Player 2 and distribute B between the two players.
c. Points on the vertical line segment between (0, 1) and r are associated with partitions that give all of $C \backslash A$ to Player 2 and distribute A between the two players.
d. Points on the horizontal line segment between (1, 0) and q are associated with partitions that give all of $C \backslash B$ to Player 1 and distribute B between the two players.

It will follow from our work in Chapter 11 that Figures 7.3a and 7.3b depict real IPSs. In other words, for each of these figures, there is a cake C and measures m_1 and m_2 on C so that the corresponding IPS is as pictured.

The only Pareto maximal partitions associated with the points on these line segments are obtained by giving all of A to Player 2 in a and by giving all of B to Player 1 in b. Thus, the only Pareto maximal points on these line segments are the points p and s. Similarly, the only Pareto minimal partitions associated with the points on these line segments are obtained by giving all of A to Player 1 in c and by giving all of B to Player 2 in d. Thus, the only Pareto minimal points on these line segments are the points q and r.

Let us focus our attention on the vertical line segment in each figure from $(1, 0)$ to p. It is clear that our previous notions (obtained under the assumption of absolute continuity) do not generalize to our present setting. In particular, the family of parallel non-negative lines given by $1x + 0y = c$, or simply $x = c$, makes first contact with the IPS at all points along the line segment between $(1, 0)$ and p, and none of these points except p is Pareto maximal. An analogous problem occurs along the horizontal line segment from $(0, 1)$ to s.

To relate this perspective to the notion of maximization of convex combinations of measures, consider the convex combination of the measures m_1 and m_2 given by $1m_1 + 0m_2 = m_1$. This is clearly maximized by any partition that gives all of $C\backslash A$ to Player 1, regardless of how A is distributed. However, the only one of these partitions that is Pareto maximal is $\langle C\backslash A, A\rangle$, the partition associated with the point p. Other partitions that give all of $C\backslash A$ to Player 1 (and are not s-equivalent to $\langle C\backslash A, A\rangle$) are associated with other points along the line segment between $(1, 0)$ and p and are not Pareto maximal. An analogous problem occurs with the convex combination $0m_1 + 1m_2 = m_2$.

Is there a theorem analogous to Theorem 7.4 that holds if the measures are not absolutely continuous with respect to each other? Since a partition that maximizes a convex combination of measures corresponds to a point of first contact of a family of parallel non-negative lines with the IPS, let us approach the question in terms of these families.

We wish to know whether we can characterize Pareto maximal points of the IPS in terms of points of first contact of families of parallel non-negative lines with the IPS. As we have just seen, Theorem 7.4 fails in our present context. The family of vertical lines and the family of horizontal lines each make first contact with the IPS at many points, only one of which, in each case, is Pareto maximal. However, there is a tempting adjustment of Theorem 7.4 that suggests itself, which we now examine. This adjustment involves ruling out certain families of non-negative lines.

We recall that a line is non-negative if and only if it does not intersect the negative x axis, the negative y axis, or the origin and that, equivalently, the line $\alpha x + \beta y = c$ is non-negative if and only if $\alpha \geq 0$, $\beta \geq 0$, and $c > 0$, where α and β are not both zero.

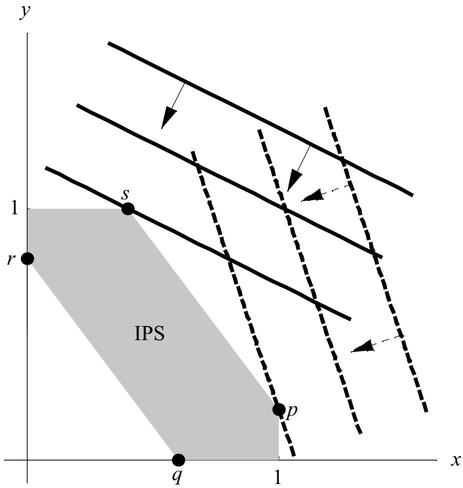


Figure 7.4

Definition 7.9 A line is *positive* if and only if it intersects the positive x axis and the positive y axis.

Thus, a line is positive if and only if it has an equation of the form $\alpha x + \beta y = c$, where $\alpha > 0$, $\beta > 0$, and $c > 0$. The effect of considering positive rather than non-negative lines is that we exclude vertical and horizontal lines from our consideration. The result we would like is the following: a point on the IPS is Pareto maximal if and only if it is the point of first contact of some family of parallel positive lines with the IPS. Or, in terms of maximization of convex combinations of measures this desired result is as follows: a partition is Pareto maximal if and only if it maximizes some convex combination of measures of the form $\alpha m_1 + \beta m_2$, where $\alpha > 0$ and $\beta > 0$.

If we examine Figure 7.3a, we see that this approach seems to work. In particular, the Pareto maximal points p and s are each a point of first contact of a family of parallel positive lines with the IPS. This is illustrated in Figure 7.4. On the other hand, this approach does not work for the IPS in Figure 7.3b. In this figure, the outer Pareto boundary has a vertical tangent line at point p and a horizontal tangent line at point s. Consequently, the only family of parallel non-negative lines that makes first contact with the IPS at p is the family of vertical lines, and the only family of parallel non-negative lines that makes first contact with the IPS at s is the family of horizontal lines. Thus, no family of parallel positive lines makes first contact with the IPS at either of these points, and therefore the approach outlined above does not work. It is not hard to see that this problem can also occur when there are more than two players.

Although the approach we have just discussed does not lead to a characterization of Pareto maximality, it does suggest a theorem. This result is a straightforward generalization of ideas already considered. It will be used in establishing Theorem 7.13, which gives the first of our two characterizations of Pareto maximality in this section. We now leave the two-player setting and consider the general n-player context. We shall refer to a convex combination of measures with all positive coefficients as a *positive convex combination of measures*. Then, a convex combination of measures is positive if and only if its sequence of coefficients corresponds to a point in the interior of the simplex.

Theorem 7.10 *Let P be a partition of C.*

a. If P is Pareto maximal, then it maximizes some convex combination of measures.

b. If P maximizes some positive convex combination of measures, then it is Pareto maximal.

Proof: For part a, we simply observe that this is precisely the forward direction of Theorem 7.4, and the proof of this direction of that theorem did not use absolute continuity. (Absolute continuity was used in the proof of the reverse direction of Theorem 7.4 in going from the assumption that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is not Pareto maximal to the assertion that there is a partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ such that, for each $i = 1, 2, \ldots, n$, $m_i(Q_i) > m_i(P_i)$.)

For part b, we assume that $P = \langle P_1, P_2, \ldots, P_n \rangle$ maximizes the positive convex combination of measures $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$. Suppose, by way of contradiction, that partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ is Pareto bigger than P. Then, $m_1(Q_1) \geq m_1(P_1), m_2(Q_2) \geq m_2(P_2), \ldots, m_n(Q_n) \geq m_n(P_n)$, with at least one of these inequalities being strict. But then, since $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$,

$$\alpha_1 m_1(Q_1) + \alpha_2 m_2(Q_2) + \cdots + \alpha_n m_n(Q_n)$$
$$> \alpha_1 m_1(P_1) + \alpha_2 m_2(P_2) + \cdots + \alpha_n m_n(P_n).$$

This contradicts the fact that P maximizes the positive convex combination of measures $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$. □

This result is very much in the spirit of Theorem 7.4. However, it is not an "if and only if" statement; hence, it does not provide a characterization of Pareto maximality. To illustrate, we return to Figure 7.3b and let

G = the set of points of the IPS that are the points of first contact with the IPS of some family of parallel non-negative lines and
H = the set of points of the IPS that are the points of first contact with the IPS of some family of parallel positive lines.

Because a positive line is certainly non-negative, it follows that $H \subseteq G$. Theorem 7.10 implies that H is a subset of the outer Pareto boundary and the outer Pareto boundary is a subset of G. Since

- the points s and p are each on the outer Pareto boundary of the IPS but neither is in H, and
- every point that is on the line segment between the points (1, 0) and p, including (1, 0) but not p, or is on the line segment between the points (0, 1) and s, including (0, 1) but not s, is in G but is not on the outer Pareto boundary of the IPS,

it follows that H is a proper subset of the outer Pareto boundary and the outer Pareto boundary is a proper subset of G. Translating this to a statement about convex combinations and Pareto maximal partitions, this tells us that

- the set of partitions that maximize some positive convex combination of the measures is a proper subset of the set of Pareto maximal partitions and
- the set of Pareto maximal partitions is a proper subset of the set of partitions that maximize some convex combination of the measures.

Thus, the converses to parts a and b of Theorem 7.10 are each false, and we see that this approach does not lead to a characterization of Pareto maximality.

The preceding discussion, together with ideas from Chapter 6, suggests how to approach a characterization of Pareto maximality using maximization of convex combinations of measures. We shall present two similar characterizations. The first (Theorem 7.13) flows directly out of Theorems 6.2, 6.4, and 7.10. The central ideas behind these characterizations are closely related to the work of E. Akin [1] and M. Dall'Aglio [19].

We need the following definition for both of our characterizations. We will also use this definition in Chapters 10 and 13.

Definition 7.11 The ordered pair (α,γ) is a *partition sequence pair* if and only if

a. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a sequence of positive numbers,
b. $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$ is a partition of $\{1, 2, \ldots, n\}$, and
c. for each $k = 1, 2, \ldots, t$, $\sum_{i \in \gamma_k} \alpha_i = 1$.

Notice that if (α, γ) is a partition sequence pair, then for each $k = 1, 2, \ldots, t$, $(\alpha_i : i \in \gamma_i)$ is an interior point of the $(|\gamma_i| - 1)$-simplex.

We call the central notion in our first characterization "*a*-maximization of a partition sequence pair," and the central notion in our second characterization "*b*-maximization of a partition sequence pair."

Definition 7.12 Let (α, γ) be a partition sequence pair with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, and let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition. We shall say that P *a-maximizes the partition sequence pair* (α, γ) if and only if the following two conditions hold:

a. For every $k = 1, 2, \ldots, t$, partition $\langle P_i : i \in \gamma_k \rangle$ of $\bigcup_{i \in \gamma_k} P_i$ maximizes the convex combination of the measures $\langle m_i : i \in \gamma_k \rangle$ corresponding to $(\alpha_i : i \in \gamma_k)$.
b. Either
   i. for every $k, k' = 1, 2, \ldots, t$ with $k < k'$, if $j \in \gamma_k$ and $j' \in \gamma_{k'}$, then $m_{j'}(P_j) = 0$ or
   ii. for every $k, k' = 1, 2, \ldots, t$ with $k < k'$, if $j \in \gamma_k$ and $j' \in \gamma_{k'}$, then $m_{j'}(P_j) = 0$.

It is easy to see that a partition P and a partition sequence pair (α, γ) satisfy condition bi of the definition, where $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, if and only if P and the partition sequence pair (α, γ'), where $\gamma' = \langle \gamma_t, \gamma_{t-1}, \ldots, \gamma_1 \rangle$, satisfy condition bii of the definition. Hence, P *a*-maximizes some partition sequence pair by satisfying condition bi of the definition if and only if P *a*-maximizes some partition sequence pair by satisfying condition bii of the definition.

There is a special case that will arise in our proof of Theorem 7.13. We have always assumed that any cake we consider is non-empty, and we certainly will assume that the cakes whose partitions we characterize in this section are non-empty. However, we will need to consider the empty cake in establishing our first characterization. Suppose that P, α, and γ are as in Definition 7.12. It may be that, for some $k = 1, 2, \ldots, t$, each player named by γ_k receives no cake (i.e., for all $i \in \gamma_k$, $P_i = \emptyset$). Then condition a of Definition 7.12 asserts that the partition of the empty cake among the players named by γ_k maximizes the convex combination of the measures $\langle m_i : i \in \gamma_k \rangle$ corresponding to $(\alpha_i : i \in \gamma_k)$. But since there is only one partition of the empty cake among the players named by γ_k, this partition maximizes this, and any, convex combination of measures.

Our first characterization of this section is the following.

Theorem 7.13 *A partition P is Pareto maximal if and only if it a-maximizes some partition sequence pair and is non-wasteful. (For the definition of "non-wasteful," see Definition 6.5.)*

Proof: Fix a partition $P = \langle P_1, P_2, \ldots, P_n\rangle$ of C. For the forward direction, we assume that P is Pareto maximal. Then P is certainly non-wasteful. We must find a partition sequence pair (α, γ) that is a-maximized by P. We shall construct $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle\gamma_1, \gamma_2, \ldots, \gamma_t\rangle$ in t stages, using part a of Theorem 7.10 at each stage.

We shall consider a different cake at each stage of the construction. For notational convenience, let $C_1 = C$. Then P is a Pareto maximal partition of C_1 and so, by part a of Theorem 7.10, we know that for some $\alpha^1 = (\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1) \in S$, P maximizes the convex combination of measure corresponding to α^1. Let $\gamma_1 = \{i \leq n : \alpha_1^1 > 0\}$ and, for each $i \in \gamma_1$, let $\alpha_i = \alpha_1^1$. If $\gamma_1 = \{1, 2, \ldots, n\}$, then let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, let $\gamma = \langle\gamma_1\rangle$, and we are done.

If $\gamma_1 \neq \{1, 2, \ldots, n\}$, let $C_2 = C\backslash(\bigcup_{i\in\gamma_1} P_i)$. We now consider C_2 to be the cake. If $C_2 = \emptyset$, let $\gamma_2 = \{1, 2, \ldots, n\}\backslash\gamma_1$ and define $(\alpha_i : i \in \gamma_2)$ arbitrarily, subject to the following two conditions:

a. for each $i \in \gamma_2, \alpha_i > 0$;
b. $\sum_{i\in\gamma_2} \alpha_i = 1$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, let $\gamma = \langle\gamma_1, \gamma_2\rangle$, and we are done.

If $C_2 \neq \emptyset$ then, by Theorem 6.2, $\langle P_i : i \in \{1, 2, \ldots, n\}\backslash\gamma_1\rangle$ is a Pareto maximal partition of C_2 among the players named by $\{1, 2, \ldots, n\}\backslash\gamma_1$. Part a of Theorem 7.10 implies that, for some $\alpha^2 = (\alpha_i^2 : i \in \{1, 2, \ldots, n\}\backslash\gamma_1)$, where the α_i^2 are non-negative numbers that sum to one, $\langle P_i : i \in \{1, 2, \ldots, n\}\backslash\gamma_1\rangle$ maximizes the convex combination of the measures $\langle m_i : i \in \{1, 2, \ldots, n\}\backslash\gamma_1\rangle$ corresponding to α^2. Let $\gamma_2 = \{i \in \{1, 2, \ldots, n\}\backslash\gamma_1 : \alpha_i^2 > 0\}$ and, for each $i \in \gamma_2$, let $\alpha_i = \alpha_i^2$. If $\gamma_1 \cup \gamma_2 = \{1, 2, \ldots, n\}$, then let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, let $\gamma = \langle\gamma_1, \gamma_2\rangle$, and we are done.

If $\gamma_1 \cup \gamma_2 \neq \{1, 2, \ldots, n\}$, we consider the cake $C_3 = C\backslash(\bigcup_{i\in(\gamma_1\cup\gamma_2)} P_i)$. If $C_3 = \emptyset$, we proceed as before, letting $\gamma_3 = \{1, 2, \ldots, n\}\backslash(\gamma_1 \cup \gamma_2)$ and defining $(\alpha_i : i \in \gamma_3)$ arbitrarily, subject to the following two conditions:

a. for each $i \in \gamma_3, \alpha_i > 0$;
b. $\sum_{i\in\gamma_3} \alpha_i = 1$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, let $\gamma = \langle\gamma_1, \gamma_2, \gamma_3\rangle$, and we are done.

If $C_3 \neq \emptyset$, we use part a of Theorem 7.10 to obtain $\alpha^3 = (\alpha_i^3 : i \in \{1, 2, \ldots, n\}\backslash(\gamma_1 \cup \gamma_2))$ and define $\gamma_3 = \{i \in \{1, 2, \ldots, n\}\backslash(\gamma_1 \cup \gamma_2) : \alpha_i^3 > 0\}$ and $\alpha_i = \alpha_i^3$ for each $i \in \gamma_3$, as before. We continue in this manner.

Notice that each γ_k is non-empty. Thus, the preceding process must eventually halt because, for some t, we must have $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t = \{1, 2, \ldots, n\}$. At this point, we will have defined a partition sequence pair (α, γ), with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$. We must show that P a-maximizes this partition sequence pair.

For condition a of Definition 7.12, fix some $k = 1, 2, \ldots, t$ and recall that $C_k = C \backslash (\bigcup_{i \in (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{k-1})} P_i)$ (where, when $k = 1$, we set $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{k-1} = \emptyset$). By construction, the partition $\langle P_i : i \in \{1, 2, \ldots, n\} \backslash (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{k-1}) \rangle$ of C_k maximizes the convex combination of the measures $\langle m_i : i \in \{1, 2, \ldots, n\} \backslash (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{k-1}) \rangle$ corresponding to α^k. (This is true whether or not $C_k = \emptyset$. See the discussion preceding the statement of the theorem.) Then, since $\gamma_k \subseteq \{1, 2, \ldots, n\} \backslash (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{k-1})$, the partition $\langle P_i : i \in \gamma_k \rangle$ of $\bigcup_{i \in \gamma_k} P_i$ maximizes the convex combination of the measures $\langle m_i : i \in \gamma_k \rangle$ corresponding to $(\alpha_i : i \in \gamma_k)$. This is so since a partition of $\bigcup_{i \in \gamma_k} P_i$ that produces a larger sum immediately yields a partition of C_k that produces a larger sum than does $\langle P_i : i \in \{1, 2, \ldots, n\} \backslash (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{k-1}) \rangle$.

Next, we show that the partition sequence pair (α, γ) satisfies condition bii of Definition 7.12. Fix $k, k' = 1, 2, \ldots, t$ with $k < k'$, and assume that $j \in \gamma_k$ and $j' \in \gamma_{k'}$. We must show that $m_j(P_{j'}) = 0$.

Consider the partition $\langle P_i : i \in \{1, 2, \ldots, n\} \backslash (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{k-1}) \rangle$. By construction, this partition maximizes the convex combination of the measures $\langle m_i : i \in \{1, 2, \ldots, n\} \backslash (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{k-1}) \rangle$ corresponding to α^k. If we change this partition by transferring $P_{j'}$ from Player j' to Player j, the resulting change in the value of this convex combination is $\alpha_j^k m_j(P_{j'}) - \alpha_{j'}^k m_{j'}(P_{j'})$. Since $k < k'$ and $j' \in \gamma_{k'}$, we know that $\alpha_{j'}^k = 0$ and, since $j \in \gamma_k$, we know that $\alpha_j^k > 0$. Hence, the resulting change in the value of the convex combination is the positive number α_j^k, times $m_j(P_{j'})$. But this change in the value of the convex combination cannot be positive, since the partition $\langle P_i : i \in \{1, 2, \ldots, n\} \backslash (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{k-1}) \rangle$ maximizes this convex combination of the measures. Thus, $m_j(P_{j'}) = 0$. This establishes that P a-maximizes the partition sequence pair (α, γ).

For the reverse direction, we assume that P is non-wasteful and that it a-maximizes the partition sequence pair (α, γ), where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$. We must show that P is Pareto maximal. We shall show that P satisfies the conditions of Theorem 6.4.

Since P is non-wasteful, condition a of Theorem 6.4 is satisfied.

To show that condition bi is satisfied, fix any $k = 1, 2, \ldots, t$. We must show that $\langle P_i : i \in \gamma_k \rangle$ is a Pareto maximal partition of $\bigcup_{i \in \gamma_k} P_i$ among the players named by γ_k. By assumption, the partition $\langle P_i : i \in \gamma_k \rangle$ of $\bigcup_{i \in \gamma_k} P_i$ maximizes the convex combination of the measures $\langle m_i : i \in \gamma_k \rangle$ corresponding to

$(\alpha_i : i \in \gamma_k)$. Since $\alpha_i > 0$ for each $i \in \gamma_k$, part b of Theorem 7.10 implies that $\langle P_i : i \in \gamma_k \rangle$ is a Pareto maximal partition of $\bigcup_{i \in \gamma_k} P_i$ among the players named by γ_k. Thus, condition bi is satisfied.

Conditions bii and biii of Theorem 6.4 are identical to conditions bi and bii, respectively, of Definition 7.12 and, hence, one of these conditions holds.

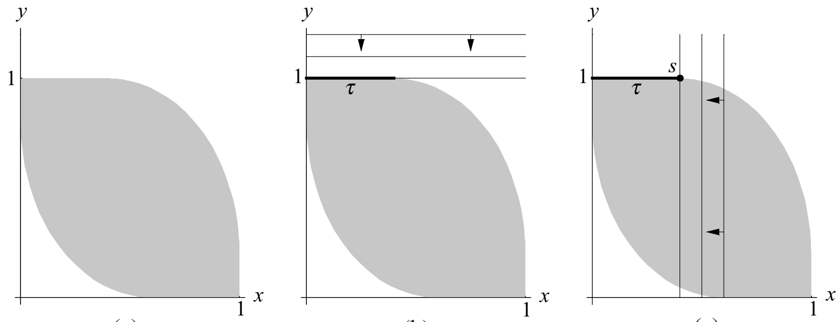
We have shown that the conditions of Theorem 6.4 are satisfied. It follows that P is Pareto maximal. This completes the proof of the theorem. □

We can think of the partition of $\{1, 2, \ldots, n\}$ given by γ as establishing a social hierarchy among the players. Suppose that some partition satisfies condition bii of Definition 7.12. Then all players named by γ_1 feel that all cake is given out at stage 1 among the players named by γ_1. Players named by γ_2 may feel that some of the cake has already been given out in stage 1, but they feel that all remaining cake is given out at stage 2 among the players named by γ_2. Players named by γ_3 may feel that some of the cake has already been given out in stages 1 and 2 among the players named by γ_1 and γ_2, but they feel that all remaining cake is given out at stage 3, and so on. Hence, we may think of players named by earlier γ_ks as having higher priority or higher social status than players named by later γ_ks. The situation would be the opposite if the partition satisfied condition bi instead of condition bii of Definition 7.12.

In Chapter 10, we shall study a different characterization of Pareto maximality (given by Theorem 10.28) and shall connect a-maximization with this new notion. This new notion will provide us with additional perspective about the iterative nature of a-maximization.

Although Theorem 7.13 is a characterization of Pareto maximality, we find it to be somewhat unsatisfying, since it involves an additional notion beyond that of maximization of convex combinations of measures, namely non-wastefulness. As we shall see, this assumption is not needed in our second characterization (Theorem 7.18).

Our second characterization is similar to our first. Let us return to Figure 7.3b, which we have repeated as Figure 7.5. (We recall that the IPS of this figure shows that our attempt at characterizing Pareto maximality using points of first contact of a single family of parallel non-negative lines, or a single family of parallel positive lines, does not work.) Consider the family of parallel lines $y = k$. As illustrated in Figure 7.5b, this family makes first contact with the IPS at all points along the line segment τ. We previously observed that the only point of τ that is Pareto maximal is its right endpoint. Next, we consider just the points of τ and ask which of these points are the points of first contact with τ of some family of parallel lines, where we consider only families that are normal to the original family $y = k$. (In general, we consider two families of parallel



(a) (b) (c)

Figure 7.5

hyperplanes to be normal if and only if any plane in one family is normal to any plane in the other family. Of course, since we are presently working in two dimensions, the only family of parallel lines that is normal to the family given by $y = k$ is the family given by $x = k$, i.e., the family of vertical lines.) As illustrated in Figure 7.5c, s is the unique such point of first contact.

Generalizing this idea to n players, and using the perspective of maximization of convex combinations of measures, rather than the equivalent perspective involving points of first contact of families of parallel hyperplanes with the IPS, we arrive at the following: a partition P of C is Pareto maximal if and only if

a. P maximizes the convex combination of measures corresponding to some $\alpha^1 \in S$;
b. among those partitions that maximize the convex combination of measures corresponding to α^1, P maximizes the convex combination of measures corresponding to some $\alpha^2 \in S$, where α^2 has zeros in any position in which α^1 has a non-zero entry; and
c. among those partitions that maximize the convex combination of measures corresponding to α^1 and to α^2, as described earlier, P maximizes the convex combination of measures corresponding to some $\alpha^3 \in S$, where α^3 has zeros in any position in which either α^1 or α^2 has non-zero entries, etc.

This process continues until there are no additional non-zero positions to be filled. Notice that requiring that each α^k has zeros in any position in which any of $\alpha^1, \alpha^2, \ldots, \alpha^{k-1}$ have non-zero entries corresponds to requiring that the family of parallel hyperplanes with coefficients given by α^k is normal to each of the families of parallel hyperplanes with coefficients given by $\alpha^1, \alpha^2, \ldots, \alpha^{k-1}$. (This correspondence uses the fact that all coefficients of each α^i are non-negative.)

We now begin making these ideas precise.

Definition 7.14 If (α, γ) is a partition sequence pair with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, we define $\alpha^k(\alpha, \gamma) = (\alpha_1^k, \alpha_2^k, \ldots, \alpha_n^k)$, for each $k = 1, 2, \ldots, t$, as follows:

$$\alpha_i^k = \begin{cases} \alpha_i & \text{if } i \in \gamma_k \\ 0 & \text{if } i \notin \gamma_k \end{cases}$$

By conditions a and c of Definition 7.11, for each $k = 1, 2, \ldots, t$, $\alpha^k(\alpha, \gamma) \in S$. To connect this definition to the preceding discussion, we recall that each γ_k tells us which coordinates are non-zero at each stage of the process and $(\alpha_i : i \in \gamma_k)$ gives the corresponding coordinates. The idea behind $\alpha^k(\alpha, \gamma)$ is simply to fill in zeros in the appropriate places in each $(\alpha_i : i \in \gamma_k)$ so that the sequence has length n and, hence, is a point in the $(n-1)$-simplex S.

We are almost ready to give our second characterization. We first need some definitions.

Definition 7.15 If Π is any collection of partitions, $P = \langle P_1, P_2, \ldots, P_n \rangle \in \Pi$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$, then we say that P *maximizes the convex combination of measures corresponding to* α *relative to* Π if and only if, for any $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle \in \Pi$, $\alpha_1 m_1(Q_1) + \alpha_2 m_2(Q_2) + \cdots + \alpha_n m_n(Q_n) \leq \alpha_1 m_1(P_1) + \alpha_2 m_2(P_2) + \cdots + \alpha_n m_n(P_n)$.

Definition 7.16 If (α, γ) is a partition sequence pair with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, we define a sequence of sets of partitions $\Pi^0(\alpha, \gamma), \Pi^1(\alpha, \gamma), \ldots, \Pi^t(\alpha, \gamma)$, as follows:

a. $\Pi^0(\alpha, \gamma) =$ Part, the set of all partitions of C among the n players, and
b. for each $k = 1, 2, \ldots, t$, $\Pi^k(\alpha, \gamma) = \{Q \in \Pi^{k-1}(\alpha, \gamma) : Q$ maximizes the convex combination of measures corresponding to $\alpha^k(\alpha, \gamma)$ relative to $\Pi^{k-1}(\alpha, \gamma)\}$.

We note that if $\Pi^0(\alpha, \gamma), \Pi^1(\alpha, \gamma), \ldots, \Pi^t(\alpha, \gamma)$ are as in the definition, then $\Pi^0(\alpha, \gamma) \supseteq \Pi^1(\alpha, \gamma) \supseteq \cdots \supseteq \Pi^t(\alpha, \gamma)$. Also, $\Pi^t(\alpha, \gamma) \neq \emptyset$.

Definition 7.17 Let (α, γ) be a partition sequence pair with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, and let P be a partition. We shall say that P *b-maximizes the partition sequence pair* (α, γ) if and only if $P \in \Pi^t(\alpha, \gamma)$.

Our second characterization of Pareto maximality is Theorem 7.18. It makes precise the ideas described earlier. Unlike our first characterization, we shall not need to explicitly assume that the partition is non-wasteful.

Theorem 7.18 *A partition P is Pareto maximal if and only if it b-maximizes some partition sequence pair.*

Proof: Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of C. For the forward direction, we shall use Theorem 7.13. For the reverse direction, we shall use the definition of Pareto maximality.

For the forward direction, we assume that P is Pareto maximal. Then, by Theorem 7.13, P a-maximizes some partition sequence pair (α, γ) and is non-wasteful. Set $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$. We will show that P b-maximizes (α, γ). In order to establish this, we must show that $P \in \Pi^t(\alpha, \gamma)$.

We may assume, without loss of generality, that P and (α, γ) satisfy condition bii of Definition 7.12 since, if they instead satisfy condition bi, we can simply reverse the order of $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$.

Suppose, by way of contradiction, that $P \notin \Pi^t(\alpha, \gamma)$ and choose any $Q \in \Pi^t(\alpha, \gamma)$. Since $P \in \Pi^0(\alpha, \gamma)$, we may let k be minimal such that $P \notin \Pi^k(\alpha, \gamma)$. Then the convex combination of measures corresponding to $\alpha^k(\alpha, \gamma)$ yields a greater sum when applied to Q than when applied to P. In other words (forgetting about terms with coefficient zero), $\sum_{i \in \gamma_k} \alpha_i m_i(Q_i) > \sum_{i \in \gamma_k} \alpha_i m_i(P_i)$.

By condition a of Definition 7.12, we know that the partition $\langle P_i : i \in \gamma_k \rangle$ of $\bigcup_{i \in \gamma_k} P_i$ maximizes the convex combination of the measures $\langle m_i : i \in \gamma_k \rangle$ corresponding to $(\alpha_i : i \in \gamma_k)$. Then, since $\sum_{i \in \gamma_k} \alpha_i m_i(Q_i) > \sum_{i \in \gamma_k} \alpha_i m_i(P_i)$, it follows that, for some $j \in \gamma_k$, there is an $A \subseteq (Q_j \backslash (\bigcup_{i \in \gamma_k} P_i))$ such that $m_j(A) > 0$. This implies that, for some $B \subseteq A$, some $k' = 1, 2, \ldots, t$ with $k' \neq k$, and some $j' \in \gamma_{k'}$, we have $B \subseteq P_{j'}$ and $m_j(B) > 0$. Condition bii of Definition 7.12 implies that $k' < k$.

Define a new partition $R = \langle R_1, R_2, \ldots, R_n \rangle$ as follows. For each $i = 1, 2, \ldots, n$,

$$R_i = \begin{cases} Q_i & \text{if } i \neq j \text{ and } i \neq j' \\ Q_j \backslash B & \text{if } i = j \\ Q_{j'} \cup B & \text{if } i = j' \end{cases}$$

We may view partition R as having been obtained from partition Q by having Player j give piece B to Player j'. Since $Q \in \Pi^t(\alpha, \gamma)$, we know that $Q \in \Pi^{k'-1}(\alpha, \gamma)$. Recall that $j \in \gamma_k$ and $j' \in \gamma_{k'}$. Then, since Q and R agree except at coordinates j and j', and $k' - 1 < k' < k$, it follows that $R \in \Pi^{k'-1}(\alpha, \gamma)$.

We claim that the convex combination of measures corresponding to $\alpha^{k'}(\alpha, \gamma)$ yields a greater sum when applied to R than when applied to Q. Partitions Q and R agree except at coordinates j and j'. Since $j \notin \gamma_{k'}$, the jth coefficient in $\alpha^{k'}(\alpha, \gamma)$ is zero. Thus, corresponding terms of the two relevant sums are equal except possibly for the j' term. Thus, we must compare $m_{j'}(Q_{j'})$ and $m_{j'}(R_{j'})$.

Recall that $B \subseteq P_{j'}$ and $m_j(B) > 0$. Since P is Pareto maximal, $m_{j'}(B) > 0$. Then, $m_{j'}(R_{j'}) = m_{j'}(Q_{j'} \cup B) = m_{j'}(Q_{j'}) + m_{j'}(B) > m_{j'}(Q_{j'})$. Hence, the convex combination of measures corresponding to $\alpha^{k'}(\alpha, \gamma)$ yields a greater sum when applied to R than when applied to Q. But $Q \in \Pi^{k'-1}(\alpha, \gamma)$, $R \in \Pi^{k'-1}(\alpha, \gamma)$, and, since $Q \in \Pi^t(\alpha, \gamma)$ and $\Pi^t(\alpha, \gamma) \subseteq \Pi^{k'}(\alpha, \gamma)$, we know that $Q \in \Pi^{k'}(\alpha, \gamma)$. This contradicts the definition of $\Pi^{k'}(\alpha, \gamma)$.

For the reverse direction, we assume that P b-maximizes the partition sequence pair (α, γ), where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$. Then $P \in \Pi^t(\alpha, \gamma)$.

Suppose, by way of contradiction, that P is not Pareto maximal, and let $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ be any partition that is Pareto bigger than P. Then, for every $i = 1, 2, \ldots, n$, $m_i(Q_i) \geq m_i(P_i)$, with at least one of these inequalities being strict. Let k be minimal such that, for some $j \in \gamma_k$, $m_j(Q_j) > m_j(P_j)$. For every $k' < k$ and $j' \in \gamma_{k'}$, $m_{j'}(Q_{j'}) = m_{j'}(P_{j'})$. Since $P \in \Pi^t(\alpha, \gamma)$ and hence, $P \in \Pi^{k-1}(\alpha, \gamma)$, it follows that $Q \in \Pi^{k-1}(\alpha, \gamma)$. But $m_i(Q_i) \geq m_i(P_i)$ for every $i \in \gamma_k$ and, for some $j \in \gamma_k$, $m_j(Q_j) > m_j(P_j)$. This implies that the convex combination of measures corresponding to $\alpha^k(\alpha, \gamma)$ yields a greater sum when applied to Q than when applied to P. But since $P \in \Pi^t(\alpha, \gamma)$, we know that $P \in \Pi^k(\alpha, \gamma)$. This contradicts the definition of $\Pi^k(\alpha, \gamma)$. □

In this section, we have made no assumptions about whether the measures are absolutely continuous with respect to each other or not. Hence, we may view Theorems 7.13 and 7.18 as general results that hold whether or not the measures are absolutely continuous with respect to each other, and Theorem 7.4 as a special case that holds if we do have absolute continuity. This is so because we may view the condition given in Theorem 7.4 (maximization of some convex combination of measures) as a special case of the conditions given in Theorems 7.13 and 7.18 (a-maximization and b-maximization, respectively, of some partition sequence pair). To see that this is so, suppose that the measures are absolutely continuous with respect to each other, and consider the partition $P = \langle P_1, P_2, \ldots, P_n \rangle$. By Theorem 7.4, P maximizes the convex combination of the measures corresponding to some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Let $\delta = \{i \leq n : \alpha_i > 0\}$. If $\delta = \{1, 2, \ldots, n\}$, then P a-maximizes and b-maximizes the partition sequence pair (α, γ), where $\gamma = \langle \{1, 2, \ldots, n\} \rangle$ (i.e., γ is the trivial partition of $\{1, 2, \ldots, n\}$ into one piece). If $\delta \neq \{1, 2, \ldots, n\}$, then, for any $i \in \{1, 2, \ldots, n\} \backslash \delta$, absolute continuity implies that P_i has measure zero. It is then straightforward to show that P a-maximizes and b-maximizes the partition sequence pair (α, γ), where $\gamma = \langle \delta, \{1, 2, \ldots, n\} \backslash \delta \rangle$.

Recall that the length of γ corresponds to the length of the iterative procedure in each of our two characterizations. Our preceding discussion tells us that if the measures are absolutely continuous with respect to each other then this iterative process has length at most two. Suppose that partition P gives a piece of cake of positive measure to each player. Then, for each $i = 1, 2, \ldots, n$, $\alpha_i > 0$. In this case, P a-maximizes and b-maximizes the partition sequence pair (α, γ), where $\gamma = \langle\{1, 2, \ldots, n\}\rangle$, and hence the iterative procedure has length one.

On the other hand, if the iterative process of Theorems 7.13 and 7.18 is of length one, it certainly need not be true that the measures are absolutely continuous with respect to each other. For example, suppose that the measures are not all absolutely continuous with respect to each other, and pick any $\alpha \in S$ with all positive coefficients. Let P be any partition that maximizes the convex combination of measures corresponding to α. By part b of Theorem 7.10, P is Pareto maximal. But then P a-maximizes and b-maximizes the partition sequence pair (α, γ), where $\gamma = \langle\{1, 2, \ldots, n\}\rangle$. We shall not carefully examine the question of when the partition sequence pair (α, γ) of Theorems 7.13 and 7.18 is such that γ is equal to the trivial partition $\langle\{1, 2, \ldots, n\}\rangle$ (and hence the iterative process has length one), but instead give a simple example that will illustrate the relevant issue.

Example 7.19 Let us again consider the IPSs in Figure 7.3. Both of these figures represent situations in which neither measure is absolutely continuous with respect to the other. For each figure, let P be a partition such that $m(P) = p$, where the point p is as indicated. (The partition P may be different for each of the figures.) In Figure 7.3b, the iterative process of Theorems 7.13 and 7.18 has length two. In particular, the partition sequence pair is (α, γ), where $\alpha = (1, 1)$ and $\gamma = \langle\{1\}, \{2\}\rangle$. For the situation represented in Figure 7.3a, this same two-step iteration and corresponding partition sequence pair will work, but so will a one-step process corresponding to the partition sequence pair (α, γ), where $\alpha = (1 - \varepsilon, \varepsilon)$ and $\gamma = \langle\{1, 2\}\rangle$, for sufficiently small $\varepsilon > 0$.

We close this section by considering chores versions of Theorems 7.10, 7.13, and 7.18. The chores version of Theorem 7.10 is the following. The proof is similar and we omit it.

Theorem 7.20 *Let P be a partition of C.*

a. *If P is Pareto minimal then it minimizes some convex combination of measures.*
b. *If P minimizes some positive convex combination of measures, then it is Pareto minimal.*

The chores versions of Theorems 7.13 and 7.18 require the notions of a-minimization and of b-minimization of a partition sequence pair (α, γ). These are defined by making the obvious adjustments to Definitions 7.12, 7.15, 7.16, and 7.17. (There is one place where it may not be clear what we mean by the "obvious adjustment." Part b of Definition 7.12 remains the same in the chores version of this definition.) Although the resulting adjustments of Theorems 7.13 and 7.18 to the chores context would be correct theorems, we shall not state or prove these results. The reason is that these results would create a misleading impression. In contrast with previous adjustments to the chores context, something very different happens when we try to adjust the iterative approaches of these theorems to the chores context. Perhaps surprisingly, the chores versions turn out to be much simpler. In particular, they do not require an iterative procedure at all.

We first consider the chores version of Theorem 7.13. We shall need the chores version of non-wasteful.

Definition 7.21 A partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is *c-non-wasteful* if and only if, for any $i = 1, 2, \ldots, n$ and $A \subseteq P_i$ with $m_i(A) > 0$, we have $m_j(A) > 0$ for every $j = 1, 2, \ldots, n$.

Clearly, any Pareto minimal partition is c-non-wasteful since, if it were not and Player i had a piece of cake that he or she considered to be of positive measure but Player j considered to have measure zero, then a transfer of that piece of cake from Player i to Player j would result in a Pareto smaller partition.

Suppose that we have made the necessary adjustments to Definition 7.12 in order to define what it means for a partition to a-minimize a partition sequence pair. Let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition, let (α, γ) be a partition sequence pair, and assume that P a-minimizes (α, γ) and is c-non-wasteful. Set $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$. Suppose that $t > 1$ (and so our iterative procedure has length greater than one) and assume, without loss of generality, that P and (α, γ) satisfy condition bi of Definition 7.12. (As noted previously, part b of this definition is the same as in the definition of a-minimization.) Fix any $k = 1, 2, \ldots, t - 1$, $j \in \gamma_k$, and $j' \in \gamma_t$. Then $m_{j'}(P_j) = 0$ and, since P is c-non-wasteful, $m_j(P_j) = 0$. Thus, we have shown that each player that is not named by γ_t believes that he or she has received a piece of cake of measure zero. This implies that the first $t - 1$ stages in our iterative process can be collapsed into a single stage. In other words, instead of $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, we have $\gamma = \langle \gamma_1', \gamma_t \rangle$, where $\gamma_1' = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{t-1}$.

We can make this characterization easier still. After collapsing the first $t - 1$ stages into a single stage in our preceding analysis, each player that receives a piece of cake in this new first stage believes that his or her piece of cake has measure zero. This observation will simplify our characterization. It provides the background for Theorem 7.23, which is our adjustment of Theorem 7.13 to the chores context. We first need a definition.

Definition 7.22 Suppose that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of C. A transfer of some $A \subseteq P_i$ from Player i to Player j is a

a. *zero-to-zero transfer* if $m_i(A) = 0$ and $m_j(A) = 0$.
b. *positive-to-positive transfer* if $m_i(A) > 0$ and $m_j(A) > 0$.
c. *positive-to-zero transfer* if $m_i(A) > 0$ and $m_j(A) = 0$.
d. *zero-to-positive transfer* if $m_i(A) = 0$ and $m_j(A) > 0$.

Notice that a partition P is non-wasteful if and only if it is not possible to perform any zero-to-positive transfers and is c-non-wasteful if and only if it is not possible to perform any positive-to-zero transfers.

Theorem 7.23 *A partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is Pareto minimal if and only if the following two conditions hold:*

a. P is c-non-wasteful.
b. The partition $\langle P_i : i \in \delta \rangle$ of $\bigcup_{i \in \delta} P_i$ minimizes some positive convex combination of the measures $\langle m_i : i \in \delta \rangle$, where $\delta = \{i \leq n : m_i(P_i) > 0\}$.

Before beginning the proof, we comment on one special case. It may be that the δ of the theorem is empty. If the measures concentrate on the complements of disjoint sets (see Definition 5.39), then there exists a partition making $\delta = \emptyset$. In this case, P is certainly Pareto minimal, and both conditions of the theorem hold. Thus, if $\delta = \emptyset$, the theorem is true.

Proof of Theorem 7.23: Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$. For the forward direction, we assume that P is Pareto minimal. Then P is certainly c-non-wasteful. Let $\delta = \{i : m_i(P_i) > 0\}$. We must show that the partition $\langle P_i : i \in \delta \rangle$ of $\bigcup_{i \in \delta} P_i$ minimizes some positive convex combination of the measures $\langle m_i : i \in \delta \rangle$.

It is easy to see that $\langle P_i : i \in \delta \rangle$ is a Pareto minimal partition of $\bigcup_{i \in \delta} P_i$ among the players named by δ, since any partition of $\bigcup_{i \in \delta} P_i$ among the players named by δ that is Pareto smaller than $\langle P_i : i \in \delta \rangle$ would immediately yield a partition of C that is Pareto smaller than P. Then, part a of Theorem 7.20 implies that $\langle P_i : i \in \delta \rangle$ minimizes some convex combination $(\alpha_i : i \in \delta)$ of the measures $\langle m_i : i \in \delta \rangle$. We claim that, for each $i \in \delta$, $\alpha_i > 0$.

Suppose, by way of contradiction, that $i \in \delta$ and $\alpha_i = 0$, and fix any $j \in \delta$ with $\alpha_j > 0$. (There must be such a j since the sequence $(\alpha_i : i \in \delta)$ consists of non-negative numbers and sums to one.) Since $j \in \delta$, we know that $m_j(P_j) > 0$.

Let Q be the partition of $\bigcup_{i \in \delta} P_i$ that results from $\langle P_i : i \in \delta \rangle$ by transferring P_j from Player j to Player i. The convex combination of the measures $\langle m_i : i \in \delta \rangle$ corresponding to $(\alpha_i : i \in \delta)$ produces a smaller sum when applied to Q than to P, contradicting the fact that $\langle P_i : i \in \delta \rangle$ is a Pareto minimal partition of $\bigcup_{i \in \delta} P_i$ among the players named by δ. Hence, for each $i \in \delta$, $\alpha_i > 0$, and we have shown that $\langle P_i : i \in \delta \rangle$ minimizes a positive convex combination of the measures $\langle m_i : i \in \delta \rangle$.

For the reverse direction of the theorem, we assume that P is c-non-wasteful and that $\langle P_i : i \in \delta \rangle$ minimizes the positive convex combination of the measures $\langle m_i : i \in \delta \rangle$ corresponding to $(\alpha_i : i \in \delta)$, where $\delta = \{i : m_i(P_i) > 0\}$. Part b of Theorem 7.20 implies that $\langle P_i : i \in \delta \rangle$ is a Pareto minimal partition of $\bigcup_{i \in \delta} P_i$ among the players named by δ. We must show that P is Pareto minimal. Suppose, by way of contradiction, that $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ is Pareto smaller than P.

Let us consider the various transfers between the players that change partition P to partition Q. We claim (using the terminology of Definition 7.22) that we need only consider positive-to-positive transfers. A zero-to-zero transfer has no effect on any of our considerations. (We note that this is *not* the case for all fairness properties. For example, a zero-to-zero transfer that some third player views as a transfer involving cake of positive measure can affect envy-freeness.) Thus, we may simply redefine Q so that any zero-to-zero transfers are not done. A zero-to-positive transfer results in a partition that is Pareto bigger. Hence, simply not doing any such transfer results in a partition that is Pareto smaller than Q, so we may, as before, assume that Q has been redefined so that no zero-to-positive transfers are done. The existence of a positive-to-zero transfer would contradict the fact that P is c-non-wasteful. Hence, we may assume that all transfers involved in the transition from P to Q are positive-to-positive transfers.

We claim that there are no such positive-to-positive transfers between a player named by δ and a player not named by δ. Recall that a player not named by δ believes that his or her piece of cake has measure zero. Hence, there can be no positive-to-positive transfer from a player not named by δ to any player. Also, there can be no positive-to-positive transfer from a player named by δ to a player not named by δ, since such a transfer would increase the receiving player's evaluation of his or her piece, contradicting the fact that Q is Pareto smaller than P.

We have shown that the transition from partition P to partition Q involves only positive-to-positive transfers, and these transfers are all done between players named by δ. It follows that $\bigcup_{i \in \delta} P_i = \bigcup_{i \in \delta} Q_i$ and hence $\langle Q_i : i \in \delta \rangle$ is a partition of $\bigcup_{i \in \delta} P_i$ among the players named by δ. Since Q is Pareto smaller than P and $m_i(P_i) = 0$ for all $i \notin \delta$, it must be that $\langle Q_i : i \in \delta \rangle$ is a Pareto smaller partition of $\bigcup_{i \in \delta} P_i$, among the players named by δ, than is $\langle P_i : i \in \delta \rangle$. This contradicts the fact that $\langle P_i : i \in \delta \rangle$ is a Pareto minimal partition of $\bigcup_{i \in \delta} P_i$ among the players named by δ. This completes the proof of the theorem. □

Next, we consider adjusting Theorem 7.18 to the chores context. We first sketch a simple example to illustrate the relevant issues. This idea here is very similar to the idea discussed in motivating Theorem 7.23. (See the two paragraphs preceding Definition 7.22.)

Assume that there are five players, Player 1, Player 2, Player 3, Player 4, and Player 5, and $P = \langle P_1, P_2, P_3, P_4, P_5 \rangle$ is a Pareto maximal partition of C. By Theorem 7.18, P b-maximizes some partition sequence pair (α, γ). Suppose that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and $\gamma = \langle \{1, 2\}, \{3\}, \{4, 5\} \rangle$. Let us review what this means:

> First look at the set of all partitions that maximize the convex combination of measures corresponding to $(\alpha_1, \alpha_2, 0, 0, 0)$. Then, among all such partitions, look at those that maximize the convex combination of measures corresponding to $(0, 0, \alpha_3, 0, 0)$. Finally, among all these partitions, single out those that maximize the convex combination of measures corresponding to $(0, 0, 0, \alpha_4, \alpha_5)$. P is one of these partitions.

It is not difficult to construct specific examples to show that it may not be possible to find a partition sequence pair that yields P as in the preceding paragraph, but where γ is a partition of $\{1, 2, 3, 4, 5\}$ into fewer than three pieces.

The situation is very different for Pareto minimality. Suppose that we have made the necessary adjustments to Definitions 7.15, 7.16, and 7.17 in order to define what it means for a partition to b-minimize a partition sequence pair. Let us continue to assume that α and γ are as before, and suppose that some partition Q is Pareto minimal and b-minimizes the partition sequence pair (α, γ). Mimicking the preceding discussion, this tells us the following:

> First look at the set of all partitions that minimize the convex combination of measures corresponding to $(\alpha_1, \alpha_2, 0, 0, 0)$. Then, among all such partitions, look at those that minimize the convex combination of measures corresponding to $(0, 0, \alpha_3, 0, 0)$. Finally, among all these partitions, single out those that minimize the convex combination of measures corresponding to $(0, 0, 0, \alpha_4, \alpha_5)$. Q is one of these partitions.

This idea is correct, but it is misleading. The set of all partitions that minimize the convex combination of measures corresponding to $(\alpha_1, \alpha_2, 0, 0, 0)$ is simply the set of all partitions that give Player 1 and Player 2 a piece of cake of measure zero (with respect to each's own measure). Similarly, among the set of all partitions that minimize the convex combination of measures corresponding to $(\alpha_1, \alpha_2, 0, 0, 0)$, the set of all partitions that minimize the convex combination of measures corresponding to $(0, 0, \alpha_3, 0, 0)$ is simply the set of all such partitions that give Player 3 a piece of cake of measure zero (with respect to his or her measure). Hence, the first two steps of this process result in the set of all partitions that give each of Player 1, Player 2, and Player 3 a piece of cake of measure zero (with respect to each's own measure). This set of partitions is the set of partitions that minimize the convex combination of measures corresponding to $(\alpha_1, \alpha_2, \alpha_3, 0, 0)$. Thus, we see that the first two stages of this iterative process can be collapsed into a single stage.

It is not hard to see that the preceding idea is general. All but the last stage of any such iteration can be collapsed into a single stage. Therefore, a partition is Pareto minimal if and only if it b-minimizes some partition sequence pair (α, γ), where γ is a partition of $\{1, 2, \ldots, n\}$ into at most two pieces. In other words, any partition that is Pareto minimal can be shown to be Pareto minimal by this type of iteration with length at most two.

It is interesting to note that this collapsing to at most two stages in our iterative procedure occurs both in the chores context and in the standard context when absolute continuity holds. (See the discussion following the proof of Theorem 7.18.) This curious connection can be carried further. We saw that, in the standard context when absolute continuity holds, the procedure can be reduced to just one stage if each player believes that he or she receives a piece of cake of positive measure. It is not hard to see that the same is true in our present context.

As in the chores version of Theorem 7.13 (i.e., Theorem 7.23), we can make this characterization easier still by noticing that, after collapsing the first two stages into one stage in our preceding example, each player that receives a piece of cake in this new first stage believes that his or her piece of cake has measure zero. Our adjustment of Theorem 7.18 to the chores context is the following.

Theorem 7.24 *Let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition and set $\delta = \{i \le n : m_i(P_i) > 0\}$. Then, P is Pareto minimal if and only if there exists $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$ such that,*

a. for each i, $\alpha_i > 0$ if and only if $i \in \delta$ and
b. P minimizes the convex combination of the measures corresponding to α, relative to $\{Q : Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ and $m_i(Q_i) = 0$ for all $i \in \delta$.

Proof: Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ and set $\delta = \{i \le n : m_i(P_i) > 0\}$. For the forward direction, we shall use Theorem 7.23. For the reverse direction, we shall use the definition of Pareto minimality.

For the forward direction, we assume that P is Pareto minimal. By Theorem 7.23, we know that $\langle P_i : i \in \delta \rangle$ minimizes the positive convex combination of the measures $\langle m_i : i \in \delta \rangle$ corresponding to some $(\alpha_i : i \in \delta)$. For each $i \in \{1, 2, \ldots, n\} \backslash \delta$, let $\alpha_i = 0$ and set $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then $\alpha_i > 0$ if and only if $i \in \delta$. We must show that P minimizes the convex combination of measures corresponding to α, relative to $\{Q : Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ and $m_i(Q_i) = 0$ for all $i \notin \delta\}$.

Suppose, by way of contradiction, that for some partition $R = \langle R_1, R_2, \ldots, R_n \rangle$ with $m_i(R_i) = 0$ for all $i \notin \delta$, the convex combination of the measures corresponding to α is smaller when applied to R than when applied to P. Consider the collection of transfers that changes partition P to partition R. Since $m_i(R_i) = 0$ for all $i \notin \delta$, any transfer in this collection from a player named by δ to a player not named by δ must be a zero-to-zero transfer or a positive-to-zero transfer (see Definition 7.22). Any zero-to-zero transfer can be ignored, since no player's measure of his or her piece changes if we simply do not do this trade, and the Pareto minimality of P implies that there are no positive-to-zero transfers. Hence, we can assume that the transition from partition P to partition R involves no transfer of cake from a player named by δ to a player not named by δ. It follows that $\bigcup_{i\in\delta} P_i \subseteq \bigcup_{i\in\delta} R_i$. This implies that the partition $\langle R_j \cap (\bigcup_{i\in\delta} P_i) : j \in \delta \rangle$ is a partition of $\bigcup_{i\in\delta} P_i$ among the players named by δ. For each $j \in \delta$, $R_j \cap (\bigcup_{i\in\delta} P_i) \subseteq R_j$ and, hence, $m_j(R_j \cap (\bigcup_{i\in\delta} P_i)) \leq m_j(R_j)$. Then, since $\alpha_i = 0$ for all $i \notin \delta$ and the convex combination of measures corresponding to α is smaller when applied to R than when applied to P, it follows that the convex combination of the measures $\langle m_i : i \in \delta \rangle$ corresponding to $(\alpha_i : i \in \delta)$ produces a smaller sum when applied to the partition $\langle R_j \cap (\bigcup_{i\in\delta} P_i) : j \in \delta \rangle$ than when applied to $\langle P_i : i \in \delta \rangle$. This contradicts the fact that $\langle P_i : i \in \delta \rangle$ minimizes this convex combination of measures.

For the reverse direction, we assume that P is not Pareto minimal, and we let R be a partition that is Pareto smaller than P. Then, for every $i = 1, 2, \ldots, n$, $m_i(R_i) \leq m_i(P_i)$, with at least one of these inequalities being strict. Let δ be as in the statement of the theorem and fix any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$ that satisfies condition a of the theorem. We must show that condition b of the theorem fails.

For each $i = 1, 2, \ldots, n$, since $m_i(R_i) \leq m_i(P_i)$, we know that if $m_i(P_i) = 0$ then $m_i(R_i) = 0$. Thus, R is in the set that appears in condition b of the theorem. Thus, it suffices to show that the convex combination of the measures corresponding to α produces a smaller sum when applied to R than when applied to P. But we know that for at least one i, $m_i(R_i) < m_i(P_i)$. For this i, $m_i(P_i) > 0$, and hence $i \in \delta$. This implies that $\alpha_i > 0$. It follows that the convex combination of the measures corresponding to α produces a smaller sum when applied to R than when applied to P, as desired. This completes the proof of the theorem. □

We close this section by providing another perspective on the lack of symmetry between Pareto maximality and Pareto minimality when absolute continuity fails. The failure of absolute continuity implies that, for some $A \subseteq C$, at least one player views A as having positive measure and at least one player views A as having measure zero. Let $\delta^{>} = \{i \leq n : m_i(A) > 0\}$ and let $\delta^{0} = \{i : m_i(A) = 0\}$. Then, $\delta^{>}$ and δ^{0} are each non-empty.

Suppose that we are interested in Pareto maximal partitions of C. Then A must be distributed among the players named by $\delta^{>}$. Precisely how A is distributed is important. Each player named by $\delta^{>}$ would like as much of A as he or she can get. And, depending on the specific measures, the distribution of the rest of the cake (i.e., of $C \backslash A$) may need to be coordinated with the distribution of A.

On the other hand, suppose now that we are interested in Pareto minimal partitions of C. Then all of A must be distributed among the players named by δ^{0}. In contrast with the previous situation, precisely how A is distributed among these players is irrelevant. Neither the players named by $\delta^{>}$ nor the players named by δ^{0} have any preferences on this matter.

8

Characterizing Pareto Optimality II

Partition Ratios

In this chapter, we introduce and study partition ratios. These ratios are numbers that can be associated with any partition. They provide us with our second approach to characterizing Pareto maximality and Pareto minimality and will be useful in future chapters. In Section 8A we consider the two-player context, and in Section 8B we establish our characterization in the general n-player context. In these sections, we assume that the measures are absolutely continuous with respect to each other. In Section 8C we consider the situation without absolute continuity. The definition of partition ratios and the corresponding characterization are very similar to a notion and result that appeared in a preliminary version of E. Akin's [1] but did not appear in the published version.

8A. Introduction: The Two-Player Context

Suppose that $P = \langle P_1, P_2 \rangle$ is a partition of C that is not Pareto maximal, and let us assume for simplicity that P_1 and P_2 are both of positive measure. (We shall drop this simplifying assumption when we consider the general n-player context in the next section.) Since P is not Pareto maximal, there is a partition $Q = \langle Q_1, Q_2 \rangle$ that is Pareto bigger than P. We can imagine the change from partition P to partition Q as being accomplished by a trade between the two players. In other words, there are sets $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ such that $Q_1 = (P_1 \backslash A_1) \cup A_2$ and $Q_2 = (P_2 \backslash A_2) \cup A_1$. Since each player is at least as happy with partition Q as with partition P, and at least one of the two players is strictly happier, we know that $m_1(A_2) \geq m_1(A_1)$ and $m_2(A_1) \geq m_2(A_2)$, with at least one of these inequalities being strict. This implies that $m_1(A_2)m_2(A_1) > m_1(A_1)m_2(A_2)$. Note that none of these four quantities can be zero, since any such trade that leads to a Pareto bigger partition must involve sets of positive measure. It follows that $(\frac{m_2(A_1)}{m_1(A_1)})(\frac{m_1(A_2)}{m_2(A_2)}) > 1$. This establishes the following:

If $P = \langle P_1, P_2\rangle$ is a partition such that, for any sets $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ of positive measure, $(\frac{m_2(A_1)}{m_1(A_1)})(\frac{m_1(A_2)}{m_2(A_2)}) \leq 1$, then P is Pareto maximal.

Consider the converse of this statement. The preceding argument cannot simply be reversed, since it is not the case that $m_1(A_2)m_2(A_1) > m_1(A_1)m_2(A_2)$ implies $m_1(A_2) \geq m_1(A_1)$ and $m_2(A_1) \geq m_2(A_2)$, with at least one of these inequalities being strict. However, we claim that the converse is true. Fix a partition $P = \langle P_1, P_2\rangle$ and assume that there are sets $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ of positive measure such that $(\frac{m_2(A_1)}{m_1(A_1)})(\frac{m_1(A_2)}{m_2(A_2)}) > 1$. Then $(\frac{m_1(A_2)}{m_1(A_1)})(\frac{m_2(A_1)}{m_2(A_2)}) > 1$. We must show that P is not Pareto maximal. We consider three cases:

Case 1: $\frac{m_1(A_2)}{m_1(A_1)} \geq 1$ and $\frac{m_2(A_1)}{m_2(A_2)} \geq 1$. At least one of these inequalities must be strict. It follows that $m_1(A_2) \geq m_1(A_1)$ and $m_2(A_1) \geq m_2(A_2)$, with at least one of these inequalities being strict. Then, a trade of A_1 and A_2 yields a partition that is Pareto bigger that P. Hence, P is not Pareto maximal.

Case 2: $\frac{m_1(A_2)}{m_1(A_1)} < 1$. By Corollary 1.6, we know that there exists a set $B_1 \subseteq A_1$ such that $m_1(B_1) = (\frac{m_1(A_2)}{m_1(A_1)})m_1(A_1) = m_1(A_2)$ and $m_2(B_1) = (\frac{m_1(A_2)}{m_1(A_1)})m_2(A_1)$. Then

$$\frac{m_2(B_1)}{m_2(A_2)} = \frac{\left(\frac{m_1(A_2)}{m_1(A_1)}\right)(m_2(A_1))}{m_2(A_2)} = \left(\frac{m_1(A_2)}{m_1(A_1)}\right)\left(\frac{m_2(A_1)}{m_2(A_2)}\right) > 1.$$

Thus, $m_2(B_1) > m_2(A_2)$. Since $m_1(B_1) = m_1(A_2)$, it follows that a trade of B_1 and A_2 yields a partition that is Pareto bigger than P. Hence P is not Pareto maximal.

Case 3: $\frac{m_2(A_1)}{m_2(A_2)} < 1$. This is similar to Case 2, and we omit it.

Thus, we have established the following:

A partition $P = \langle P_1, P_2\rangle$ is a Pareto maximal if and only if, for any sets $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$, $(\frac{m_2(A_1)}{m_1(A_1)})(\frac{m_1(A_2)}{m_2(A_2)}) \leq 1$.

Consider the following two quantities:

$$\mathrm{pr}_{12} = \sup\left\{\frac{m_2(A)}{m_1(A)} : A \subseteq P_1 \text{ and } A \text{ has positive measure}\right\}$$
$$\mathrm{pr}_{21} = \sup\left\{\frac{m_1(A)}{m_2(A)} : A \subseteq P_2 \text{ and } A \text{ has positive measure}\right\}$$

We continue to assume that P gives a piece of cake of positive measure to each player. Then each of the preceding sets is non-empty. We note that $0 < \mathrm{pr}_{12} \leq \infty$ and $0 < \mathrm{pr}_{21} \leq \infty$. (That each is greater than zero follows from absolute continuity.)

The following two conditions are easily seen to be equivalent:

- for any sets $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$, $(\frac{m_2(A_1)}{m_1(A_1)})(\frac{m_1(A_2)}{m_2(A_2)}) \leq 1$, and
- $\mathrm{pr}_{12}\mathrm{pr}_{21} \leq 1$,

where we set $(\kappa)(\infty) = (\infty)(\kappa) = (\infty)(\infty) = \infty > 1$ for any $\kappa > 0$. Our preceding work establishes the following:

A partition $P = \langle P_1, P_2 \rangle$ is Pareto maximal if and only if $\mathrm{pr}_{12}\mathrm{pr}_{21} \leq 1$.

The letters "pr" are meant to denote "partition ratio." We shall give the general n-player definition in the next section.

8B. The Characterization

In this section, we generalize the analysis of the previous section to the n-player context. We now drop our simplifying assumption that the partitions we consider are partitions into sets of positive measure.

We have shown that for two players, a partition $P = \langle P_1, P_2 \rangle$ into sets of positive measure is Pareto maximal if and only if, for any sets $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ of positive measure, $(\frac{m_2(A_1)}{m_1(A_1)})(\frac{m_1(A_2)}{m_2(A_2)}) \leq 1$. We then restated this in terms of partition ratios. Theorem 8.9 generalizes this result to the n-player context. The argument for the forward direction of the theorem is a generalization of the argument given in the previous section. However, the reverse direction needs an additional result, Theorem 8.2. This result is trivially true for two players, but is not at all obvious for more than two players. We first need a definition.

Definition 8.1 Suppose that $P = \langle P_1, P_2, \dots, P_n \rangle$ is a partition, $\{i_1, i_2, \dots, i_t\} \subseteq \{1, 2, \dots, n\}$, and for each $j = 1, 2, \dots, t$, $A_{i_j} \subseteq P_{i_j}$. Then, $CT(\langle i_1, i_2, \dots, i_t \rangle | \langle A_{i_1}, A_{i_2}, \dots, A_{i_t} \rangle)$ denotes the trade in which Player i_1 gives A_{i_1} to Player i_2, Player i_2 gives A_{i_2} to Player $i_3, \dots,$ Player i_{t-1} gives $A_{i_{t-1}}$ to Player i_t, and Player i_t gives A_{i_t} to Player i_1. A trade of this form is a *cyclic trade*. A *positive cyclic trade* is a cyclic trade in which each A_{i_j} has positive measure.

Let us re-examine our proof in the previous section of the following statement:

If $P = \langle P_1, P_2 \rangle$ is a partition such that, for any sets $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ of positive measure, $(\frac{m_2(A_1)}{m_1(A_1)})(\frac{m_1(A_2)}{m_2(A_2)}) \leq 1$, then P is Pareto maximal.

We assumed that P is not Pareto maximal and then asserted (now using the terminology of Definition 8.1) that there must be a cyclic trade yielding a Pareto bigger partition. If this cyclic trade involves the sets A_1 and A_2, then $(\frac{m_2(A_1)}{m_1(A_1)})(\frac{m_1(A_2)}{m_2(A_2)}) > 1$. If we try to generalize this argument to more than two players, we find that there is a gap. We can show that if there is a positive cyclic trade that yields a Pareto bigger partition, then we obtain the desired inequality. Certainly, if P is not Pareto maximal, then there is a collection of transfers that yields a Pareto bigger partition. However, the question is this: If P is not Pareto maximal, does there exist a positive cyclic trade that yields a Pareto bigger partition? This is the gap, and it is filled by the following theorem.

Theorem 8.2 *If P is a partition that is not Pareto maximal, then there is a positive cyclic trade that yields a partition Pareto bigger than P.*

This result was proved by D. Weller ([43]). Our proof is a slight variation of his.

We shall prove two lemmas before beginning the proof of Theorem 8.2. The following lemma will be used in the proofs of Theorem 8.2 and Lemma 8.8 and will also be used in Chapters 13 and 14.

Lemma 8.3 *Suppose that $A_1, A_2, \ldots, A_t \subseteq C$ all have positive measure, where $t > 1$. Then there exist positive-measure sets $B_1, B_2, \ldots, B_t \subseteq C$ that satisfy the following:*

a. For every $i = 1, 2, \ldots, t$, $B_i \subseteq A_i$.
b. $(\frac{m_1(B_t)}{m_1(B_1)})(\frac{m_2(B_1)}{m_2(B_2)})\cdots(\frac{m_t(B_{t-1})}{m_t(B_t)}) = (\frac{m_1(A_t)}{m_1(A_1)})(\frac{m_2(A_1)}{m_2(A_2)})\cdots(\frac{m_t(A_{t-1})}{m_t(A_t)})$.
c. $m_1(B_t) = m_1(B_1), m_2(B_1) = m_2(B_2), \ldots, m_{t-1}(B_{t-2}) = m_{t-1}(B_{t-1})$.
d. For at least one $i = 1, 2, \ldots, t$, $B_i = A_i$.

Although this shall not be used, we note that condition c allows us to rewrite condition b as

$$\frac{m_t(B_{t-1})}{m_t(B_t)} = \left(\frac{m_1(A_t)}{m_1(A_1)}\right)\left(\frac{m_2(A_1)}{m_2(A_2)}\right)\cdots\left(\frac{m_t(A_{t-1})}{m_t(A_t)}\right).$$

Proof of Lemma 8.3: Fix positive-measure sets $A_1, A_2, \ldots, A_t \subseteq C$. For each $k = 1, 2, \ldots, t-1$, we shall show how to obtain sets $B_1^k, B_2^k, \ldots, B_t^k$ that satisfy, or partially satisfy, the four conditions of the lemma, plus one additional condition. In particular, for each such k, we shall define $B_1^k, B_2^k, \ldots, B_t^k$ satisfying the following:

i. Condition a of the lemma.
ii. Condition b of the lemma.
iii. $m_1(B_t^k) = m_1(B_1^k), m_2(B_1^k) = m_2(B_2^k), \ldots, m_k(B_{k-1}^k) = m_k(B_k^k)$.
iv. For at least one $i = t, 1, 2, \ldots, k$, $B_i^k = A_i$.
v. For each $i = k+1, k+2, \ldots, t-1$, $B_i^k = A_i$.

Then, $B_1^{t-1}, B_2^{t-1}, \ldots, B_t^{t-1}$ will satisfy the four conditions of the lemma. We begin by defining $B_1^1, B_2^1, \ldots, B_t^1$. We consider two cases:

Case 1: $m_1(A_t) \geq m_1(A_1)$. Then $\frac{m_1(A_1)}{m_1(A_t)} \leq 1$ and so, by Corollary 1.6, we may let $B_t^1 \subseteq A_t$ be such that, for each $j = 1, 2, \ldots, t$, $m_j(B_t^1) = (\frac{m_1(A_1)}{m_1(A_t)})m_j(A_t)$. For each $i = 1, 2, \ldots, t-1$, set $B_i^1 = A_i$. Then $B_1^1, B_2^1, \ldots, B_t^1$ clearly satisfy conditions i and v. Condition ii is satisfied since

$$\left(\frac{m_1(B_t^1)}{m_1(B_1^1)}\right)\left(\frac{m_2(B_1^1)}{m_2(B_2^1)}\right)\cdots\left(\frac{m_{t-1}(B_{t-2}^1)}{m_{t-1}(B_{t-1}^1)}\right)\left(\frac{m_t(B_{t-1}^1)}{m_t(B_t^1)}\right)$$

$$= \left(\frac{\left(\frac{m_1(A_1)}{m_1(A_t)}\right)m_1(A_t)}{m_1(A_1)}\right)\left(\frac{m_2(A_1)}{m_2(A_2)}\right)\cdots$$

$$\left(\frac{m_{t-1}(A_{t-2})}{m_{t-1}(A_{t-1})}\right)\left(\frac{m_t(A_{t-1})}{\left(\frac{m_1(A_1)}{m_1(A_t)}\right)m_t(A_t)}\right)$$

$$= \left(\frac{m_1(A_t)}{m_1(A_1)}\right)\left(\frac{m_2(A_1)}{m_2(A_2)}\right)\cdots\left(\frac{m_{t-1}(A_{t-2})}{m_{t-1}(A_{t-1})}\right)\left(\frac{m_t(A_{t-1})}{m_t(A_t)}\right),$$

condition iii is satisfied since $m_1(B_t^1) = (\frac{m_1(A_1)}{m_1(A_t)})m_1(A_t) = m_1(A_1) = m_1(B_1^1)$, and condition iv is satisfied since $B_1^1 = A_1$.

Case 2: $m_1(A_t) < m_1(A_1)$. Then, since $\frac{m_1(A_t)}{m_1(A_1)} < 1$, Corollary 1.6 tells us that there exists $B_1^1 \subseteq A_1$ such that, for all $j = 1, 2, \ldots, t$, $m_j(B_1^1) = (\frac{m_1(A_t)}{m_1(A_1)})m_j(A_1)$. For each $i = 2, 3, \ldots, t$, set $B_i^1 = A_i$. Then, as in Case 1, $B_1^1, B_2^1, \ldots, B_t^1$ clearly satisfy conditions i and v. Condition ii is satisfied since

$$\left(\frac{m_1(B_t^1)}{m_1(B_1^1)}\right)\left(\frac{m_2(B_1^1)}{m_2(B_2^1)}\right)\left(\frac{m_3(B_2^1)}{m_3(B_3^1)}\right)\cdots\left(\frac{m_t(B_{t-1}^1)}{m_t(B_t^1)}\right)$$

$$= \left(\frac{m_1(A_t)}{\left(\frac{m_1(A_t)}{m_1(A_1)}\right)m_1(A_1)}\right)\left(\frac{\left(\frac{m_1(A_t)}{m_1(A_1)}\right)m_2(A_1)}{m_2(A_2)}\right)\left(\frac{m_3(A_2)}{m_3(A_3)}\right)\cdots$$

$$\left(\frac{m_t(A_{t-1})}{m_t(A_t)}\right)$$

$$= \left(\frac{m_1(A_t)}{m_1(A_1)}\right)\left(\frac{m_2(A_1)}{m_2(A_2)}\right)\left(\frac{m_3(A_2)}{m_3(A_3)}\right)\cdots\left(\frac{m_t(A_{t-1})}{m_t(A_t)}\right),$$

condition iii is satisfied since $m_1(B_t^1) = m_1(A_t) = (\frac{m_1(A_t)}{m_1(A_1)})m_1(A_1) = m_1(B_1^1)$, and condition iv is satisfied since $B_t^1 = A_t$.

Next, fix any $s = 1, 2, \ldots, t-2$ and suppose that we have defined $B_1^s, B_2^s, \ldots, B_t^s$ satisfying conditions i, ii, iii, iv, and v. We wish to define $B_1^{s+1}, B_2^{s+1}, \ldots, B_t^{s+1}$ satisfying these conditions. As before, we consider two cases.

Case 1: $m_{s+1}(B_s^s) \geq m_{s+1}(B_{s+1}^s)$. Then $\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)} \leq 1$. By Corollary 1.6, for each $i = t, 1, 2, \ldots, s$, we let $B_i^{s+1} \subseteq B_i^s$ be such that, for each $j = 1, 2, \ldots, t, m_j(B_i^{s+1}) = (\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)})m_j(B_i^s)$. For each $i = s+1, s+2, \ldots, t-1$, set $B_i^{s+1} = B_i^s$. Then, $B_1^{s+1}, B_2^{s+1}, \ldots, B_t^{s+1}$ clearly satisfy conditions i and v. (Satisfaction of these conditions uses our assumption that $B_1^s, B_2^s, \ldots, B_t^s$ satisfy condition v.) For condition ii, we must show that $(\frac{m_1(B_t^{s+1})}{m_1(B_1^{s+1})})(\frac{m_2(B_1^{s+1})}{m_2(B_2^{s+1})}) \cdots (\frac{m_t(B_{t-1}^{s+1})}{m_t(B_t^{s+1})}) = (\frac{m_1(A_t)}{m_1(A_1)})(\frac{m_2(A_1)}{m_2(A_2)}) \cdots (\frac{m_t(A_{t-1})}{m_t(A_t)})$. We establish this as follows:

$$\left(\frac{m_1(B_t^{s+1})}{m_1(B_1^{s+1})}\right)\left(\frac{m_2(B_1^{s+1})}{m_2(B_2^{s+1})}\right)\cdots\left(\frac{m_t(B_{t-1}^{s+1})}{m_t(B_t^{s+1})}\right)$$

$$= \left[\left(\frac{m_1(B_t^{s+1})}{m_1(B_1^{s+1})}\right)\left(\frac{m_2(B_1^{s+1})}{m_2(B_2^{s+1})}\right)\cdots\left(\frac{m_s(B_{s-1}^{s+1})}{m_s(B_s^{s+1})}\right)\left(\frac{m_{s+1}(B_s^{s+1})}{m_{s+1}(B_{s+1}^{s+1})}\right)\right]$$

$$\times\left[\left(\frac{m_{s+2}(B_{s+1}^{s+1})}{m_{s+2}(B_{s+2}^{s+1})}\right)\left(\frac{m_{s+3}(B_{s+2}^{s+1})}{m_{s+3}(B_{s+3}^{s+1})}\right)\cdots\right.$$

$$\left.\left(\frac{m_{t-1}(B_{t-2}^{s+1})}{m_{t-1}(B_{t-1}^{s+1})}\right)\left(\frac{m_t(B_{t-1}^{s+1})}{m_t(B_t^{s+1})}\right)\right]$$

$$= \left[\left(\frac{\left(\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)}\right)m_1(B_t^s)}{\left(\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)}\right)m_1(B_1^s)}\right)\left(\frac{\left(\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)}\right)m_2(B_1^s)}{\left(\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)}\right)m_2(B_2^s)}\right)\cdots\right.$$

$$\left.\left(\frac{\left(\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)}\right)m_s(B_{s-1}^s)}{\left(\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)}\right)m_s(B_s^s)}\right)\left(\frac{\left(\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)}\right)m_{s+1}(B_s^s)}{m_{s+1}(B_{s+1}^s)}\right)\right]$$

$$\times\left[\left(\frac{m_{s+2}(B_{s+1}^s)}{m_{s+2}(B_{s+2}^s)}\right)\left(\frac{m_{s+3}(B_{s+2}^s)}{m_{s+3}(B_{s+3}^s)}\right)\cdots\right.$$

$$\left.\left(\frac{m_{t-1}(B_{t-2}^s)}{m_{t-1}(B_{t-1}^s)}\right)\left(\frac{m_t(B_{t-1}^s)}{\left(\frac{m_{s+1}(B_{s+1}^s)}{m_{s+1}(B_s^s)}\right)m_t(B_t^s)}\right)\right]$$

$$= \left(\frac{m_1\left(B_t^s\right)}{m_1\left(B_1^s\right)}\right)\left(\frac{m_2\left(B_1^s\right)}{m_2\left(B_2^s\right)}\right)\cdots\left(\frac{m_t\left(B_{t-1}^s\right)}{m_t\left(B_t^s\right)}\right)$$
$$= \left(\frac{m_1(A_t)}{m_1(A_1)}\right)\left(\frac{m_2(A_1)}{m_2(A_2)}\right)\cdots\left(\frac{m_t(A_{t-1})}{m_t(A_t)}\right)$$

The last equality follows from our assumption that the sets B_1^s, B_2^s, ..., B_t^s satisfy condition ii.

For condition iii, we must show that $m_1(B_t^{s+1}) = m_1(B_1^{s+1})$, $m_2(B_1^{s+1}) = m_2(B_2^{s+1}), \ldots, m_{s+1}(B_s^{s+1}) = m_{s+1}(B_{s+1}^{s+1})$. By our assumption that the sets B_1^s, B_2^s, ..., B_t^s satisfy condition iii, we know that $m_1(B_t^s) = m_1(B_1^s)$, $m_2(B_1^s) = m_2(B_2^s), \cdots, m_s(B_{s-1}^s) = m_s(B_s^s)$ and, hence,

$$m_1\left(B_t^{s+1}\right) = \left(\frac{m_{s+1}\left(B_{s+1}^s\right)}{m_{s+1}\left(B_s^s\right)}\right) m_1\left(B_t^s\right)$$
$$= \left(\frac{m_{s+1}\left(B_{s+1}^s\right)}{m_{s+1}\left(B_s^s\right)}\right) m_1\left(B_1^s\right) = m_1\left(B_1^{s+1}\right)$$
$$m_2\left(B_1^{s+1}\right) = \left(\frac{m_{s+1}\left(B_{s+1}^s\right)}{m_{s+1}\left(B_s^s\right)}\right) m_2\left(B_1^s\right)$$
$$= \left(\frac{m_{s+1}\left(B_{s+1}^s\right)}{m_{s+1}\left(B_s^s\right)}\right) m_2\left(B_2^s\right) = m_2\left(B_2^{s+1}\right)$$
$$\cdots$$
$$m_s\left(B_{s-1}^{s+1}\right) = \left(\frac{m_{s+1}\left(B_{s+1}^s\right)}{m_{s+1}\left(B_s^s\right)}\right) m_s\left(B_{s-1}^s\right)$$
$$= \left(\frac{m_{s+1}\left(B_{s+1}^s\right)}{m_{s+1}\left(B_s^s\right)}\right) m_s\left(B_s^s\right) = m_s\left(B_s^{s+1}\right).$$

To show that condition iii holds, it remains for us to show that $m_{s+1}(B_s^{s+1}) = m_{s+1}(B_{s+1}^{s+1})$. We establish this as follows:

$$m_{s+1}\left(B_s^{s+1}\right) = \left(\frac{m_{s+1}\left(B_{s+1}^s\right)}{m_{s+1}\left(B_s^s\right)}\right) m_{s+1}\left(B_s^s\right) = m_{s+1}\left(B_{s+1}^s\right)$$
$$= m_{s+1}\left(B_{s+1}^{s+1}\right)$$

For condition iv, we must show that, for at least one $i = t, 1, 2, \ldots, s+1$, $B_i^{s+1} = A_i$. Our assumption that B_1^s, B_2^s, ..., B_t^s satisfy condition v implies that $B^s_{s+1} = A_{s+1}$. Hence, since $B^{s+1}_{s+1} = B^s_{s+1}$, it follows that $B^{s+1}_{s+1} = A_{s+1}$.

Case 2: $m_{s+1}(B^s_s) < m_{s+1}(B^s_{s+1})$. Then $\frac{m_{s+1}(B^s_s)}{m_{s+1}(B^s_{s+1})} < 1$ and by Corollary 1.6 we let $B^{s+1}_{s+1} \subseteq B^s_{s+1}$ be such that, for each $j = 1, 2, \ldots, t$, $m_j(B^{s+1}_{s+1}) = (\frac{m_{s+1}(B^s_s)}{m_{s+1}(B^s_{s+1})}) m_j(B^s_{s+1})$. For each $i = 1, 2, \ldots, s, s+2, \ldots, t$, set $B^{s+1}_i = B^s_i$. Then $B^{s+1}_1, B^{s+1}_2, \ldots, B^{s+1}_t$ satisfy conditions i and v.

For condition ii, we must show that $(\frac{m_1(B^{s+1}_t)}{m_1(B^{s+1}_1)})(\frac{m_2(B^{s+1}_1)}{m_2(B^{s+1}_2)}) \cdots (\frac{m_t(B^{s+1}_{t-1})}{m_t(B^{s+1}_t)}) = (\frac{m_1(A_t)}{m_1(A_1)})(\frac{m_2(A_1)}{m_2(A_2)}) \cdots (\frac{m_t(A_{t-1})}{m_t(A_t)})$. We establish this as follows:

$$\begin{aligned}
&\left(\frac{m_1\left(B^{s+1}_t\right)}{m_1\left(B^{s+1}_1\right)}\right)\left(\frac{m_2\left(B^{s+1}_1\right)}{m_2\left(B^{s+1}_2\right)}\right)\cdots\left(\frac{m_t\left(B^{s+1}_{t-1}\right)}{m_t\left(B^{s+1}_t\right)}\right)\\
&= \left(\frac{m_1\left(B^{s+1}_t\right)}{m_1\left(B^{s+1}_1\right)}\right)\left(\frac{m_2\left(B^{s+1}_1\right)}{m_2\left(B^{s+1}_2\right)}\right)\cdots\left(\frac{m_s\left(B^{s+1}_{s-1}\right)}{m_s\left(B^{s+1}_s\right)}\right)\left(\frac{m_{s+1}\left(B^{s+1}_s\right)}{m_{s+1}\left(B^{s+1}_{s+1}\right)}\right)\\
&\quad\times\left(\frac{m_{s+2}\left(B^{s+1}_{s+1}\right)}{m_{s+2}\left(B^{s+1}_{s+2}\right)}\right)\left(\frac{m_{s+3}\left(B^{s+1}_{s+2}\right)}{m_{s+3}\left(B^{s+1}_{s+3}\right)}\right)\cdots\left(\frac{m_{t-1}\left(B^{s+1}_{t-2}\right)}{m_{t-1}\left(B^{s+1}_{t-1}\right)}\right)\left(\frac{m_t\left(B^{s+1}_{t-1}\right)}{m_t\left(B^{s+1}_t\right)}\right)\\
&= \left(\frac{m_1\left(B^s_t\right)}{m_1\left(B^s_1\right)}\right)\left(\frac{m_2\left(B^s_1\right)}{m_2\left(B^s_2\right)}\right)\cdots\left(\frac{m_s\left(B^s_{s-1}\right)}{m_s\left(B^s_s\right)}\right)\left(\frac{m_{s+1}\left(B^s_s\right)}{\left(\frac{m_{s+1}(B^s_s)}{m_{s+1}(B^s_{s+1})}\right)m_{s+1}\left(B^s_{s+1}\right)}\right)\\
&\quad\times\left(\frac{\left(\frac{m_{s+1}(B^s_s)}{m_{s+1}(B^s_{s+1})}\right)m_{s+2}\left(B^s_{s+1}\right)}{m_{s+2}\left(B^s_{s+2}\right)}\right)\left(\frac{m_{s+3}\left(B^s_{s+2}\right)}{m_{s+3}\left(B^s_{s+3}\right)}\right)\cdots\\
&\quad\left(\frac{m_{t-1}\left(B^s_{t-2}\right)}{m_{t-1}\left(B^s_{t-1}\right)}\right)\left(\frac{m_t\left(B^s_{t-1}\right)}{m_t\left(B^s_t\right)}\right)\\
&= \left(\frac{m_1\left(B^s_t\right)}{m_1\left(B^s_1\right)}\right)\left(\frac{m_2\left(B^s_1\right)}{m_2\left(B^s_2\right)}\right)\cdots\left(\frac{m_t\left(B^s_{t-1}\right)}{m_t\left(B^s_t\right)}\right)\\
&= \left(\frac{m_1(A_t)}{m_1(A_1)}\right)\left(\frac{m_2(A_1)}{m_2(A_2)}\right)\cdots\left(\frac{m_t(A_{t-1})}{m_t(A_t)}\right)
\end{aligned}$$

The last equality follows from our assumption that $B^s_1, B^s_2, \ldots, B^s_t$ satisfy condition ii.

For condition iii, we must show that $m_1(B^{s+1}_t) = m_1(B^{s+1}_1)$, $m_2(B^{s+1}_1) = m_2(B^{s+1}_2), \ldots, m_{s+1}(B^{s+1}_s) = m_{s+1}(B^{s+1}_{s+1})$. By assumption, $B^s_1, B^s_2, \ldots, B^s_t$ satisfy condition iii and, hence, $m_1(B^s_t) = m_1(B^s_1), m_2(B^s_1) = m_2(B^s_2), \ldots, m_s(B^s_{s-1}) = m_s(B^s_s)$. This implies that $m_1(B_t^{s+1}) = m_1(B_1^{s+1}), m_2(B_1^{s+1}) = m_2(B_2^{s+1}), \ldots, m_s(B_{s-1}^{s+1}) = m_s(B_s^{s+1})$. It remains for us to show that $m_{s+1}(B_s^{s+1}) = m_{s+1}(B_{s+1}^{s+1})$. We establish this as follows:

$$m_{s+1}\left(B_s^{s+1}\right) = m_{s+1}\left(B_s^s\right) = \left(\frac{m_{s+1}\left(B_s^s\right)}{m_{s+1}\left(B_{s+1}^s\right)}\right) m_{s+1}\left(B_{s+1}^s\right) = m_{s+1}\left(B_{s+1}^{s+1}\right)$$

Finally, we show that condition iv holds. Our assumption that $B_1^s, B_2^s, \ldots, B_t^s$ satisfy condition iv implies that, for at least one $i = t, 1, 2, \ldots, s$, $B_i^s = A_i$. But, for any such i, $B_i^{s+1} = B_i^s$. Hence, $B_i^{s+1} = A_i$ for at least one such i, and so condition iv is satisfied.

This establishes that, for each $k = 1, 2, \ldots, t-1$, there are sets $B_1^k, B_2^k, \ldots, B_t^k$ satisfying conditions i, ii, iii, iv, and v. Set $B_1 = B_1^{t-1}$, $B_2 = B_2^{t-1}, \ldots, B_t = B_t^{t-1}$. Then $B_1, B_2, \ldots, B_t$ satisfy conditions a, b, c, and d. This completes the proof of the lemma. □

Next, we introduce some additional structure and prove a lemma about this structure.

Definition 8.4 Suppose that P and Q are partitions. The *directed graph associated with the transition from P to Q*, denoted by $G(P, Q)$, is the graph given as follows:

a. The vertices of $G(P, Q)$ are labeled $1, 2, \ldots, n$, with each vertex corresponding to one of the n players.
b. There is an arrow from vertex i to vertex j if and only if the transition from partition P to partition Q involves a positive-measure transfer from Player i to Player j.

A closed path in $G(P, Q)$ corresponds to a positive cyclic trade. Theorem 8.2 asserts that if P is a partition that is not Pareto maximal, then there is a partition Q that is Pareto bigger than P and is such that the directed graph $G(P, Q)$ contains exactly one closed path and no arrows besides those in this path. Because all graphs we shall consider will have the same n vertices, we shall use the notation "$G(P, R) \subseteq G(P, Q)$" to mean that every arrow in $G(P, R)$ is in $G(P, Q)$.

The following lemma will be used in the proof of Theorem 8.2.

Lemma 8.5 *Suppose that P and Q are partitions and Q is Pareto bigger than P. Then $G(P, Q)$ contains at least one closed path. Or, equivalently, the transition from P to Q involves at least one positive cyclic trade.*

Proof: Suppose that P and Q are partitions, that Q is Pareto bigger than P, and assume, by way of contradiction, that $G(P, Q)$ contains no closed paths.

Since Q is Pareto bigger that P, we know that P and Q are not p-equivalent, and so the transition from P to Q involves some positive-measure transfer of cake. Hence, $G(P, Q)$ contains at least one arrow. Since $G(P, Q)$ contains no closed paths, there must be at least one vertex in $G(P, Q)$ such that an arrow begins at this vertex but no arrow ends at this vertex. This tells us that in the transition from partition P to partition Q at least one player gives up some piece of cake of positive measure but receives no piece of cake of positive measure. This player is less happy with partition Q than with partition P, contradicting the fact that partition Q is Pareto bigger than partition P. □

Suppose that partition P is not Pareto maximal. The lemma tells us that the transition from P to any Pareto bigger partition includes at least one positive cyclic trade. Theorem 8.2 asserts that some positive cyclic trade, by itself, changes P into a Pareto bigger partition.

Proof of Theorem 8.2: Assume that partition P is not Pareto maximal and suppose, by way of contradiction, that no positive cyclic trade yields a partition that is Pareto bigger than P. We will obtain our desired result by repeatedly applying the following claim.

Claim Suppose Q is a partition that is Pareto bigger than P. Then there is a partition R satisfying that

i. R is p-equivalent to or Pareto bigger than Q (and hence is Pareto bigger than P),
ii. $G(P, R) \subseteq G(P, Q)$, and
iii. $G(P, R)$ contains at least one fewer closed path than does $G(P, Q)$.

Proof of Claim: We assume that Q is Pareto bigger than P. Lemma 8.5 tells us that $G(P, Q)$ contains at least one closed path or, equivalently, that the transition from P to Q involves at least one positive cyclic trade. For ease of notation, we assume (by renumbering if necessary) that for some $t > 1$ this cyclic trade involves Players 1 through t, in order. In particular, suppose that $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle A_1, A_2, \ldots, A_t\rangle)$ is a positive cyclic trade that is maximal with respect to these players. In other words, in changing from partition P to partition Q, Player 1 gives A_1 (and no proper superset of A_1) to Player 2, Player 2 gives A_2 (and no proper superset of A_2) to Player 3, etc. Since $A_1, A_2, \ldots, A_t$ are all subsets of C of positive measure, we can apply Lemma 8.3 to obtain positive-measure sets $B_1, B_2, \ldots, B_t$, such that

a. for every $i = 1, 2, \ldots, t$, $B_i \subseteq A_i$.
b. $(\frac{m_1(B_t)}{m_1(B_1)})(\frac{m_2(B_1)}{m_2(B_2)}) \cdots (\frac{m_t(B_{t-1})}{m_t(B_t)}) = (\frac{m_1(A_t)}{m_1(A_1)})(\frac{m_2(A_1)}{m_2(A_2)}) \cdots (\frac{m_t(A_{t-1})}{m_t(A_t)})$.

c. $m_1(B_t) = m_1(B_1), m_2(B_1) = m_2(B_2), \ldots, m_{t-1}(B_{t-2}) = m_{t-1}(B_{t-1})$.
d. for at least one $i = 1, 2, \ldots, t$, $B_i = A_i$.

We shall use only conditions a, c, and d here.

By condition a, $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle B_1, B_2, \ldots, B_t\rangle)$ is a positive cyclic trade. Condition c implies that Player 1, Player 2, ..., Player $t-1$, are equally happy before and after this trade. What about Player t? Suppose that $m_t(B_{t-1}) > m_t(B_t)$. In other words, suppose that Player t is happier after the trade. Then $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle B_1, B_2, \ldots, B_t\rangle)$ is a positive cyclic trade that transforms partition P into a Pareto bigger partition. This is contrary to our assumption that no positive cyclic trade yields a partition that is Pareto bigger than P. Hence, $m_t(B_{t-1}) \le m_t(B_t)$.

We are now ready to describe the partition R. We begin with an informal description. Every transfer of cake in $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle B_1, B_2, \ldots, B_t\rangle)$ is contained within a transfer of cake in $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle A_1, A_2, \ldots, A_t\rangle)$, and every transfer of cake in $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle A_1, A_2, \ldots, A_t\rangle)$ is in the set of transfers that changes partition P into partition Q. Thus, the transfers making up $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle B_1, B_2, \ldots, B_t\rangle)$ constitute part of the transfers that change partition P to partition Q. To obtain partition R, we simply start with P and do the transfers that change partition P to partition Q, except that we leave out all of the transfers in $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle B_1, B_2, \ldots, B_t\rangle)$.

In the precise definition of the partition R that follows, it will be convenient to imagine starting with partition P, doing all of the transfers that change P to Q, and then undoing those transfers given by $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle B_1, B_2, \ldots, B_t\rangle)$. The positive cyclic trade that undoes these transfers is $\mathrm{CT}(\langle t, t-1, \ldots, 2, 1\rangle | \langle B_{t-1}, B_{t-2}, \ldots, B_1, B_t\rangle)$. Hence, R is the partition obtained from Q by completing all of transfers given by $\mathrm{CT}(\langle t, t-1, \ldots, 2, 1\rangle | \langle B_{t-1}, B_{t-2}, \ldots, B_1, B_t\rangle)$. If we set $Q = \langle Q_1, Q_2, \ldots, Q_n\rangle$, then $R = \langle R_1, R_2, \ldots, R_n\rangle$ where, for each $i = 1, 2, \ldots, n$, we define R_i as follows:

$$R_i = \begin{cases} (Q_i \cup B_i) \setminus B_{i-1} & \text{if } i = 1, 2, \ldots, t \\ Q_i & \text{if } i = t+1, t+2, \ldots, n \end{cases}$$

where we set $B_0 = B_t$.

If $i = t+1, t+2, \ldots, n$, then $R_i = Q_i$, and so Player i is indifferent between partition Q and partition R. By condition c of Lemma 8.3, we know that $m_1(B_t) = m_1(B_1), m_2(B_1) = m_2(B_2), \ldots, m_{t-1}(B_{t-2}) = m_{t-1}(B_{t-1})$, and so Player 1, Player 2, ..., Player $t-1$ are also indifferent between partition Q and R. Recalling that $m_t(B_{t-1}) \le m_t(B_t)$, we see that Player t is at least as happy with partition R as with partition Q. Hence, partition R is p-equivalent to or Pareto bigger than partition Q and, since Q is Pareto bigger than P, it follows that R is Pareto bigger than P, and so condition i of the claim is satisfied.

Concerning condition ii, we must show that every arrow that appears in $G(P, R)$ also appears in $G(P, Q)$). In other words, we must show that, for distinct $i, j = 1, 2, \ldots, n$, if the transition from partition P to partition R involves a positive-measure transfer from Player i to Player j, then so does the transition from partition P to partition Q. Since we have portrayed the change from partition P to partition R as consisting of first changing partition P to partition Q, and then completing the transfers given by $\mathrm{CT}(\langle t, t-1, \ldots, 2, 1\rangle | \langle B_{t-1}, B_{t-2}, \ldots, B_1, B_t\rangle)$, it suffices to show that completing these transfers does not introduce any positive-measure transfers that were not already present in the change from P to Q. This is clear from our definition of the B_i, since each transfer in $\mathrm{CT}(\langle t, t-1, \ldots, 2, 1\rangle | \langle B_{t-1}, B_{t-2}, \ldots, B_1, B_t\rangle)$ partially or completely undoes a transfer involved in the transition from partition P to partition Q. Hence, this cyclic trade adds no new arrows to the directed graph.

For condition iii, we first note that, by condition ii, there are no closed paths in $G(P, R)$ that are not also in $G(P, Q)$. We must show that some closed path in $G(P, Q)$ is not present in $G(P, R)$. Consider the closed path in $G(P, Q)$ associated with $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle A_1, A_2, \ldots, A_t\rangle)$. This is the closed path consisting of an arrow from vertex 1 to vertex 2, an arrow from vertex 2 to vertex 3, ..., an arrow from vertex $t-1$ to vertex t, and an arrow from vertex t to vertex 1. We need only show that one of these arrows does not exist in $G(P, R)$. By condition d of Lemma 8.3, for at least one $i = 1, 2, \ldots, t$, $B_i = A_i$. Fix such an i and note that the transfer of A_i from Player i to Player $i+1$ in $\mathrm{CT}(\langle 1, 2, \ldots, t\rangle | \langle A_1, A_2, \ldots, A_t\rangle)$ is precisely undone by the transfer of B_i from Player $i+1$ to Player i in $\mathrm{CT}(\langle t, t-1, \ldots, 2, 1\rangle | \langle B_{t-1}, B_{t-2}, \ldots, B_1, B_t\rangle)$. This tells us that $G(P, R)$ does not contains an arrow from vertex i to vertex $i+1$, and so the closed path in $G(P, Q)$ under consideration does not exist in $G(P, R)$. (This uses our assumption that, in changing from partition P to partition Q, Player i gives nothing besides A_i to Player $i+1$.) This completes the proof of the claim.

We return to the proof of the theorem. We have assumed that P is not Pareto maximal and, by way of contradiction, that no positive cyclic trade yields a partition that is Pareto bigger than P. Let Q be a partition that is Pareto bigger than P. Since the number of closed paths in any directed graph on n vertices is certainly finite, we can apply the claim repeatedly to arrive at a partition R that is Pareto bigger than P and is such that $G(P, R)$ contains no closed paths. This contradicts Lemma 8.5 and thus completes the proof of the theorem. □

We are now ready to generalize the two-player analysis from Section 8A.

Definition 8.6 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$. For distinct $i, j = 1, 2, \ldots, n$, we define pr_{ij}, the *ij partition ratio*, as follows:

$$\mathrm{pr}_{ij} = \sup \left\{ \frac{m_j(A)}{m_i(A)} : A \subseteq P_i \text{ and } A \text{ has positive measure} \right\}$$

If, for some $i = 1, 2, \ldots, n$, P_i has measure zero, then for every $j = 1, 2, \ldots, n$ with $j \neq i$, $\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i$ and A has positive measure$\} = \emptyset$. In this case, we shall consider pr_{ij} to be undefined for every such j. It is possible that $\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i$ and A has positive measure$\}$ is unbounded for some i and j, in which case we set $\mathrm{pr}_{ij} = \infty$. Also, absolute continuity guarantees that, for all distinct $i, j = 1, 2, \ldots, n$, if pr_{ij} is defined, then $\mathrm{pr}_{ij} > 0$. Thus, for any such i and j, either pr_{ij} is undefined or else $0 < \mathrm{pr}_{ij} \leq \infty$.

Definition 8.7 Suppose that $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, n\}$ are distinct.

a. A sequence of the form $\langle \mathrm{pr}_{i_t i_1}, \mathrm{pr}_{i_1 i_2}, \ldots, \mathrm{pr}_{i_{t-2} i_{t-1}}, \mathrm{pr}_{i_{t-1} i_t} \rangle$, where each such $\mathrm{pr}_{i_j i_k}$ is defined, is called a *cyclic sequence*.
b. *CS(k)* denotes the set of all cyclic sequences of length k.
c. *CS* denotes the set of all cyclic sequences.
d. If $\varphi = \langle \mathrm{pr}_{i_t i_1}, \mathrm{pr}_{i_1 i_2}, \ldots, \mathrm{pr}_{i_{t-2} i_{t-1}}, \mathrm{pr}_{i_{t-1} i_t} \rangle \in \mathrm{CS}$, then the *cyclic product* of φ, denoted by *CP*(φ), is the product $\mathrm{pr}_{i_t i_1} \mathrm{pr}_{i_1 i_2} \cdots \mathrm{pr}_{i_{t-2} i_{t-1}} \mathrm{pr}_{i_{t-1} i_t}$.

Of course, different partitions will have different partition ratios and different sets CS associated with them. In what follows, it should always be clear by context, whenever we mention a partition ratio or the set CS, to which partition we are referring.

Since one or more of the terms in a cyclic product can be infinite, but none of the terms can be zero, it follows that, for any $\varphi \in \mathrm{CS}$, $0 < \mathrm{CP}(\varphi) \leq \infty$.

Our characterization of Pareto maximality, Theorem 8.9, will follow easily from Theorem 8.2 and the following lemma.

Lemma 8.8 *Let P be a partition and fix any $k = 2, 3, \ldots, n$. There exists $\varphi \in CS(k)$ such that $CP(\varphi) > 1$ if and only if there exists a positive cyclic trade of length k that produces a partition Pareto bigger than P.*

Proof: Fix a partition P and $k = 2, 3, \ldots, n$.

For the forward direction, suppose that, for some $\varphi \in \mathrm{CS}(k)$, $\mathrm{CP}(\varphi) > 1$. By renumbering, if necessary, we may assume that $\varphi = \langle \mathrm{pr}_{k1}, \mathrm{pr}_{12}, \ldots, \mathrm{pr}_{k-2,k-1}, \mathrm{pr}_{k-1,k} \rangle$. By the definition of the partition ratios (regardless of whether or not any involved partition ratio is equal to infinity), this implies that, for each

$i = 1, 2, \ldots, k$, there is a set $A_i \subseteq P_i$ of positive measure such that the following inequality holds:

$$\left(\frac{m_1(A_k)}{m_k(A_k)}\right)\left(\frac{m_2(A_1)}{m_1(A_1)}\right)\cdots\left(\frac{m_{k-1}(A_{k-2})}{m_{k-2}(A_{k-2})}\right)\left(\frac{m_k(A_{k-1})}{m_{k-1}(A_{k-1})}\right) > 1$$

Rearranging terms gives us the following inequality:

$$\left(\frac{m_1(A_k)}{m_1(A_1)}\right)\left(\frac{m_2(A_1)}{m_2(A_2)}\right)\cdots\left(\frac{m_{k-1}(A_{k-2})}{m_{k-1}(A_{k-1})}\right)\left(\frac{m_k(A_{k-1})}{m_k(A_k)}\right) > 1$$

Since each A_i has positive measure, we can apply Lemma 8.3 to obtain sets $B_1, B_2, \ldots, B_k$, each of positive measure, such that

a. for every $i = 1, 2, \ldots, k$, $B_i \subseteq A_i$.
b. $(\frac{m_1(B_k)}{m_1(B_1)})(\frac{m_2(B_1)}{m_2(B_2)})\cdots(\frac{m_k(B_{k-1})}{m_k(B_k)}) = (\frac{m_1(A_k)}{m_1(A_1)})(\frac{m_2(A_1)}{m_2(A_2)})\cdots(\frac{m_k(A_{k-1})}{m_k(A_k)})$.
c. $m_1(B_k) = m_1(B_1), m_2(B_1) = m_2(B_2), \ldots, m_{k-1}(B_{k-2}) = m_{k-1}(B_{k-1})$.
d. for at least one $i = 1, 2, \ldots, k$, $B_i = A_i$.

In contrast with our previous application of Lemma 8.3 in the proof of Theorem 8.2, we shall use only conditions a, b, and c here.

By condition b, we know that $(\frac{m_1(B_k)}{m_1(B_1)})(\frac{m_2(B_1)}{m_2(B_2)})\cdots(\frac{m_k(B_{k-1})}{m_k(B_k)}) > 1$. This, together with condition c, implies that $\frac{m_k(B_{k-1})}{m_k(B_k)} > 1$ and, hence, $m_k(B_{k-1}) > m_k(B_k)$.

For each $i = 1, 2, \ldots, k$, $B_i \subseteq A_i \subseteq P_i$ and, hence, $B_i \subseteq P_i$. Consider the positive cyclic trade $\mathrm{CT}(\langle 1, 2, \ldots, k\rangle | \langle B_1, B_2, \ldots, B_k\rangle)$. Let Q be the partition obtained from partition P by completing this trade. Condition c tells us that Player 1, Player 2, ..., Player $k-1$ are each indifferent between partitions P and Q. Since $m_k(B_{k-1}) > m_k(B_k)$, we know that Player k is happier with partition Q than with partition P. Thus, $\mathrm{CT}(\langle 1, 2, \ldots, k\rangle | \langle B_1, B_2, \ldots, B_k\rangle)$ is a positive cyclic trade of length k that produces a partition that is Pareto bigger than P.

For the reverse direction, we assume that there is a positive cyclic trade of length k that produces a partition that is Pareto bigger than P. By renumbering, if necessary, let such a cyclic trade be given by $\mathrm{CT}(\langle 1, 2, \ldots, k\rangle | \langle A_1, A_2, \ldots, A_k\rangle)$. Then, $m_1(A_k) \geq m_1(A_1)$, $m_2(A_1) \geq m_2(A_2)$, ..., $m_k(A_{k-1}) \geq m_k(A_k)$, with at least one of these inequalities being strict. It follows that

$$\left(\frac{m_1(A_k)}{m_1(A_1)}\right)\left(\frac{m_2(A_1)}{m_2(A_2)}\right)\cdots\left(\frac{m_k(A_{k-1})}{m_k(A_k)}\right) > 1.$$

Rearranging terms gives us the following inequality:

$$\left(\frac{m_2(A_1)}{m_1(A_1)}\right)\left(\frac{m_3(A_2)}{m_2(A_2)}\right)\cdots\left(\frac{m_k(A_{k-1})}{m_{k-1}(A_{k-1})}\right)\left(\frac{m_1(A_k)}{m_k(A_k)}\right) > 1$$

Each of the fractions on the left-hand side of this inequality is one of the terms over which the supremum is taken in the definition of the corresponding partition ratio. Thus, $\mathrm{pr}_{12}\mathrm{pr}_{23}\ldots\mathrm{pr}_{k-1,k}\mathrm{pr}_{k1} > 1$, and we have shown that there exists $\varphi \in \mathrm{CS}(k)$ with $\mathrm{CP}(\varphi) > 1$. This completes the proof of the lemma. □

Besides its use in the proof of Theorem 8.9, Lemma 8.8 will also be used in Chapter 13 when we consider a possible hierarchy involving the failure of Pareto maximality. Our characterization of Pareto maximality using Partition ratios is the following.

Theorem 8.9 *A partition P is Pareto maximal if and only if for every* $\varphi \in$ CS, $\mathrm{CP}(\varphi) \leq 1$.

Proof: For the forward direction, suppose that P is a partition and that, for some $\varphi \in \mathrm{CS}$, $\mathrm{CP}(\varphi) > 1$. By Lemma 8.8, we know that there exists a positive cyclic trade that produces a partition Pareto bigger than P. Hence, P is not Pareto maximal.

For the reverse direction, suppose that P is not Pareto maximal. Theorem 8.2 implies that there is a positive cyclic trade that yields a partition Pareto bigger than P. Therefore, by Lemma 8.8, there exists a $\varphi \in \mathrm{CS}$ such that $\mathrm{CP}(\varphi) > 1$. □

Next, we present an application of the theorem. Example 8.10 applies the theorem first to show that a certain partition is not Pareto maximal, and then to show that another partition is Pareto maximal. The cake, measures, and partitions in this example are the same as in Example 6.3.

Example 8.10 Let C be the interval $[0, 3)$ on the real number line and let m_L be Lebesgue measure on this set. Suppose that there are three players, Player 1, Player 2, and Player 3, with corresponding measures m_1, m_2, and m_3, respectively, defined as follows: for any $A \subseteq C$,

$$m_1(A) = .3m_L(A \cap [0,1)) + .1m_L(A \cap [1,2)) + .6m_L(A \cap [2,3))$$
$$m_2(A) = .6m_L(A \cap [0,1)) + .3m_L(A \cap [1,2)) + .1m_L(A \cap [2,3))$$
$$m_3(A) = .1m_L(A \cap [0,1)) + .6m_L(A \cap [1,2)) + .3m_L(A \cap [2,3))$$

As noted in Example 6.3, it is easy to verify that $m_1(C) = m_2(C) = m_3(C) = 1$, and so m_1, m_2, and m_3 are measures on C. It is also easy to verify that these measures are absolutely continuous with respect to each other.

Consider the partition $P = \langle[0, 1), [1, 2), [2, 3)\rangle$. The corresponding partition ratios are as follows:

$$\mathrm{pr}_{12} = \sup\left\{\frac{m_2(A)}{m_1(A)} : A \subseteq [0, 1) \text{ and } A \text{ has positive measure}\right\} = \frac{.6}{.3} = 2$$
$$\mathrm{pr}_{21} = \sup\left\{\frac{m_1(A)}{m_2(A)} : A \subseteq [1, 2) \text{ and } A \text{ has positive measure}\right\} = \frac{.1}{.3} = \frac{1}{3}$$
$$\mathrm{pr}_{13} = \sup\left\{\frac{m_3(A)}{m_1(A)} : A \subseteq [0, 1) \text{ and } A \text{ has positive measure}\right\} = \frac{.1}{.3} = \frac{1}{3}$$
$$\mathrm{pr}_{31} = \sup\left\{\frac{m_1(A)}{m_3(A)} : A \subseteq [2, 3) \text{ and } A \text{ has positive measure}\right\} = \frac{.6}{.3} = 2$$
$$\mathrm{pr}_{23} = \sup\left\{\frac{m_3(A)}{m_2(A)} : A \subseteq [1, 2) \text{ and } A \text{ has positive measure}\right\} = \frac{.6}{.3} = 2$$
$$\mathrm{pr}_{32} = \sup\left\{\frac{m_2(A)}{m_3(A)} : A \subseteq [2, 3) \text{ and } A \text{ has positive measure}\right\} = \frac{.1}{.3} = \frac{1}{3}$$

Since $\mathrm{pr}_{12}\mathrm{pr}_{23}\mathrm{pr}_{31} = (2)(2)(2) = 8 > 1$, Theorem 8.9 tells us that P is not Pareto maximal. We also note that each of the three cyclic sequences of length two (i.e., $\langle\mathrm{pr}_{12}, \mathrm{pr}_{21}\rangle$, $\langle\mathrm{pr}_{13}, \mathrm{pr}_{31}\rangle$, and $\langle\mathrm{pr}_{23}, \mathrm{pr}_{32}\rangle$) have product equal to $2/3$. By Lemma 8.8, it follows that there is no cyclic trade of length two that produces a partition Pareto bigger than P.

With cake C and measures m_1, m_2, and m_3, as before, let us now consider the partition $Q = \langle[2, 3), [0, 1), [1, 2)\rangle$. The corresponding partition ratios are as follows:

$$\mathrm{pr}_{12} = \sup\left\{\frac{m_2(A)}{m_1(A)} : A \subseteq [2, 3) \text{ and } A \text{ has positive measure}\right\} = \frac{.1}{.6} = \frac{1}{6}$$
$$\mathrm{pr}_{21} = \sup\left\{\frac{m_1(A)}{m_2(A)} : A \subseteq [0, 1) \text{ and } A \text{ has positive measure}\right\} = \frac{.3}{.6} = \frac{1}{2}$$
$$\mathrm{pr}_{13} = \sup\left\{\frac{m_3(A)}{m_1(A)} : A \subseteq [2, 3) \text{ and } A \text{ has positive measure}\right\} = \frac{.3}{.6} = \frac{1}{2}$$
$$\mathrm{pr}_{31} = \sup\left\{\frac{m_1(A)}{m_3(A)} : A \subseteq [1, 2) \text{ and } A \text{ haspositive measure}\right\} = \frac{.1}{.6} = \frac{1}{6}$$
$$\mathrm{pr}_{23} = \sup\left\{\frac{m_3(A)}{m_2(A)} : A \subseteq [0, 1) \text{ and } A \text{ has positive measure}\right\} = \frac{.1}{.6} = \frac{1}{6}$$
$$\mathrm{pr}_{32} = \sup\left\{\frac{m_2(A)}{m_3(A)} : A \subseteq [1, 2) \text{ and } A \text{ has positive measure}\right\} = \frac{.3}{.6} = \frac{1}{2}$$

Computing all cyclic products, we have:

$$\mathrm{pr}_{12}\mathrm{pr}_{21} = \mathrm{pr}_{13}\mathrm{pr}_{31} = \mathrm{pr}_{23}\mathrm{pr}_{32} = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \frac{1}{12}$$
$$\mathrm{pr}_{12}\mathrm{pr}_{23}\mathrm{pr}_{31} = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{216}$$
$$\mathrm{pr}_{32}\mathrm{pr}_{21}\mathrm{pr}_{13} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

Since all cyclic products are less than one, Theorem 8.9 tells us that Q is Pareto maximal.

We recall from Chapter 6 (see Definition 6.1) that a partition $P = \langle P_1, P_2, \ldots, P_n\rangle$ is proper subpartition Pareto maximal if and only if, for any proper and non-empty $\delta \subseteq \{1, 2, \ldots, n\}$, $\langle P_i : i \in \delta\rangle$ is Pareto maximal partition of $\bigcup_{i\in\delta} P_i$ among the players named by δ. Also (see Theorem 6.2), if P is Pareto maximal, then P is proper subpartition Pareto maximal. We presented Example 6.3 to show that the converse is false. We are now in a position to give some additional perspective on this idea by connecting it with partition ratios.

Let P be a partition. We observe that the partition ratios associated with a proper subpartition of P are the same as the corresponding partition ratios for P. Also, if $\delta \subseteq \{1, 2, \ldots, n\}$ is proper and non-empty, and $\varphi \in \mathrm{CS}$ refers only to players named by δ, then $\varphi \in \mathrm{CS}(k)$ for some $k \le |\delta|$. This, together with Theorem 8.9, implies that

a. P is proper subpartition Pareto maximal if and only if, for any $\varphi \in \mathrm{CS}$ of length less than n, $\mathrm{CP}(\varphi) \le 1$ and
b. P is proper subpartition Pareto maximal but not Pareto maximal if and only if, for any $\varphi \in \mathrm{CS}$ of length less than n, $\mathrm{CP}(\varphi) \le 1$ and, for some $\varphi \in \mathrm{CS}$ of length n, $\mathrm{CP}(\varphi) > 1$.

Partition $P = \langle[0, 1), [1, 2), [2, 3)\rangle$ in Example 8.10 is precisely as described in statement b. We shall revisit this example in Chapter 10 (where we will gain some additional geometric perspective) and in Chapter 13 (in our study of a possible hierarchy involving the failure of Pareto maximality).

Next, we consider chores versions of the ideas presented in this section. The definitions of cyclic trade and of positive cyclic trade, as given in Definition 8.1, do not refer to Pareto maximality. Hence, these definitions are appropriate for our discussion of chores. The chores version of Theorem 8.2 is the following. The proof is analogous and we omit it.

Theorem 8.11 *If P is a partition that is not Pareto minimal, then there is a positive cyclic trade that yields a partition Pareto smaller than P.*

Our definition of chores partition ratio, which we denote by qr_{ij}, is analogous to the definition of partition ratio.

Definition 8.12 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$. For distinct $i, j = 1, 2, \ldots, n$, we define qr_{ij}, the ij *chores partition ratio*, as follows:

$$\mathrm{qr}_{ij} = \inf\left\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i \text{ and } A \text{ has positive measure}\right\}$$

As in the standard context, if P_i has measure zero for some $i = 1, 2, \ldots, n$, then we shall consider qr_{ij} to be undefined for every $j = 1, 2, \ldots, n$ with $j \neq i$. Also, for distinct $i, j = 1, 2, \ldots, n$, $0 \leq \mathrm{qr}_{ij} < \infty$. This is in contrast with the fact that, for such i and j, $0 < \mathrm{pr}_{ij} \leq \infty$.

Chores cyclic sequences are defined in the obvious way. We shall let $CCS(k)$ and CCS denote the set of all chores cyclic sequences of length k and the set of all chores cyclic sequences, respectively. If $\varphi = \langle \mathrm{qr}_{i_t i_1}, \mathrm{qr}_{i_1 i_2}, \ldots, \mathrm{qr}_{i_{t-2} i_{t-1}}, \mathrm{qr}_{i_{t-1} i_t} \rangle \in \mathrm{CCS}$, then the *chores cyclic product* of φ, denoted by $CCP(\varphi)$, is the product $\mathrm{qr}_{i_t i_1} \mathrm{qr}_{i_1 i_2} \ldots \mathrm{qr}_{i_{t-2} i_{t-1}} \mathrm{qr}_{i_{t-1} i_t}$.

Since one or more of the terms in a chores cyclic product can be zero, but none of the terms can equal infinity, it follows that, for any $\varphi \in \mathrm{CCS}$, $0 \leq \mathrm{CP}(\varphi) < \infty$. The chores versions of Lemma 8.8 and the characterization theorem, Theorem 8.9, are the following.

Lemma 8.13 *Let P be a partition and fix $k = 2, 3, \ldots, n$. There exists $\varphi \in$* $\mathrm{CCS}(k)$ *such that* $\mathrm{CP}(\varphi) < 1$ *if and only if there exists a positive cyclic trade of length k that produces a partition Pareto smaller than P.*

Theorem 8.14 *A partition P is Pareto minimal if and only if, for every $\varphi \in$* CCS, $\mathrm{CP}(\varphi) \geq 1$.

The proofs are analogous to the proofs of Lemma 8.8 and Theorem 8.9 and we omit them.

We close this section by noting that, in contrast with our general theme, we have presented no geometric perspective on the partition ratio characterizations of Pareto maximality and Pareto minimality. In the next chapter, we will consider a new geometric framework for studying cake division, and then, in Chapter 10, we will present a characterization of Pareto maximality and Pareto minimality using this framework. In Chapter 13, armed with this material, we will look back and supply geometric perspective for what we have done in this section.

8C. The Situation Without Absolute Continuity

In this section we make no general assumptions about absolute continuity. Recall that a partition P is non-wasteful (see Definition 6.5) if and only if it is not possible to perform any zero-to-positive transfer and is c-non-wasteful (see Definition 7.21) if and only if it is not possible to perform any positive-to-zero transfer. We shall frequently use the notions of zero-to-positive transfer, positive-to-zero transfer, etc. (see Definition 7.22).

Our presentation in this section parallels that of the previous section, but there will be important differences. We begin by re-examining Definition 8.1. The definition and notation for cyclic trade needs no change. However, the definition of positive cyclic trade given in Definition 8.1 does not make sense in our present context, since a subset of C may have positive measure for some players but not for others. The appropriate adjustment of this part of Definition 8.1 is the following.

Definition 8.15 The cyclic trade $\mathrm{CT}(\langle i_1, i_2, \ldots, i_t\rangle | \langle A_{i_1}, A_{i_2}, \ldots, A_{i_t}\rangle)$ is a *positive cyclic trade* if each transfer of cake in this cyclic trade is a positive-to-positive transfer.

In other words, a cyclic trade is positive if and only if, for each transfer that is part of the cyclic trade, both the player giving up the piece of cake and the player receiving it give positive value to that piece. If the measures are absolutely continuous with respect to each other, then certainly the condition given in Definition 8.15 implies the condition given in Definition 8.1, and so these definitions are consistent.

Next, we consider Theorem 8.2. This result is easily seen to be false if absolute continuity fails. Suppose, for example, that A and B are disjoint subsets of C, $m_1(A) = 0$, $m_1(B) = 0$, $m_2(A) > 0$, and $m_2(B) > 0$, and consider the partitions $P = \langle C\backslash A, A\rangle$ and $Q = \langle C\backslash(A \cup B), A \cup B\rangle$. Q is Pareto bigger than P, and hence P is not Pareto maximal. However, it is clear that no positive cyclic trade, starting with P, yields a partition that is Pareto bigger than P. This example uses the fact that P is wasteful (see Definition 6.5). We need only add the assumption of non-wastefulness to Theorem 8.2 to obtain a correct result.

Theorem 8.16 *If P is a partition that is non-wasteful and is not Pareto maximal, then there is a positive cyclic trade that produces a partition Pareto bigger than P.*

Before beginning the proof, we state and prove two lemmas. These results, Lemmas 8.17 and 8.17, are the appropriate adjustments of Lemmas 8.3 and 8.5, respectively, which were used in the proof of Theorem 8.2.

Certainly Lemma 8.3 cannot be applied directly in the proof of Theorem 8.16, since it refers to sets of positive measure, and that concept does not make sense without absolute continuity. The appropriate adjustment of Lemma 8.3 to our present setting is the following.

Lemma 8.17 *Suppose that $A_1, A_2, \ldots, A_t \subseteq C$, where $t > 1$, are such that, for every $i = 1, 2, \ldots, t$, $m_i(A_i) > 0$ and $m_i(A_{i-1}) > 0$ (where we set $A_0 = A_t$). Then there exist $B_1, B_2, \ldots, B_t \subseteq C$ that satisfy the following:*

a. For every $i = 1, 2, \ldots, t$, $B_i \subseteq A_i$.
b. $(\frac{m_1(B_t)}{m_1(B_1)})(\frac{m_2(B_1)}{m_2(B_2)})\cdots(\frac{m_t(B_{t-1})}{m_t(B_t)}) = (\frac{m_1(A_t)}{m_1(A_1)})(\frac{m_2(A_1)}{m_2(A_2)})\cdots(\frac{m_t(A_{t-1})}{m_t(A_t)})$.
c. $m_1(B_t) = m_1(B_1), m_2(B_1) = m_2(B_2), \ldots, m_{t-1}(B_{t-2}) = m_{t-1}(B_{t-1})$.
d. For at least one $i = 1, 2, \ldots, t$, $B_i = A_i$.
e. For every $i = 1, 2, \ldots, n$, $m_i(B_i) > 0$ and $m_i(B_{i-1}) > 0$ (where we set $B_0 = B_t$).

The proof is the same as the proof of Lemma 8.3, and we omit it.

Next, we need an appropriate adjustment of Definition 8.4 to our present setting.

Definition 8.18 Suppose that P and Q are partitions. The *directed graph associated with the transition from P to Q*, denoted by $G(P, Q)$, is the graph given as follows:

a. The vertices of $G(P, Q)$ are labeled $1, 2, \ldots, n$, with each vertex corresponding to one of the n players.
b. There is an arrow from vertex i to vertex j if and only if the transition from partition P to partition Q involves either a positive-to-positive, a positive-to-zero, or a zero-to-positive transfer from Player i to Player j.

We shall refer to an arrow in $G(P, Q)$ as a *positive-to-positive arrow*, a *positive-to-zero arrow*, or a *zero-to-positive arrow* depending on whether the transfer to which it corresponds is positive-to-positive, positive-to-zero, or zero-to-positive, respectively. Then, a closed path in $G(P, Q)$ consisting of positive-to-positive arrows corresponds to a positive cyclic trade. Theorem 8.16 implies that if P is a partition that is non-wasteful and is not Pareto maximal then there is a partition Q that is Pareto bigger than P and is such that the directed graph $G(P, Q)$ contains exactly one closed path consisting of positive-to-positive arrows (and no arrows besides those in this path).

As in the previous section, we shall write "$G(P, R) \subseteq G(P, Q)$" to mean that every arrow in $G(P, R)$ is in $G(P, Q)$, but now we shall also mean that every arrow in $G(P, R)$ is of the same type (i.e., positive-to-positive, positive-to-zero, or zero-to-positive) as it is in $G(P, Q)$.

The following lemma is the appropriate adjustment of Lemma 8.5.

Lemma 8.19 *Suppose that P and Q are partitions and Q is Pareto bigger than P. Then at least one of the following two conditions holds:*

a. $G(P, Q)$ contains at least one zero-to-positive arrow (or, equivalently, the transition from P to Q involves at least one zero-to-positive transfer).

b. $G(P, Q)$ contains at least one closed path consisting of positive-to-positive arrows (or, equivalently, the transition from P to Q involves at least one positive cyclic trade).

Proof: The proof is similar to the proof of Lemma 8.5. Suppose that P and Q are partitions and that Q is Pareto bigger than P. Assume, by way of contradiction, that $G(P, Q)$ contains no zero-to-positive arrows and no closed paths consisting of positive-to-positive arrows. We consider two cases.

Case 1: $G(P, Q)$ contains at least one positive-to-positive arrow. By assumption, $G(P, Q)$ contains no closed paths consisting of positive-to-positive arrows. It follows that for some i there is a positive-to-positive arrow that begins at vertex i but no positive-to-positive arrow that ends at vertex i. Since we have assumed that $G(P, Q)$ contains no zero-to-positive arrows, it follows that either no arrows of $G(P, Q)$ end at vertex i or else only (one or more) positive-to-zero arrows end at vertex i. In either case, in the transition from partition P to partition Q, Player i believes that he or she has given up a piece of cake of positive measure and has received no cake of positive measure. Thus, Player i is less happy with partition Q than with partition P. This contradicts the fact that partition Q is Pareto bigger than partition P.

Case 2: $G(P, Q)$ contains no positive-to-positive arrows. Since partition Q is Pareto bigger than partition P, we know that P and Q are not p-equivalent and hence the transition from P to Q involves at least one transfer that is positive-to-positive, positive-to-zero, or zero-to-positive. We have assumed that $G(P, Q)$ contains no positive-to-positive arrows and no zero-to-positive arrows. Hence, all arrows in $G(P, Q)$ are positive-to-zero arrows. Suppose such an arrow exists from Player i to Player j. Then, in the transition from partition P to partition Q, Player i believes that he or she has given up a piece of cake of positive measure and received no cake of positive measure. Thus, Player i is less happy with partition Q than with partition P. This contradicts the fact that partition Q is Pareto bigger than partition P.

Proof of Theorem 8.16: The proof is similar to the proof of Theorem 8.2. Suppose that partition P is non-wasteful and is not Pareto maximal and assume, by way of contradiction, that there does not exist a positive cyclic trade that yields a partition Pareto bigger than P. Notice that, since P is non-wasteful, $G(P, Q)$ contains no zero-to-positive transfers for any partition Q. The following is the appropriate adjustment to our present setting of the claim used in the proof of Theorem 8.2. As in that proof, the desired result will follow easily by repeatedly applying the claim. □

Claim Suppose Q is a partition that is Pareto bigger than P. Then there is a partition R satisfying that

a. R is p-equivalent to or Pareto bigger than Q (and hence is Pareto bigger than P),
b. $G(P, R) \subseteq G(P, Q)$, and
c. $G(P, R)$ contains at least one fewer closed path consisting of positive-to-positive arrows than does $G(P, Q)$.

The proof of the claim is almost identical to the proof of the claim found in the proof of Theorem 8.2, and we omit it. The only differences are the following:

- Lemma 8.17 is used in place of Lemma 8.3.
- Lemma 8.19 is used in place of Lemma 8.5. (If Q is Pareto bigger than P, then Lemma 8.19, together with our assumption that P is non-wasteful, implies that $G(P, Q)$ contains at least one closed path consisting of positive-to-positive arrows.)

Returning to the proof of the theorem, we let Q be a partition that is Pareto bigger than P. We repeatedly apply the claim to arrive at a partition R that is Pareto bigger than P and is such that $G(P, R)$ contains no closed paths consisting of positive-to-positive arrows. This contradicts Lemma 8.19 and our assumption that P is non-wasteful.

The definition of partition ratio in the previous section refers to sets of positive measure, and this is not meaningful in our present setting. The appropriate definition of partition ratio when absolute continuity is not assumed is the following.

Definition 8.20 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$. For distinct $i, j = 1, 2, \ldots, n$, we define pr_{ij}, the *ij partition ratio*, as follows:

$$\mathrm{pr}_{ij} = \sup \left\{ \frac{m_j(A)}{m_i(A)} : A \subseteq P_i \text{ andeither } m_i(A) > 0 \text{ or } m_j(A) > 0 \right\}$$

We shall say that pr_{ij} is undefined if $m_i(P_i) = 0$ and $m_j(P_i) = 0$.

The conditions given in the definition rule out the possibility that any term $\frac{m_j(A)}{m_i(A)}$ in the preceding set can be of the form $\frac{0}{0}$. However, in contrast with the situation in the previous section, there can be elements of this set of the form $\frac{0}{\kappa}$, where $\kappa > 0$, and elements of this set of the form $\frac{\kappa}{0}$, where $\kappa > 0$. We set $\frac{\kappa}{0}$, where $\kappa > 0$, equal to infinity. In this case, pr_{ij} is infinite. We recall that in the absolute continuity context a partition ratio is the supremum of a collection of (finite) numbers and is infinite if and only if this collection is unbounded. In the present context, a partition ratio can be infinite if either this occurs or if at least one term in the set over which the supremum is taken is infinite. This difference will be important in what follows, and we therefore introduce notation to make this distinction.

Notation 8.21 Let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition and fix distinct $i, j = 1, 2, \ldots, n$.

a. We write $pr_{ij} = \infty^*$ if and only if
   i. for each $A \subseteq P_i$, if $m_j(A) > 0$, then $m_i(A) > 0$ and
   ii. $\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i$ and either $m_i(A) > 0$ or $m_j(A) > 0\}$ is unbounded.
b. We write $pr_{ij} = \infty^{**}$ if and only if, for some $A \subseteq P_i$, $m_j(A) > 0$ and $m_i(A) = 0$.

After stating condition ai, the set in condition aii could have been written more simply as "$\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i$ and $m_i(A) > 0\}$." We note that, for any partition P and distinct $i, j = 1, 2, \ldots, n$, $\text{pr}_{ij} = \infty^{**}$ if and only if there is a zero-to-positive transfer from Player i to Player j. Then,

P is non-wasteful
if and only if
for all distinct $i, j = 1, 2, \ldots, n$, there is no zero-to-positive transfer from Player i to Player j
if and only if
for all distinct $i, j = 1, 2, \ldots, n$, $\text{pr}_{ij} \neq \infty^{**}$.

Definition 8.22 is the same as Definition 8.7, except for the description of the multiplication rules for our two different types of infinities.

Definition 8.22 Suppose that $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, n\}$ are distinct.

a. A sequence of the form $\langle \text{pr}_{i_t i_1}, \text{pr}_{i_1 i_2}, \ldots, \text{pr}_{i_{t-2} i_{t-1}}, \text{pr}_{i_{t-1} i_t} \rangle$, where each such $\text{pr}_{i_j i_k}$ is defined, is called a *cyclic sequence*.
b. $CS(k)$ denotes the set of all cyclic sequences of length k.
c. CS denotes the set of all cyclic sequences.

d. If $\varphi = \langle \mathrm{pr}_{i_t i_1}, \mathrm{pr}_{i_1 i_2}, \ldots, \mathrm{pr}_{i_{t-2} i_{t-1}}, \mathrm{pr}_{i_{t-1} i_t} \rangle \in \mathrm{CS}$, then the *cyclic product* of φ, denoted by $CP(\varphi)$, is the product $\mathrm{pr}_{i_t i_1} \mathrm{pr}_{i_1 i_2} \ldots \mathrm{pr}_{i_{t-2} i_{t-1}} \mathrm{pr}_{i_{t-1} i_t}$ where we set
   i. $(0)(\infty^*) = (\infty^*)(0) = 0$,
   ii. $(0)(\infty^{**}) = (\infty^{**})(0) = \infty^{**}$,
   iii. (positive number)$(\infty^*) = (\infty^*)$(positive number) $= \infty^*$,
   iv. (positive number)$(\infty^{**}) = (\infty^{**})$(positive number) $= \infty^{**}$,
   v. $(\infty^*)(\infty^*) = \infty^*$,
   vi. $(\infty^*)(\infty^{**}) = (\infty^{**})(\infty^*) = \infty^{**}$, and
   vii. $(\infty^{**})(\infty^{**}) = \infty^{**}$.

It follows from the definition that, for any $\varphi \in \mathrm{CS}$, $\mathrm{CP}(\varphi) \geq 0$ and we can have $\mathrm{CP}(\varphi) = \infty^*$ or $\mathrm{CP}(\varphi) = \infty^{**}$.

We give some informal perspective on our conventions for arithmetic involving ∞^* and ∞^{**}. We first note that ∞^{**} corresponds to an achieved infinity, rather than a limit. Or, in other words, $\mathrm{pr}_{ij} = \infty^{**}$ corresponds to the existence of a zero-to-positive transfer from Player i to Player j. The existence of such a transfer implies that the given partition is not Pareto maximal, and thus we define any product involving ∞^{**} to be ∞^{**}, to be consistent with our upcoming characterization theorem, Theorem 8.24. On the other hand, ∞^* is not an achieved infinity, but a limit. We can think of $\mathrm{pr}_{ij} = \infty^*$ as indicating the existence of transfers from Player i to Player j such that the value of the piece of cake received by Player j (according to m_j) divided by the value of the piece of cake given up by Player i (according to m_i) is as large as we want, but still finite. This as-large-as-we-want-but-still-finite situation, when combined with a partition ratio that is zero, should give us zero, and when combined with a positive number, should give us "as large as we want but still finite," i.e., ∞^*.

Our characterization of Pareto maximality is Theorem 8.24. It will follow easily from Theorem 8.16 and the following lemma, which is our adjustment of Lemma 8.8.

Lemma 8.23 *Let P be a partition that is non-wasteful and fix any $k = 2, 3, \ldots, n$. There exists $\varphi \in \mathrm{CS}(k)$ such that $\mathrm{CP}(\varphi) > 1$ if and only if there is a positive cyclic trade of length k that produces a partition Pareto bigger than P.*

Proof: Fix a partition P that is non-wasteful and any $k = 2, 3, \ldots, n$.

For the forward direction, suppose that for some $\varphi \in \mathrm{CS}(k)$, $\mathrm{CP}(\varphi) > 1$. The proof is precisely as in the forward direction of the proof of Lemma 8.8. In that proof we noted that some of the partition ratios might be equal to ∞, and that this causes no difficulty. Now we note that, since P is non-wasteful, no partition ratio is equal to ∞^{**}. However, some of the partition ratios might equal ∞^*.

As in the case of partition ratios equal to ∞ in the proof of Lemma 8.8, this causes no difficulty.

The reverse direction is precisely as in the proof of the reverse direction of Lemma 8.8. □

The statement of our characterization theorem is the same as Theorem 8.9, where we assumed that the measures were absolutely continuous with respect to each other.

Theorem 8.24 *A partition P is Pareto maximal if, and only if, for every $\varphi \in CS$, $CP(\varphi) \leq 1$.*

Proof: The proof is similar to the proof of Theorem 8.9.

For the forward direction, we suppose that P is a partition and that for some $\varphi \in \text{CS}$, $\text{CP}(\varphi) > 1$. We must show that P is not Pareto maximal. We consider two cases.

Case 1: P is wasteful. Then there exists a zero-to-positive transfer from some Player i to some Player j. The partition that results from doing just this one transfer is Pareto bigger than P. Hence, P is not Pareto maximal.

Case 2: P is non-wasteful. By Lemma 8.23, there is a positive cyclic trade that produces a partition Pareto bigger than P. Hence, P is not Pareto maximal.

For the reverse direction, suppose that P is not Pareto maximal. We must show that, for some $\varphi \in \text{CS}$, $\text{CP}(\varphi) > 1$. We consider the same two cases as before.

Case 1: P is wasteful. This implies that, for some i and j, $\text{pr}_{ij} = \infty^{**}$. Let φ be any cyclic sequence that includes pr_{ij}. Our arithmetic rules (see Definition 8.22) imply that $\text{CP}(\varphi) = \infty^{**} > 1$.

Case 2: P is non-wasteful. Theorem 8.16 tells us that there is a positive cyclic trade that produces a partition Pareto bigger than P. It follows from Lemma 8.23 that there exists $\varphi \in \text{CS}$ such that $\text{CP}(\varphi) > 1$.

This completes the proof of the theorem. □

Next, we present an application. We use Theorem 8.24 first to show that a certain partition is Pareto maximal and then to show that another partition is not Pareto maximal.

Example 8.25 Let C be the interval $[0, 3)$ on the real number line and let m_L be Lebesgue measure on this set. Suppose that there are three players, Player 1, Player 2, and Player 3, with corresponding measures m_1, m_2, and m_3, respectively, defined as follows: for any $A \subseteq C$,

$$\begin{aligned}
m_1(A) &= .7m_L(A \cap [0, 1)) + .3m_L(A \cap [1, 2)) \\
m_2(A) &= .2m_L(A \cap [0, 1)) + .8m_L(A \cap [1, 2)) \\
m_3(A) &= .3m_L(A \cap [0, 1)) + .1m_L(A \cap [1, 2)) + .6m_L(A \cap [2, 3))
\end{aligned}$$

It is easy to verify that $m_1(C) = m_2(C) = m_3(C) = 1$, and so m_1, m_2, and m_3 are measures on C. These measures are not absolutely continuous with respect to each other since, for example, $m_1([2, 3)) = 0$ but $m_3([2, 3)) = .6$.

Consider the partition $P = \langle [0, 1), [1, 2), [2, 3) \rangle$. We will use Theorem 8.24 to show that P is Pareto maximal. The partition ratios are as follows:

$$\begin{aligned}
\text{pr}_{12} &= \sup \left\{ \frac{m_2(A)}{m_1(A)} : A \subseteq [0, 1) \text{ and either } m_1(A) \neq 0 \text{ or } m_2(A) \neq 0 \right\} \\
&= \frac{.2}{.7} = \frac{2}{7} \\
\text{pr}_{21} &= \sup \left\{ \frac{m_1(A)}{m_2(A)} : A \subseteq [1, 2) \text{ and either } m_2(A) \neq 0 \text{ or } m_1(A) \neq 0 \right\} \\
&= \frac{.3}{.8} = \frac{3}{8} \\
\text{pr}_{13} &= \sup \left\{ \frac{m_3(A)}{m_1(A)} : A \subseteq [0, 1) \text{ and either } m_1(A) \neq 0 \text{ or } m_3(A) \neq 0 \right\} \\
&= \frac{.3}{.7} = \frac{3}{7} \\
\text{pr}_{31} &= \sup \left\{ \frac{m_1(A)}{m_3(A)} : A \subseteq [2, 3) \text{ and either } m_3(A) \neq 0 \text{ or } m_1(A) \neq 0 \right\} \\
&= \frac{0}{.6} = 0 \\
\text{pr}_{23} &= \sup \left\{ \frac{m_3(A)}{m_2(A)} : A \subseteq [1, 2) \text{ and either } m_2(A) \neq 0 \text{ or } m_3(A) \neq 0 \right\} \\
&= \frac{.1}{.8} = \frac{1}{8} \\
\text{pr}_{32} &= \sup \left\{ \frac{m_2(A)}{m_3(A)} : A \subseteq [2, 3) \text{ and either } m_3(A) \neq 0 \text{ or } m_2(A) \neq 0 \right\} \\
&= \frac{0}{.6} = 0
\end{aligned}$$

Next, we compute all cyclic products:

$$\begin{aligned}
\mathrm{pr}_{12}\mathrm{pr}_{21} &= \left(\frac{2}{7}\right)\left(\frac{3}{8}\right) = \frac{3}{28} \\
\mathrm{pr}_{13}\mathrm{pr}_{31} &= \left(\frac{3}{7}\right)(0) = 0 \\
\mathrm{pr}_{23}\mathrm{pr}_{32} &= \left(\frac{1}{8}\right)(0) = 0 \\
\mathrm{pr}_{12}\mathrm{pr}_{23}\mathrm{pr}_{31} &= \left(\frac{2}{7}\right)\left(\frac{1}{8}\right)(0) = 0 \\
\mathrm{pr}_{32}\mathrm{pr}_{21}\mathrm{pr}_{13} &= (0)\left(\frac{3}{8}\right)\left(\frac{3}{7}\right) = 0
\end{aligned}$$

Since all cyclic product are less than one, Theorem 8.24 implies that P is Pareto maximal. We observe that every cyclic product involving Player 3 is equal to zero. This corresponds to the fact that Player 1 and Player 2 each put value zero on Player 3's piece, [2, 3).

With cake C and measures m_1, m_2, and m_3, as before, we now consider the partition $Q = \langle [1, 2), [2, 3), [0, 1), \rangle$. The corresponding partition ratios are as follows:

$$\begin{aligned}
\mathrm{pr}_{12} &= \sup\left\{\frac{m_2(A)}{m_1(A)} : A \subseteq [1, 2) \text{ and either } m_1(A) \neq 0 \text{ or } m_2(A) \neq 0\right\} \\
&= \frac{.8}{.3} = \frac{8}{3} \\
\mathrm{pr}_{21} &= \sup\left\{\frac{m_1(A)}{m_2(A)} : A \subseteq [2, 3) \text{ andeither } m_2(A) \neq 0 \text{ or } m_1(A) \neq 0\right\} \\
&\quad \text{undefined} \\
\mathrm{pr}_{13} &= \sup\left\{\frac{m_3(A)}{m_1(A)} : A \subseteq [1, 2) \text{ and either } m_1(A) \neq 0 \text{ or } m_3(A) \neq 0\right\} \\
&= \frac{.1}{.3} = \frac{1}{3} \\
\mathrm{pr}_{31} &= \sup\left\{\frac{m_1(A)}{m_3(A)} : A \subseteq [0, 1) \text{ and either } m_3(A) \neq 0 \text{ or } m_1(A) \neq 0\right\} \\
&= \frac{.7}{.3} = \frac{7}{3} \\
\mathrm{pr}_{23} &= \sup\left\{\frac{m_3(A)}{m_2(A)} : A \subseteq [2, 3) \text{ and either } m_2(A) \neq 0 \text{ or } m_3(A) \neq 0\right\} \\
&= \frac{.6}{0} = \infty^{**} \\
\mathrm{pr}_{32} &= \sup\left\{\frac{m_2(A)}{m_3(A)} : A \subseteq [0, 1) \text{ and either } m_3(A) \neq 0 \text{ or } m_2(A) \neq 0\right\} \\
&= \frac{.2}{.3} = \frac{2}{3}
\end{aligned}$$

Since, for example, $\text{pr}_{23}\text{pr}_{32} = (\infty^{**})(\frac{2}{3}) = \infty^{**}$, Theorem 8.24 implies that Q is not Pareto maximal.

In this example, and in our example from the last section (Example 8.10), all sets over which suprema were taken were singletons. We chose these examples for simplicity. In Chapter 13, we shall present some perspective on partition ratios when these sets are not singletons.

We close this section by presenting chores versions of the main ideas of this section. The definitions of zero-to-zero, positive-to-positive, positive-to-zero, and zero-to-positive transfers, and of positive cyclic trade, as in Definitions 7.22 and 8.15, remain appropriate for our present setting. The chores version of Theorem 8.16 is the following. The proof is entirely analogous and we omit it.

Theorem 8.26 *If P is a partition that is c-non-wasteful and is not Pareto minimal, then there is a positive cyclic trade that produces a partition Pareto smaller than P.*

Our definition of the chores partition ratios for the non-absolute continuity context is the natural combination of Definition 8.12 (chores partition ratios with absolute continuity) and Definition 8.20 (partition ratios without absolute continuity).

Definition 8.27 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$. For distinct $i, j = 1, 2, \ldots, n$, we define qr_{ij}, the ij *chores partition ratio*, as follows:

$$\text{qr}_{ij} = \inf \left\{ \frac{m_j(A)}{m_i(A)} : A \subseteq P_i \text{ and either } m_i(A) > 0 \text{ or } m_j(A) > 0 \right\}$$

As we did for pr_{ij}, we shall say that qr_{ij} is undefined if $m_i(P_i) = 0$ and $m_j(P_i) = 0$.

In contrast with the standard context where there were two different ways that a pr_{ij} could be infinite, there is only one way that a qr_{ij} could be infinite. For some i and j, qr_{ij} is infinite if and only if $m_i(P_i) = 0$ and $m_j(P_i) > 0$. We shall simply write "$\text{qr}_{ij} = \infty$" in this case, and shall not need our "∞^{*}" or "∞^{**}" notation. We note that, for any i and j, $0 \leq \text{qr}_{ij} \leq \infty$.

However, we do need to make a distinction between two different ways that a qr_{ij} can be equal to zero. This distinction is analogous to the distinction we made between $\text{pr}_{ij} = \infty^{*}$ and $\text{pr}_{ij} = \infty^{**}$. That distinction was based on the fact that if the set $\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i \text{ and either } m_i(A) > 0 \text{ or } m_j(A) > 0\}$ has an infinite supremum, it might or might not have an element that is infinite. Similarly, if the set $\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i \text{ and either } m_i(A) > 0 \text{ or } m_j(A) > 0\}$ has an infimum that is zero, it might or might not have an element that is zero. Thus, we make the following distinction.

Notation 8.28 Let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition and fix distinct $i, j = 1, 2, \ldots, n$.

a. We write $qr_{ij} = 0^*$ if and only if
   i. for each $A \subseteq P_i$, if $m_i(A) > 0$, then $m_j(A) > 0$ and
   ii. $\inf\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i$ andeither $m_i(A) > 0$ or $m_j(A) > 0\} = 0$.
b. We write $qr_{ij} = 0^{**}$ if and only if, for some $A \subseteq P_i$, $m_i(A) > 0$ and $m_j(A) = 0$.

In analogy with what was the case for Notation 8C, we note that, after stating condition ai, the set in condition aii could have been written more simply as $\inf\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i$ and $m_j(A) > 0\}$. Also, we observe that, for any partition P and distinct $i, j = 1, 2, \ldots, n$, $\text{qr}_{ij} = 0^{**}$ if and only if there is a positive-to-zero transfer from Player i to Player j. Then,

P is c-non-wasteful
if and only if
for all distinct $i, j = 1, 2, \ldots, n$, there is no positive-to-zero transfer from Player i to Player j
if and only if
for all distinct $i, j = 1, 2, \ldots, n$, $\text{qr}_{ij} \neq 0^{**}$.

Next, we wish to define chores cyclic sequences and chores cyclic products. In analogy with what we did in Definition 8.22, where we defined the relevant arithmetic for ∞^* and for ∞^{**}, we make the following definition.

Definition 8.29 Suppose that $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, n\}$ are distinct.

a. A sequence of the form $\langle \text{qr}_{i_t i_1}, \text{qr}_{i_1 i_2}, \ldots, \text{qr}_{i_{t-2} i_{t-1}}, \text{qr}_{i_{t-1} i_t} \rangle$, where each such $\text{qr}_{i_j i_k}$ is defined, is called a *chores cyclic sequence*.
b. *CCS*(k) denotes the set of all chores cyclic sequences of length k.
c. *CCS* denotes the set of all chores cyclic sequences.
d. If $\varphi = \langle \text{qr}_{i_t i_1}, \text{qr}_{i_1 i_2}, \ldots, \text{qr}_{i_{t-2} i_{t-1}}, \text{qr}_{i_{t-1} i_t} \rangle \in \text{CCS}$, then the *chores cyclic product* of φ, denoted by *CCP*(φ), is the product $\text{qr}_{i_t i_1} \text{qr}_{i_1 i_2} \ldots \text{qr}_{i_{t-2} i_{t-1}} \text{qr}_{i_{t-1} i_t}$ where we set
   i. $(0^*)(\infty) = (\infty)(0^*) = \infty$,
   ii. $(0^{**})(\infty) = (\infty)(0^{**}) = 0^{**}$,
   iii. (positive number)$(0^*) = (0^*)$(positivenumber) $= 0^*$,
   iv. (positive number)$(0^{**}) = (0^{**})$(positive number) $= 0^{**}$,
   v. $(0^*)(0^*) = 0^*$,

vi. $(0^{**})(0^{*}) = (0^{*})(0^{**}) = 0^{**}$, and
vii. $(0^{**})(0^{**}) = 0^{**}$.

It follows from the definition that, for any $\varphi \in \text{CCS}$, $\text{CCP}(\varphi) \leq \infty$ and we can have $\text{CCP}(\varphi) = 0^{*}$ or $\text{CCP}(\varphi) = 0^{**}$. The motivation for our conventions for arithmetic involving 0^{*} and 0^{**} is analogous to the motivation that we discussed for arithmetic involving ∞^{*} and ∞^{**}.

The chores version of Lemma 8.23 is the following. The proof is analogous and we omit it.

Lemma 8.30 *Let P be a partition that is c-non-wasteful and fix any $k = 2, 3, \ldots, n$. There exists $\varphi \in \text{CCS}(k)$ such that $\text{CCP}(\varphi) < 1$ if and only if there is a positive cyclic trade of length k that produces a partition Pareto smaller than P.*

The statement of our characterization theorem is the same as Theorem 8.14, where we assumed that the measures were absolutely continuous with respect to each other.

Theorem 8.31 *A partition P is Pareto minimal if and only if, for every $\varphi \in \text{CCS}$, $\text{CCP}(\varphi) \geq 1$.*

The proof uses Theorem 8.26 and Lemma 8.30 in the same way that the proof of Theorem 8.24 used Theorem 8.16 and Lemma 8.23. We omit the details.

9

Geometric Object #2

The Radon–Nikodym Set (RNS)

In this chapter, we introduce the second of the two geometric objects that we associate with cake division. We call this object the Radon–Nikodym Set, or RNS. For our first geometric object, the IPS (or, more generally, the FIPS), we were interested in a geometric perspective on the set of all partitions of C. Our present goal is quite different. When we introduced the IPS, we started with the cake C, we considered the set of all partitions of C, and then we formed a geometric object, the IPS (or the FIPS), that contains useful information about this set. Now, we start with the cake C, we form a geometric object, the RNS, and then we use this new geometric object to construct partitions having desired properties. In the next chapter, we will use the RNS to obtain a new characterization of Pareto maximality and Pareto minimality. In Chapter 12, we will study the relationship between the IPS and the RNS.

In Section 9A, we assume that the measures are absolutely continuous with respect to each other. In Section 9B, we consider what happens when absolute continuity fails. Much of the material in this chapter is attributable to D. Weller ([43]).

9A. The RNS

We have made no geometric assumptions about the cake C. C is simply a set on which a σ-algebra and measures have been defined, and we have not assumed that it exists in some $\mathbf{R}^n$ or in any other geometric framework. A useful perspective on the new geometric object we are about to define is that what we are going to do is to take the cake apart and reassemble it in a natural geometric structure so that the location of each "bit" of cake in this structure conveys useful information. In particular, the location of a "bit" of cake will correspond to the relative worth that the different players assign to it.

To further develop this intuition, we present a perspective attributable to E. Akin (see [1]). We imagine that all of the players are seated around some "table" and the cake is taken apart and reassembled on this table in such a way that each "bit" of cake is placed close to players that value it highly and far from players that do not value it highly. More specifically, if, for example, a particular "bit" of cake is valued three times as much by Player i as by Player j, then this "bit" of cake will be placed on the "table" three times as far from Player j as from Player i.

In moving toward making these ideas precise, let us examine more closely two words that were obviously just used quite informally: "bit" and "table." We shall be taking the cake apart bit by bit and reassembling it. Thus, each "bit" of cake is a single point of cake. But what does it mean to compare the value that different players assign to a bit of cake, since (by our assumption that all measures are non-atomic) all players give measure zero to a single point of cake? This question will be answered by considering a density function for each measure, rather than the measure itself. We shall define these density functions shortly, using the Radon–Nikodym theorem.

The most convenient "table" to have our players sit at is the simplex (of the appropriate size). We may imagine each player sitting at a vertex of the simplex, with the reassembled bits of cake most desirable to each player placed close to that vertex and the least desirable bits of cake placed far away.

We begin to make these notions precise by defining a new measure $\mu = m_1 + m_2 + \cdots + m_n$. Notice that each m_i is absolutely continuous with respect to μ. This is true regardless of whether or not the m_i are absolutely continuous with respect to each other. The measure μ is absolutely continuous with respect to every m_i if and only if the m_i are all absolutely continuous with respect to each other. Hence, in our present setting in which we are assuming that $m_1, m_2, \ldots, m_n$ are absolutely continuous with respect to each other, $\mu, m_1, m_2, \ldots, m_n$ are all absolutely continuous with respect to each other. Thus, in using the terms "positive measure," "measure zero," and "almost every $a \in C$," we need not specify to which measure we are referring. Notice that (assuming $n \geq 2$) $\mu(C) > 1$ and, hence, μ is not a probability measure.

The Radon–Nikodym theorem (see, for example [38]) tells us that, for each $i = 1, 2, \ldots, n$, there is a function f_i from the cake C to the non-negative real numbers so that, for any $A \subseteq C$, $m_i(A) = \int_A f_i \, d\mu$. Each such f_i is called the *Radon–Nikodym derivative* of the m_i with respect to μ. Such functions are also often called *density functions*, and we shall refer to them as such. (The measure μ will always be as defined above, and hence there will be no ambiguity in using the term "density function" without referring to μ.)

Lemma 9.1

a. *For almost every* $a \in C$, $f_1(a) + f_2(a) + \cdots + f_n(a) = 1$.
b. *For any* $i = 1, 2, \ldots, n$ *and almost every* $a \in C$, $0 < f_i(a) < 1$.

Proof: For part a suppose, by way of contradiction, that $\{a \in C : f_1(a) + f_2(a) + \cdots + f_n(a) \neq 1\}$ has positive measure. Let $B^< = \{a \in C : f_1(a) + f_2(a) + \cdots + f_n(a) < 1\}$ and let $B^> = \{a \in C : f_1(a) + f_2(a) + \cdots + f_n(a) > 1\}$. Then, either $B^<$ or $B^>$ has positive measure.

If $B^<$ has positive measure, then

$$\begin{aligned}\mu(B^<) &= m_1(B^<) + m_2(B^<) + \cdots + m_n(B^<)\\ &= \int_{B^<} f_1\, d\mu + \int_{B^<} f_2\, d\mu + \cdots + \int_{B^<} f_n\, d\mu\\ &= \int_{B^<} (f_1 + f_2 + \cdots + f_n) d\mu < \int_{B^<} 1\, d\mu = \mu(B^<).\end{aligned}$$

This is a contradiction. The proof if $B^>$ has positive measure is similar. Hence, for almost every $a \in C$, $f_1(a) + f_2(a) + \cdots + f_n(a) = 1$.

For part b, fix any $i = 1, 2, \ldots, n$. We first show that, for almost every $a \in C$, $0 < f_i(a)$. Let $D^\leq = \{a \in C : f_i(a) \leq 0\}$ and suppose, by way of contradiction, that $D^\leq$ has positive measure. Then

$$0 < m_i(D^\leq) = \int_{D^\leq} f_i\, d\mu \leq 0.$$

This is a contradiction. Hence, $D^\leq$ has measure zero and so $0 < f_i(a)$ for almost every $a \in C$. This and part a imply that, for almost every $a \in C$, $f_i(a) < 1$. □

The lemma justifies the following:

> By redefining some or all of the f_i on a set of measure zero, if necessary, we shall assume from now on that for each $i = 1, 2, \ldots, n$ and *every* $a \in C$,
> $f_1(a) + f_2(a) + \cdots + f_n(a) = 1$ and $0 < f_i(a) < 1$.

This will simplify our analysis.

It follows from Lemma 9.1 and the preceding assumptions that, for every $a \in C$, $(f_1(a), f_2(a), \ldots, f_n(a))$ is a point in the interior of the simplex. We are now ready to define the RNS.

Definition 9.2

a. For each $a \in C$, let $f(a) = (f_1(a), f_2(a), \ldots, f_n(a))$.
b. The *Radon–Nikodym Set*, or *RNS*, is $\{f(a) : a \in C\}$.

By the preceding discussion, we see that the RNS is a subset of the interior of the simplex.

We shall frequently identify points of C with their images under f. So, for example, if we say that "almost every point of the RNS lies on the line ℓ," we mean that, for almost every $a \in C$, $f(a) \in \ell$. Also, we shall apply the function f both to elements of C and to subsets of C.

To connect the definition of the RNS with our previous informal discussion, we may think of the simplex as the table. Player i's vertex is the vertex of the simplex with ith coordinate one and zeros in all other positions. A point of the RNS corresponds to one or more bits of cake and the distance from a point of the RNS to a given player's vertex tells us how much that player values (the bit of cake corresponding to) that point, relative to the other players (where "closer" means "more valued").

Next, we consider examples to illustrate some of the possibilities for the RNS. In Example 9.3, we examine an RNS for two, three, and four players. In each of Examples 9.4, 9.5, 9.6, and 9.7, we consider an RNS for three players. When there are three players, we think of the points (1, 0, 0), (0, 1, 0), and (0, 0, 1) as being the vertices of Player 1, Player 2, and Player 3, respectively. A similar correspondence holds when there are other numbers of players. In these examples, we give just the RNSs and not the corresponding cake and measures. In the next section, we shall give examples in which we start by defining a cake and measures and then give the corresponding RNS.

Example 9.3 A cake with all measures identical. This situation is illustrated in Figure 9.1. If all of the measures are identical, then all of the density functions are identical, except possibly on a set of measure zero. We assume that these functions have been redefined on a set of measure zero, if necessary, so that they are identical on all of C. This, together with part a of Lemma 9.1, tells us that each density function is a constant function with value $\frac{1}{n}$ (where n is the number of players). This implies that the RNS consists of a single point. For two players, this is the point $(\frac{1}{2}, \frac{1}{2})$; for three players, this is the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; and for four players, this is the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. These three situations are illustrated in Figures 9.1a, 9.1b, and 9.1c.

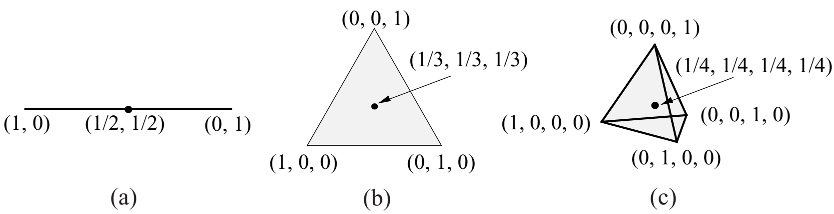


Figure 9.1

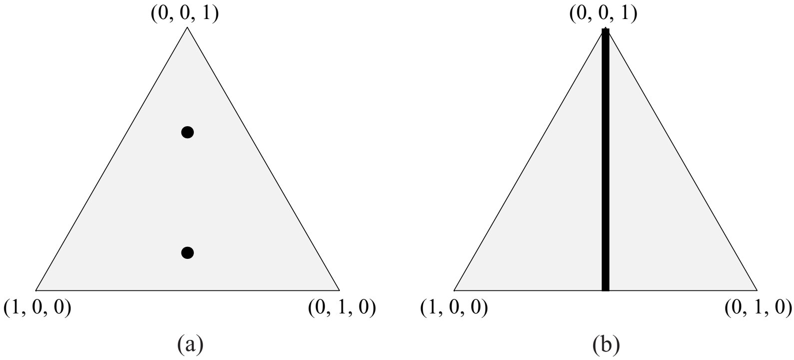


Figure 9.2

Let us connect this with our previous informal discussion. Since the measures are identical, any bit of cake is equally valued by all players and, hence, any bit of cake corresponds to a point of the RNS that is equidistant from the vertex of each player. For two players this point is $(\frac{1}{2}, \frac{1}{2})$, for three players this point is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and for four players this point is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. We also note that for two players the RNS is a one-dimensional object in $\mathbf{R}^2$, for three players the RNS is a two-dimensional object in $\mathbf{R}^3$, and for four players the RNS is a three-dimensional object in $\mathbf{R}^4$.

Example 9.4 There are three players and $m_1 = m_2$. This assumption implies that, for almost every $a \in C$, $f_1(a) = f_2(a)$. We assume that this holds for *every* $a \in C$. Then every point of the RNS lies on the line $x = y$. Of course, one possibility is that $m_1 = m_2 = m_3$, as in the previous example (and as illustrated in Figure 9.1b). Two other possibilities are illustrated in Figure 9.2.

In Figure 9.2a, the RNS consists of two points, $(\frac{3}{7}, \frac{3}{7}, \frac{1}{7})$ and $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$. This corresponds to a cake C that can be partitioned into two pieces, A and B, such that

- Player 1 and Player 2 value each bit of cake equally,
- Player 1 and Player 2 each value each bit in A three times as much as does Player 3, and
- Player 3 values each bit in B three times and much as do Player 1 and Player 2.

In Figure 9.2b, the RNS is smoothly spread out along the part of the line $x = y$ that lies inside the simplex. This corresponds to a continuous distribution in the ratio of values that Player 1 and Player 2 assign to bits of C, as compared to the values assigned to bits of C by Player 3.

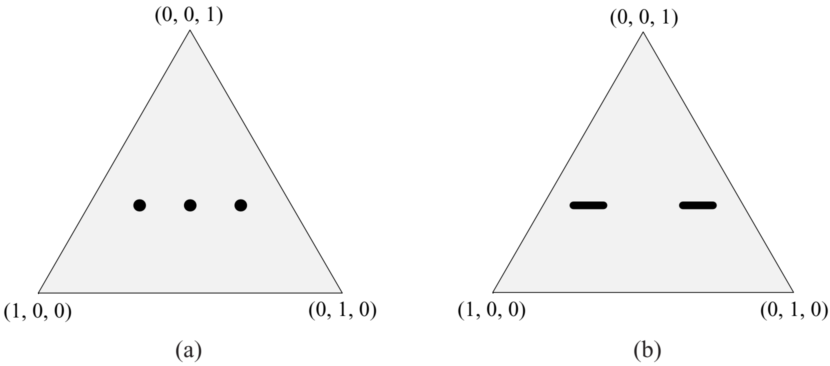


Figure 9.3

Example 9.5 There are three players and $m_3 = \frac{1}{2}m_1 + \frac{1}{2}m_2$. This assumption implies that, for almost every $a \in C$, $f_3(a) = \frac{1}{2}f_1(a) + \frac{1}{2}f_2(a)$. We assume that this holds for *every* $a \in C$. We know that, for every a, $f_1(a) + f_2(a) + f_3(a) = 1$. These two equations imply that, for every $a \in C$, $f_3(a) = 1 - (f_1(a) + f_2(a)) = 1 - 2f_3(a)$. Thus, for every such $a, 3f_3(a) = 1$ and so $f_3(a) = \frac{1}{3}$. This tells us that every point of the RNS lies on the line $z = \frac{1}{3}$. Of course, one possibility is that $m_1 = m_2 = m_3$, as given in Example 9.3. Two other possibilities are illustrated in Figure 9.3.

The situation pictured in Figures 9.3a and 9.3b is similar to the situation described in the previous example and pictured in Figures 9.2a and 9.2b, respectively. Of course, in our present example, the RNS lies on the line $z = \frac{1}{3}$ instead of on the line $x = y$, as in the previous example. In Figure 9.3a, we have chosen to have the RNS consist of three points instead of two. In Figure 9.3b, the continuous distribution involves the ratio of values that Player 1 assigns to bits of cake compared to the values assigned to bits of cake by Player 2. (Player 3 assigns the same value to every bit of cake.) Also, in this case, we have chosen to have the range of this ratio extend only over the two line segments in the figure.

Example 9.6 There are three players and $m_3 = \frac{1}{3}m_1 + \frac{2}{3}m_2$. This assumption implies that, for almost every $a \in C$, $f_3(a) = \frac{1}{3}f_1(a) + \frac{2}{3}f_2(a)$. We assume that this holds for *every* $a \in C$. Since, for every $a \in C$, $f_1(a) + f_2(a) + f_3(a) = 1$, it follows that, for every such a, $1 = f_1(a) + f_2(a) + f_3(a) = f_1(a) + f_2(a) + \frac{1}{3}f_1(a) + \frac{2}{3}f_2(a) = \frac{4}{3}f_1(a) + \frac{5}{3}f_2(a)$. This tells us that all of the points of the RNS lie on the line $\frac{4}{3}x + \frac{5}{3}y = 1$. Two possibilities are illustrated in Figure 9.4.

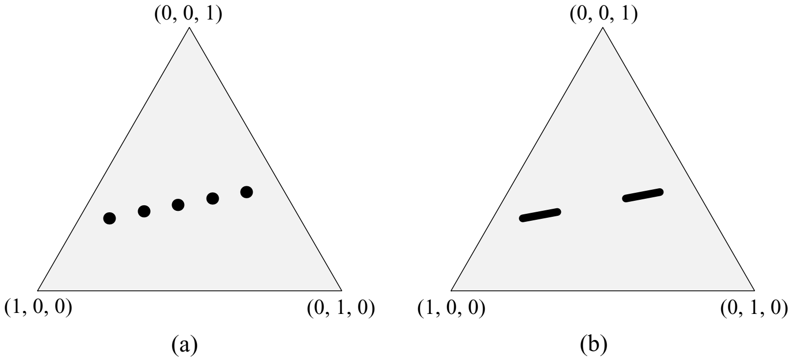


Figure 9.4

Example 9.7 There are three players and there is a partition $\langle A_1, A_2, A_3\rangle$ of C such that, for distinct $i, j = 1, 2, 3$ and any $B \subseteq A_i$, $\frac{m_i(B)}{m_j(B)} > 17$. (We do not assume that any particular player receives any particular piece of cake. We simply assert that such a partition exists, and we consider what this says about the RNS.) This situation can be described as the near failure of absolute continuity, since A_1 and every subset of A_1 has far greater value to Player 1 than to Player 2 or Player 3. An analogous statement holds for A_2 and A_3 and subsets of these sets. In other words, the measures "almost" concentrate on disjoint sets. In this case, it is not hard to see that, for distinct $i, j = 1, 2, 3$, $\frac{f_i(a)}{f_j(a)} > 17$ for almost every $a \in A_i$. We assume that this holds for *every* $a \in C$.

Fix $a \in A_1$ and set $f(a) = (a_1, a_2, a_3)$. Then $\frac{a_1}{a_2} > 17$, $\frac{a_1}{a_3} > 17$, and $a_1 + a_2 + a_3 = 1$. This implies that $2a_1 > 17a_2 + 17a_3 = 17(a_2 + a_3) = 17(1 - a_1) = 17 - 17a_1$, and thus $a_1 > \frac{17}{19}$. Also, we have $17a_2 < a_1 \le a_1 + a_3 = 1 - a_2$, and thus $a_2 < \frac{1}{18}$. Similarly, $a_3 < \frac{1}{18}$. Analogous arguments apply to points chosen from A_2 and A_3. This situation is illustrated in Figure 9.5. For each $i = 1, 2, 3$, $\{f(a) : a \in A_i\}$ is very close to the vertex associated with Player i.

In the last example, in contrast with the earlier examples, the measures are linearly independent. To see this, suppose, by way of contradiction, that this is not the case. The given condition certainly implies that no two of the measures are equal. Then one of the measures must be a positive weighted average of the other two. Assume, by renumbering if necessary, that $m_3 = \alpha_1 m_1 + \alpha_2 m_2$, where α_1 and α_2 are both positive. Then

$$\alpha_1 + \alpha_2 = \alpha_1 m_1(C) + \alpha_2 m_2(C) = m_3(C) = 1$$

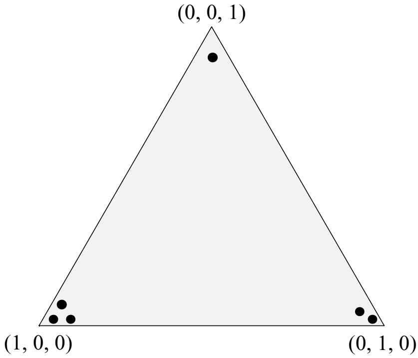


Figure 9.5

and, hence,

$$m_3(P_3) = \alpha_1 m_1(P_3) + \alpha_2 m_2(P_3) < \alpha_1 \frac{m_3(P_3)}{17} + \alpha_2 \frac{m_3(P_3)}{17}$$
$$= \frac{m_3(P_3)}{17}(\alpha_1 + \alpha_2) = \frac{m_3(P_3)}{17}.$$

This is a contradiction and, thus, the measures are linearly independent.

Also, in the last example, the RNS does not lie on a line, as in the earlier examples. These observations illustrate the following fact: there is a line containing almost all of the points of the RNS if and only if the measures are linearly dependent. This fact about the three-player context is a special case of the Theorem 9.8. The theorem uses the notion of dimension of a subset of $\mathbf{R}^n$. Although the intuitive notion of dimension is quite familiar, a precise definition is non-trivial, and we shall not give one. For our purposes, we shall only require three facts. For any $G \subseteq \mathbf{R}^n$, let $dim(G)$ denote the dimension of G. Then, for any $G \subseteq \mathbf{R}^n$,

a. $\dim(G) \leq n - 1$ if and only if $G \subseteq H$ for some hyperplane H in $\mathbf{R}^n$;
b. $\dim(G) \leq n - 2$ if and only if $G \subseteq H_1 \cap H_2$ for some two distinct hyperplanes H_1 and H_2 $\mathbf{R}^n$; and
c. if $\dim(G) \leq n - 2$ and $p \in \mathbf{R}^n$, then $\dim(G \cup \{p\}) \leq n - 1$.

Theorem 9.8 *There is a lower-dimensional subset of the simplex that contains almost all of the RNS if and only if the measures are linearly dependent.*

To clarify any confusion about the terminology "lower-dimensional subset of the simplex," we recall that the simplex in $\mathbf{R}^n$ has dimension $n - 1$. Then, a "lower-dimensional subset of the simplex" is a subset of the simplex that has dimension less than $n - 1$. In Examples 9.4, 9.5, 9.6, and 9.7, there are three players and hence the simplex is the two-simplex in $\mathbf{R}^3$, which is a two-dimensional subset of $\mathbf{R}^3$. In each of Examples 9.4, 9.5, and 9.6, the measures are linearly dependent, and in each of these we saw that there is a one-dimensional subset of the simplex (i.e., a line segment in the standard two-simplex in $\mathbf{R}^3$) that contains almost all of the points of the RNS. On the other hand, in Example 9.7, the measures are linearly independent and there is no one-dimensional subset of the simplex that contains almost all of the points of the RNS.

Proof of Theorem 9.8: For the forward direction, we assume that there is a lower-dimensional subset G of the simplex that contains almost all of the RNS. Then $\dim(G) \leq n - 2$. By the preceding condition c, $\dim(G \cup \{(0, 0, \ldots, 0)\}) \leq n - 1$. Then, by condition a, $G \cup \{(0, 0, \ldots, 0)\}) \subseteq H$ for some hyperplane H in $\mathbf{R}^n$. Suppose that H is given by $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = k$ for some constants $\alpha_1, \alpha_2, \ldots, \alpha_n, k$, where not all of the α_i are equal to zero. Since $(0, 0, \ldots, 0) \in H$, it follows that $k = 0$, and so H is given by $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$. We claim that $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n = 0$.

Since almost every element of the RNS lies on G, and $G \subseteq H$, we know that almost every element of the RNS lies on H. Hence, for almost every $a \in C$, $\alpha_1 f_1(a) + \alpha_2 f_2(a) + \cdots + \alpha_n f_n(a) = 0$. We must show that, for every $A \subseteq C$, $\alpha_1 m_1(A) + \alpha_2 m_2(A) + \cdots + \alpha_n m_n(A) = 0$. We establish this as follows. For any such A,

$$\begin{aligned}
&\alpha_1 m_1(A) + \alpha_2 m_2(A) + \cdots + \alpha_n m_n(A) \\
&\quad = \alpha_1 \int_A f_1 \, d\mu + \alpha_2 \int_A f_2 \, d\mu + \cdots + \alpha_n \int_A f_n \, d\mu \\
&\quad = \int_A (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_2 f_2) d\mu = \int_A 0 \, d\mu = 0 \,.
\end{aligned}$$

This tells us that $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n = 0$. Therefore, the measures are linearly dependent.

For the reverse direction, we assume that the measures are linearly dependent. In particular, suppose that constants $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, are such that $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n = 0$. We claim that, for almost every $a \in C$, $\alpha_1 f_1(a) + \alpha_2 f_2(a) + \cdots + \alpha_n f_n(a) = 0$. Let $B^< = \{a \in C : \alpha_1 f_1(a) + \alpha_2 f_2(a) + \cdots + \alpha_n f_n(a) < 0\}$, let $B^> = \{a \in C : \alpha_1 f_1(a) + \alpha_2 f_2(a) + \cdots + \alpha_n f_n(a) > 0\}$, and assume, by way of contradiction, that either $B^<$ or $B^>$ has positive measure.

If $B^<$ has positive measure, then

$$\begin{aligned}
&\alpha_1 m_1(B^<) + \alpha_2 m_2(B^<) + \cdots + \alpha_n m_n(B^<) \\
&\quad = \alpha_1 \int_{B^<} f_1\, d\mu + \alpha_2 \int_{B^<} f_2\, d\mu + \cdots + \alpha_n \int_{B^<} f_n\, d\mu \\
&\quad = \int_{B^<} (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n) d\mu < 0
\end{aligned}$$

and if $B^>$ has positive measure, then

$$\begin{aligned}
&\alpha_1 m_1(B^>) + \alpha_2 m_2(B^>) + \cdots + \alpha_n m_n(B^>) \\
&\quad = \alpha_1 \int_{B^>} f_1\, d\mu + \alpha_2 \int_{B^>} f_2\, d\mu + \cdots + \alpha_n \int_{B^>} f_n\, d\mu \\
&\quad = \int_{B^>} (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n) d\mu > 0 .
\end{aligned}$$

In either case, we have a contradiction, since we assumed that $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n = 0$. Hence, for almost every $a \in C$, $\alpha_1 f_1(a) + \alpha_2 f_2(a) + \cdots + \alpha_n f_n(a) = 0$. This tells us that the hyperplane $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$ contains almost all of the RNS. Let G be the intersection of the RNS with this hyperplane. Then almost all of the RNS lies on G, and G is a subset of this hyperplane.

We know that G is a subset of the RNS, the RNS is a subset of the simplex, and the simplex is a subset of the hyperplane $x_1 + x_2 + \cdots + x_n = 1$. Hence, G is a subset of the hyperplane $x_1 + x_2 + \cdots + x_n = 1$. Clearly the hyperplanes $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$ and $x_1 + x_2 + \cdots + x_n = 1$ are distinct (since, for example, the first of these hyperplanes contains the origin, but the second does not). Then, by the preceding condition b, $\dim(G) \le n - 2$. This establishes that there is a lower-dimensional subset of the simplex that contains almost all of the RNS, as desired. This completes the proof of the theorem. □

We recall from Chapter 5 (see Corollary 5.7) that a super envy-free partition exists if and only if the measures are linearly independent. This, together with Theorem 9.8, yields the following result.

Corollary 9.9 *There exists a super envy-free partition if and only if no lower-dimensional subset of the simplex contains almost all of the RNS.*

9B. The Situation Without Absolute Continuity

In this section we make no general assumptions about absolute continuity. We begin by returning to our informal view of the players sitting at a table, with each bit of cake positioned closer to players who value that bit more. If absolute continuity fails, then there are bits of cake that have no value to at least one player and positive value to at least one player. Suppose a is such a bit of cake, assume that Player i places no value on a, and that some other player places positive value on a. Then a will be located at a point on the table as far away from Player i as possible.

To make this idea more precise, let us suppose that for some $i, j = 1, 2, \ldots, n$, there exists $A \subseteq C$ such that $m_i(A) = 0$, and $m_j(A) > 0$. Then, $\{a \in A : f_i(a) = 0 \text{ and } f_j(a) > 0\}$ has positive measure with respect to μ. For any a in this set, $f(a) = (f_1(a), f_2(a), \ldots, f_n(a))$ is on the boundary of the simplex. In particular, it lies on the face of the simplex that is farthest from Player i's vertex.

For an additional example, we suppose that some $A \subseteq C$, has positive value to Player j, and no value to any other player. Then, $\{a \in A : f_j(a) > 0$ and, for each $i = 1, 2, \ldots, n$ with $i \neq j$, $f_i(a) = 0\}$ has positive measure with respect to μ. For any a in this set, $f(a) = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the "1" occurs in the jth position. This is Player j's vertex. It is as far as possible from all other players.

For the remainder of this section, we adopt the convention that expressions such as "almost every" or "positive measure" refer to the measure μ. The appropriate revision of Lemma 9.1 is Lemma 9.10.

Lemma 9.10
a. *For almost every* $a \in C$, $f_1(a) + f_2(a) + \cdots + f_n(a) = 1$.
b. *For any* $i = 1, 2, \ldots, n$ *and almost every* $a \in C$, $0 \leq f_i(a) \leq 1$.
c. *If the measures are not all absolutely continuous with respect to each other, then, for some* $i = 1, 2, \ldots, n$, $\{a \in C : f_i(a) = 0\}$ *has positive measure.*

Proof: The proof for part a is precisely as in the proof of part a of Lemma 9.1, since that proof did not use absolute continuity.

For part b, fix any $i = 1, 2, \ldots, n$. We first show that, for almost every $a \in C$, $0 \leq f_i(a)$. Let $B^< = \{a \in C : f_i(a) < 0\}$. We must show that $\mu(B^<) > 0$, then

$$0 \leq m_i(B^<) = \int_{B^<} f_i \, d\mu < 0 .$$

This is a contradiction and, hence, $\mu(B^<) = 0$. It follows that, for almost every $a \in C$, $0 \leq f_i(a)$. This and part a imply that for almost every $a \in C$, $f_i(a) \leq 1$.

For part c, we note that if absolute continuity fails, then for some $i, j = 1, 2, \ldots, n$ and some $A \subseteq C$, $m_i(A) = 0$ and $m_j(A) > 0$. This implies that $\mu(A) > 0$ and, since $\int_A f_i \, d\mu = m_i(A) = 0$, it follows that $\{a \in A : f_i(a) = 0\}$ has positive measure. Hence, $\{a \in C : f_i(a) = 0\}$ has positive measure. □

As we did in the previous section (using Lemma 9.1), this lemma allows us to make a simplifying assumption:

> By redefining some or all of the f_i on a set of measure zero, if necessary, we shall assume from now on that, for each $i = 1, 2, \ldots, n$ and *every* $a \in C$,
> $f_1(a) + f_2(a) + \cdots + f_n(a) = 1$ and $0 \leq f_i(a) \leq 1$.

Corollary 9.11 *The boundary of the simplex is associated with a piece of cake of positive measure if and only if the measures are not all absolutely continuous with respect to each other.*

Proof: This follows immediately from Lemmas 9.1 and 9.10. □

In particular for any $i = 1, 2, \ldots, n$ and any $A \subseteq C$ of positive measure, the following holds: $f(a)$ lies on the face of the simplex farthest from Player i's vertex for almost every $a \in A$ if and only if $m_i(A) = 0$.

We close this section by considering two possibilities for the RNS if absolute continuity fails. In Examples 9.12 and 9.13, we return to Examples 5.43 and 5.44, respectively. In each of these examples in Chapter 5, we specified a cake and measures on the cake, and pictured the corresponding IPS. Here, we repeat the descriptions of this cake and these measures, and picture the corresponding RNS.

Example 9.12 Let C be the interval $[0, 3)$ on the real number line and let m_L be Lebesgue measure on this set. Suppose that there are three players, Player 1, Player 2, and Player 3, with corresponding measures m_1, m_2, and m_3, respectively, defined as follows: for any $A \subseteq C$,

$$m_1(A) = \frac{1}{2} m_L(A \cap [0, 2))$$
$$m_2(A) = \frac{1}{2} m_L(A \cap [1, 3))$$
$$m_3(A) = m_L(A \cap [1, 2))$$

We wish to determine the corresponding RNS. It is not hard to see that all points of C in the interval $[0, 1)$ correspond to the same point in the RNS, as will all points in the interval $[1, 2)$ and all points in the interval $[2, 3)$.

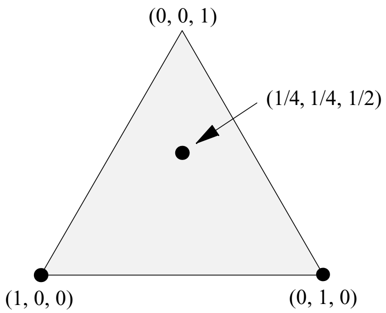


Figure 9.6

We first consider the points in the interval [0, 1). Since $m_1([0, 1)) > 0$ and $m_2([0, 1)) = m_3([0, 1)) = 0$, it follows that every point in the interval [0, 1) corresponds, via f, to the point (1, 0, 0). Similarly, every point in the interval [2, 3) corresponds, via f, to the point (0, 1, 0).

Finally, we consider points in the interval $[1, 2)$. For any $A \subseteq [1, 2)$, $m_1(A) = m_2(A)$ and $m_3(A) = 2m_1(A) = 2m_2(A)$. This implies that, for almost every $a \in [1, 2)$, $f_1(a) = f_2(a)$ and $f_3(a) = 2f_1(a) = 2f_2(a)$. Let us assume that this is so for *every* such a. Then, for every $a \in A$, since $f_1(a) + f_2(a) + f_3(a) = 1$, it follows that $f(a) = (f_1(a), f_2(a), f_3(a)) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Hence, every point in the interval $[1, 2)$ corresponds, via f, to the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. We have established that the RNS consists of the three points $(1, 0, 0)$, $(0, 1, 0)$, and $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. It is pictured in Figure 9.6.

We make the following observations concerning absolute continuity and the RNS for this situation:

- The existence of the point $(1, 0, 0)$ in the RNS corresponds to the fact that m_1 is not absolutely continuous with respect to m_2 or m_3.
- The existence of the point $(0, 1, 0)$ in the RNS corresponds to the fact that m_2 is not absolutely continuous with respect to m_1 or m_3.
- The fact that, except for the point $(1, 0, 0)$, there are no points on the line segment connecting Player 1's vertex and Player 3's vertex that are in the RNS corresponds to the fact that m_3 is absolutely continuous with respect to m_2.
- The fact that, except for the point $(0, 1, 0)$, there are no points on the line segment connecting Player 2's vertex and Player 3's vertex that are in the RNS corresponds to the fact that m_3 is absolutely continuous with respect to m_1.

- Since the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ of the RNS is an interior point of the simplex, this point corresponds to a piece of cake (namely the interval $[1, 2)$) on which all of the measures are absolutely continuous with respect to each other.

Example 9.13 Let C be the interval $[0, 3)$ on the real number line and let m_L be Lebesgue measure on this set. Suppose that there are three players, Player 1, Player 2, and Player 3, with corresponding measures m_1, m_2, and m_3, respectively, defined as follows: for any $A \subseteq C$,

$$\begin{aligned} m_1(A) &= \frac{2}{3}m_L(A \cap [0, 1)) + \frac{1}{3}m_L(A \cap [1, 2)) \\ m_2(A) &= \frac{1}{3}m_L(A \cap [0, 1)) + \frac{2}{3}m_L(A \cap [1, 2)) \\ m_3(A) &= \frac{1}{3}m_L(A \cap [0, 3)) \end{aligned}$$

We wish to determine the corresponding RNS. As in the previous example, it is easy to see that the RNS consists of three points, one corresponding to each of the intervals $[0, 1)$, $[1, 2)$, and $[2, 3)$. We begin by considering the interval $[0, 1)$. For any $A \subseteq [0, 1)$ that has positive measure, $m_2(A) = m_3(A)$ and $m_1(A) = 2m_2(A) = 2m_3(A)$. Thus, for almost every $a \in [0, 1)$, $f_2(a) = f_3(a)$ and $f_1(a) = 2f_2(a) = 2f_3(a)$. We may assume that this is so for *every* such a. Then, for every $a \in A$, since $f_1(a) + f_2(a) + f_3(a) = 1$, it follows that $f(a) = (f_1(a), f_2(a), f_3(a)) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Hence, every point in the interval $[0, 1)$ corresponds, via f, to the point $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Similarly, we see that every point in the interval $[1, 2)$ corresponds, via f, to the point $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

Finally, we consider the interval $[2, 3)$. Since $m_1([2, 3)) = m_2([2, 3)) = 0$ and $m_3([2, 3)) > 0$, it follows that almost every point in the interval $[2, 3)$ corresponds, via f, to the point $(0, 0, 1)$. We may assume that this is so for *every* such a. Hence, we see that the RNS consists of the three points $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, and $(0, 0, 1)$. It is pictured in Figure 9.7.

We make the following observations concerning absolute continuity and the RNS for this situation:

- The existence of the point $(0, 0, 1)$ in the RNS corresponds to the fact that m_3 is not absolutely continuous with respect to m_1 or m_2.
- The fact that, except for the point $(0, 0, 1)$, there are no points on the line segment connecting Player 1's vertex and Player 3's vertex that are in the RNS corresponds to the fact that m_1 is absolutely continuous with respect to m_2.

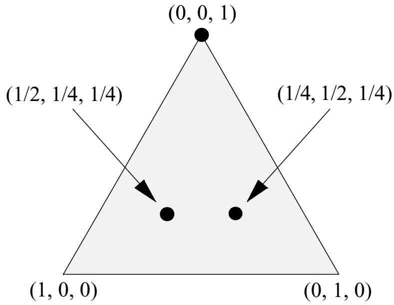


Figure 9.7

- The fact that, except for the point (0, 0, 1), there are no points on the line segment connecting Player 2's vertex and Player 3's vertex that are in the RNS corresponds to the fact that m_2 is absolutely continuous with respect to m_1.
- Since the points $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ of the RNS are interior points of the simplex, these points correspond to a piece of cake (namely the interval $[0, 2)$) on which all of the measures are absolutely continuous with respect to each other.

The reader may wish to consider the relationship between the IPSs corresponding to the two previous examples, given in Figures 5.2 and 5.3, and their RNSs, given in Figures 9.6 and 9.7. We shall not pursue this topic here, since we shall discuss the general issue of the relationship between IPSs and RNSs in Chapter 12.

We close this chapter by describing two possible problems with the definition, and with our intuitive understanding, of the RNS. We have defined the RNS to be $\{f(a) : a \in C\}$ where, for each $a \in C$, $f(a) = (f_1(a), f_2(a), \ldots, f_n(a))$. For any cake and corresponding measures, we can simply change the values of the density functions on a set of measure zero and make any desired point of the simplex become part of the RNS. For instance, in Example 9.13, we could take a single point $a \in C$, set $f_1(a) = \frac{1}{7}$, $f_2(a) = \frac{2}{7}$, $f_3(a) = \frac{4}{7}$, and leave all other values of these functions unchanged. Then f_1, f_2, and f_3 are still density functions for m_1, m_2, and m_3, respectively, the RNS still contains the points $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, and (0, 0, 1), but it now also contains the point $(\frac{1}{7}, \frac{2}{7}, \frac{4}{7})$. Clearly, we want to avoid this situation. It is tempting to try to do so by simply declaring that no point of the RNS can be associated with a piece of cake of measure zero. That this approach does not work can be seen by considering the RNS of Figure 9.2b. In this situation, the RNS is smoothly spread out along a line segment. It may well be that no point of this RNS corresponds to a piece of cake of positive measure, and we certainly cannot eliminate all of these points from the RNS. This example suggests a second problem. By redefining the density functions on a set of measure zero, we can remove one point from this RNS, place it at another point on the line segment, and thus have the RNS consist of all points on this line segment except for one. We certainly want to avoid this situation too. Although it is possible to state precise rules involving the redefining of the density functions on sets of measure zero so as to avoid these problems, we shall not do so. Instead, we shall be content with a slightly informal perspective. We shall always assume that the density functions have been redefined on a set of measure zero, if necessary, so that the following two conditions hold:

- If p is a point in the simplex satisfying that, for some $\varepsilon > 0$, the set of all bits of cake associated with points of the simplex that are within ε of p (including p itself) has measure zero, then p is not in the RNS.
- If p is an interior point of a curve τ that lies in the simplex, and every point of $\tau\backslash\{p\}$ is in the RNS, then p is in the RNS.

10

Characterizing Pareto Optimality III

The RNS, Weller's Construction, and w-Association

In this chapter, we use the structure introduced in the previous chapter (i.e., the RNS) to develop our third approach to characterizing Pareto maximality and Pareto minimality. We begin in Section 10A by examining the two-player context. In Section 10B, we show how to use the RNS to associate one or more partitions with each point in the interior of the simplex, and then we use this idea to characterize Pareto maximality and Pareto minimality. In Sections 10A and 10B, we assume that the measures are absolutely continuous with respect to each other. In Section 10C, we consider what happens when absolute continuity fails.

10A. Introduction: The Two-Player Context

We begin this section with a brief discussion and three examples in the two-player context. This will provide motivation for the general situation.

We assume that there are two players, Player 1 and Player 2, whom we shall refer to as "she" and "he" respectively, and we consider the RNS associated with these players' measures. Since there are two players, the setting for the RNS is the one-simplex, which is the line segment between $(1, 0)$ and $(0, 1)$. The closer a point of the RNS is to $(1, 0)$, the more it is valued by Player 1 (in comparison with Player 2) and the closer a point of the RNS is to $(0, 1)$, the more it is valued by Player 2 (in comparison with Player 1). We consider three examples to illustrate how the RNS is a useful structure in the study of Pareto maximal partitions and then we will see how these examples illustrate the characterization of Pareto maximality that is the main focus of this chapter.

Example 10.1 If we wish to obtain a Pareto maximal partition of C among the two players, it makes sense to give Player 1 bits of cake that are more valued by her and to give Player 2 bits of cake that are more valued by him. Thus, let us consider a partition $P = \langle P_1, P_2 \rangle$ of the cake in which Player 1 receives all bits of cake that are associated with points of the RNS between $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$, and Player 2 receives all bits of cake that are associated with points of the RNS between $(\frac{1}{2}, \frac{1}{2})$ and $(0, 1)$. Thus, $f(P_1)$ is a subset of the interval from $(1, 0)$ to $(\frac{1}{2}, \frac{1}{2})$, and $f(P_2)$ is a subset of the interval from $(\frac{1}{2}, \frac{1}{2})$ to $(0, 1)$. For the present, we assume that the RNS does not contain the point $(\frac{1}{2}, \frac{1}{2})$.

We claim that P is Pareto maximal. To see this, we first note that if A_1 and A_2 are both sets of positive measure with $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ then, since every bit of cake in P_1 is more valued by Player 1 than by Player 2 and every bit of cake in P_2 is more valued by Player 2 than by Player 1, it follows that $m_1(A_1) > m_2(A_1)$ and $m_2(A_2) > m_1(A_2)$. Each of the following two slightly different approaches establishes the Pareto maximality of P.

P is Pareto maximal if and only if any trade that leaves one player better off must leave the other player worse off. Fix pieces of cake $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$, both of positive measure, and suppose first that a trade of A_1 and A_2 between Player 1 and Player 2 makes Player 1 better off. We must show that this trade makes Player 2 worse off. Since Player 1 is made better off by this trade, $m_1(A_2) > m_1(A_1)$. But then $m_2(A_2) > m_1(A_2) > m_1(A_1) > m_2(A_1)$, and so Player 2 is made worse off by this trade. The argument that if Player 2 is made better off by a trade, then Player 1 must be made worse off, is similar. Hence, P is Pareto maximal.

A second approach to showing that P is Pareto maximal involves the use of partition ratios and Theorem 8.9. Since $m_1(A_1) > m_2(A_1)$ for every $A_1 \subseteq P_1$ of positive measure, it follows that $\mathrm{pr}_{12} = \sup\{\frac{m_2(A)}{m_1(A)} : A \subseteq P_1$ and A has positive measure$\} \leq 1$. Similarly, since $m_2(A_2) > m_1(A_2)$ for every $A_2 \subseteq P_2$ of positive measure, it follows that $\mathrm{pr}_{21} = \sup\{\frac{m_1(A)}{m_2(A)} : A \subseteq P_2$ and A has positive measure$\} \leq 1$. Thus $\mathrm{pr}_{12}\mathrm{pr}_{21} \leq 1$, and it follows from Theorem 8.9 that P is Pareto maximal.

Is the aforementioned partition the only Pareto maximal partition? The answer is: definitely not. We can easily name two additional partitions. The partition obtained by giving all of the cake to Player 1 and the partition obtained by giving all of the cake to Player 2 are both Pareto maximal. Are there others? It turns out that, in general, there are many. In Example 10.1, there was nothing special about the point $(\frac{1}{2}, \frac{1}{2})$.

Example 10.2 Fix κ with $0 < \kappa < 1$ and let us consider the partition $P = \langle P_1, P_2 \rangle$ of the cake in which Player 1 receives all bits of cake that are associated with points of the RNS between $(1, 0)$ and $(\kappa, 1 - \kappa)$, and Player 2 receives all bits of cake that are associated with points of the RNS between $(\kappa, 1 - \kappa)$ and

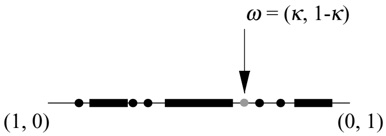


Figure 10.1

(0, 1). As in the previous example for $(\frac{1}{2}, \frac{1}{2})$, we assume, for the present, that the RNS does not contain the point $(\kappa, 1-\kappa)$.

For any $a \in P_1$, $f(a)$ is between $(1, 0)$ and $(\kappa, 1-\kappa)$. This tells us that $\kappa < f_1(a) < 1$ and $0 < f_2(a) < 1-\kappa$. Hence, for any such a, $\frac{f_1(a)}{f_2(a)} > \frac{\kappa}{1-\kappa}$, and hence $f_1(a) > (\frac{\kappa}{1-\kappa})f_2(a)$. This implies that, for any $A_1 \subseteq P_1$ of positive measure, $m_1(A_1) > (\frac{\kappa}{1-\kappa})m_2(A_1)$. Similarly, it can be shown that, for any $A_2 \subseteq P_2$ of positive measure, $m_2(A_2) > (\frac{1-\kappa}{\kappa})m_1(A_2)$. We claim that these facts imply that P is Pareto maximal. We consider both of the approaches used in the previous example.

Suppose that A_1 and A_2 are sets of positive measure with $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$ and assume that a trade of A_1 and A_2 makes Player 1 better off. Then, $m_1(A_2) > m_1(A_1)$. This, and the inequalities developed in the previous paragraph, imply that $m_2(A_2) > (\frac{1-\kappa}{\kappa})m_1(A_2) > (\frac{1-\kappa}{\kappa})m_1(A_1) > (\frac{1-\kappa}{\kappa})(\frac{\kappa}{1-\kappa})m_2(A_1) = m_2(A_1)$. This tells us that Player 2 is made worse off by this trade. Similarly, it can be shown that any trade that makes Player 2 better off must make Player 1 worse off. This establishes that P is Pareto maximal.

Or, using partition ratios, we see that, since $m_1(A_1) > (\frac{\kappa}{1-\kappa})m_2(A_1)$ for every $A_1 \subseteq P_1$ of positive measure, it follows that $\text{pr}_{12} \leq (\frac{1-\kappa}{\kappa})$. Similarly, since $m_2(A_2) > (\frac{1-\kappa}{\kappa})m_1(A_2)$ for every $A_2 \subseteq P_2$ of positive measure, it follows that $\text{pr}_{21} \leq (\frac{\kappa}{1-\kappa})$. Hence, $\text{pr}_{12}\text{pr}_{21} \leq (\frac{1-\kappa}{\kappa})(\frac{\kappa}{1-\kappa}) = 1$ and therefore, by Theorem 8.9, it follows that P is Pareto maximal.

The ideas discussed in the previous example are illustrated in Figure 10.1. In the figure, we have darkened the RNS. (The RNS in this illustration consists of three line segments plus five additional points.) We may choose any point ω that is in the interior of the simplex, give all bits of cake (associated with points) to the left of ω to Player 1, give all bits of cake (associated with points) to the right of ω to Player 2, and the resulting partition is Pareto maximal.

There is one gap in our preceding discussion. We have not said what to do with pieces of cake that are associated with $(\frac{1}{2}, \frac{1}{2})$ in our first example, or with $(\kappa, 1-\kappa)$ in our second example. We address this point now.

Suppose that in Example 10.2 there is a piece of cake of positive measure that is associated with the point $(\kappa, 1-\kappa)$. In this case, we divide this piece arbitrarily between Player 1 and Player 2. It is not hard to verify, using either

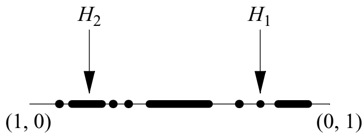


Figure 10.2

of the two approaches we used in the two examples, that any such partition (where pieces of cake not associated with ω are distributed as described in Example 10.2) is Pareto maximal. In contrast with the situation when there is no piece of cake of positive measure associated with $(\kappa, 1-\kappa)$, this situation has the following two properties:

a. There are infinitely many non-p-equivalent Pareto maximal partitions corresponding (as described earlier) to the same point, $(\kappa, 1-\kappa)$.
b. Given a Pareto maximal partition corresponding to the point $(\kappa, 1-\kappa)$, there may be trades between the two players that yield partitions p-equivalent but not s-equivalent to the original partition.

In particular, there exist trades as in property b whenever the cake associated with the point $(\kappa, 1-\kappa)$ has positive measure, except in the case when all of the cake associated with point $(\kappa, 1-\kappa)$ is given to one of the two players. We shall examine these two properties in more detail later. (Property a is related to the idea of the RNS being "concentrated," which we study in Section 12C. Property b is related to the notion of "strong Pareto maximality," which we study in Chapter 14.)

Example 10.3 Consider Figure 10.2. This is the same RNS as in Figure 10.1. Let A_1 be the set of all bits of cake associated with the point H_1 and let A_2 be the set of all bits of cake associated with points on the line segment H_2. Let $P = \langle P_1, P_2 \rangle$ be any partition of C such that $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$. It is clear that $\frac{m_2(A_1)}{m_1(A_1)} > \frac{m_2(A_2)}{m_1(A_2)}$. Then, we have

$$\begin{aligned} p_{12} &= \sup\left\{\frac{m_2(A)}{m_1(A)} : A \subseteq P_1 \text{ and } A \text{ has positive measure}\right\} \\ &\geq \frac{m_2(A_1)}{m_1(A_1)} > \frac{m_2(A_2)}{m_1(A_2)} = \frac{1}{\left(\frac{m_1(A_2)}{m_2(A_2)}\right)} \\ &\geq \frac{1}{\sup\left\{\frac{m_1(A)}{m_2(A)} : A \subseteq P_2 \text{ and } A \text{ has positive measure}\right\}} = \frac{1}{p_{21}}. \end{aligned}$$

This implies that $p_{12}p_{21} > 1$. Hence, by Theorem 8.9, we conclude that P is not Pareto maximal.

What is the difference between the RNS in Examples 10.2 and 10.3 that makes for Pareto maximality in one case but not in the other? Intuitively, the difference is that in Example 10.2 it is possible to identify a point, namely $(\kappa, 1-\kappa)$, such that all points associated with P_1 lie to the left of this point and all points associated with P_2 lie to the right of this point, whereas in Example 10.3 there is no such point. This suggests that the issue here involves the relationship between some sort of right limit point of P_1 and left limit point of P_2. This idea shall be made explicit in Definition 10.4 and Theorem 10.9.

10B. The Characterization

We are now ready to begin the formal treatment of the ideas introduced in the previous section. We do so in the general n-player context. Recall that S^+ denotes the interior of the simplex S. The construction we present is essentially attributable to D. Weller [43]. We have made some small modifications and simplifications.

Definition 10.4 Suppose that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition and $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. We shall say that P is *w-associated with* ω if and only if the following holds for all distinct $i, j = 1, 2, \ldots, n$:

$$\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j} \text{ for almost every } a \in P_i$$

The "w" in "w-associated" denotes "Weller." Later in this section, we will discuss our reasons for insisting that ω not be on the boundary of the simplex.

It follows easily from the definition that, if $P = \langle P_1, P_2, \ldots, P_n \rangle$ is w-associated with ω for some $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$, then for distinct $i, j = 1, 2, \ldots, n$, and any $A \subseteq P_i$, $\frac{m_i(A)}{m_j(A)} \geq \frac{\omega_i}{\omega_j}$.

There is helpful geometric perspective due to Weller that illustrates the relationship between the partition P and the point ω in Definition 10.4 when $n = 3$. This perspective is given in Figure 10.3 and is the three-player version of the perspective given in Example 10.2 and Figure 10.1 in the previous section for two players. Fix any $\omega \in S^+$, as in Figure 10.3, and consider the regions H_1, H_2, and H_3, as in the figure. We intend H_1, H_2, and H_3 to denote closed regions that intersect on their common boundaries. For any partition $P = \langle P_1, P_2, P_3 \rangle$ of C, P is w-associated with ω if and only if, for each $i = 1, 2, 3$, points in the simplex corresponding to points in P_i are all contained in H_i (i.e., $f(P_i) \subseteq H_i$).

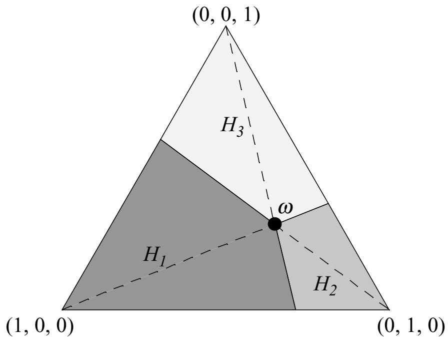


Figure 10.3

To put it another way, a partition is w-associated with ω if and only if, for each $i = 1, 2, 3$, the points in C corresponding to interior points of an H_i go to Player i and points of C corresponding to boundary points of some H_i go to any of the players associated with this boundary. (We recall that since the measures are absolutely continuous with respect to each other, the RNS contains no points on the boundary of the simplex.) Thus, we see that there may be many partitions w-associated with ω. This is an issue mentioned in the previous section that we shall study in detail in Section 12C.

Theorem 10.9 will provide a characterization of Pareto maximality using the notion of w-associated. Its proof shall use a natural correspondence that exists between points $\alpha \in S^+$ that provide coefficients for a convex combination of measures and points $\omega \in S^+$ that are to be used as in Definition 10.4. This correspondence is given by the following definition and theorem.

Definition 10.5 For any $p = (p_1, p_2, \ldots, p_n) \in S^+$, let $\mathrm{RD}(p) = \left(\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}}\right)\left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right)$.

It is straightforward to verify that RD is a bijection from S^+ to S^+. Also, for any $p \in S^+$, $\mathrm{RD}(\mathrm{RD}(p)) = p$. The letters "RD" are meant to denote "take the *reciprocals* and then *divide* to make a sum of one." This defines RD for the $(n-1)$-simplex. We shall use this same name, RD, for this function for any n.

Our characterization theorem will follow easily from the following result.

Theorem 10.6 *Fix partition $P = \langle P_1, P_2, \ldots, P_n\rangle$, $\omega \in S^+$, and $\alpha \in S^+$, with $\alpha = \mathrm{RD}(\omega)$ (and thus $\omega = \mathrm{RD}(\alpha)$). P is w-associated with ω if and only if P maximizes the convex combination of measures corresponding to α.*

Proof: Fix partition $P = \langle P_1, P_2, \ldots, P_n \rangle$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S^+$ with $\alpha = \mathrm{RD}(\omega)$.

For the forward direction, we assume that P does not maximize the convex combination of measures corresponding to α. Then we can choose some partition $R = \langle R_1, R_2, \ldots, R_n \rangle$ such that

$$\begin{aligned} &\alpha_1 m_1(P_1) + \alpha_2 m_2(P_2) + \cdots + \alpha_n m_n(P_n) \\ &\quad < \alpha_1 m_1(R_1) + \alpha_2 m_2(R_2) + \cdots + \alpha_n m_n(R_n). \end{aligned}$$

Since $\alpha = \mathrm{RD}(\omega)$, this implies that

$$\begin{aligned} &\left(\frac{1}{\frac{1}{\omega_1} + \frac{1}{\omega_2} + \cdots + \frac{1}{\omega_n}}\right)\left(\left(\frac{1}{\omega_1}\right) m_1(P_1) + \left(\frac{1}{\omega_2}\right) m_2(P_2)\right. \\ &\left. + \cdots + \left(\frac{1}{\omega_n}\right) m_n(P_n)\right) \\ &< \left(\frac{1}{\frac{1}{\omega_1} + \frac{1}{\omega_2} + \cdots + \frac{1}{\omega_n}}\right)\left(\left(\frac{1}{\omega_1}\right) m_1(R_1)\right. \\ &\left. + \left(\frac{1}{\omega_2}\right) m_2(R_2) + \cdots + \left(\frac{1}{\omega_n}\right) m_n(R_n)\right) \end{aligned}$$

or

$$\begin{aligned} &\left(\frac{1}{\omega_1}\right) m_1(P_1) + \left(\frac{1}{\omega_2}\right) m_2(P_2) + \cdots + \left(\frac{1}{\omega_n}\right) m_n(P_n) \\ &< \left(\frac{1}{\omega_1}\right) m_1(R_1) + \left(\frac{1}{\omega_2}\right) m_2(R_2) + \cdots + \left(\frac{1}{\omega_n}\right) m_n(R_n). \end{aligned}$$

We can view the partition R as arising from the partition P by a finite number (at most $n(n-1)$) of transfers between the players. Each of these transfers contributes to changing

$$\left(\frac{1}{\omega_1}\right) m_1(P_1) + \left(\frac{1}{\omega_2}\right) m_2(P_2) + \cdots + \left(\frac{1}{\omega_n}\right) m_n(P_n)$$

to

$$\left(\frac{1}{\omega_1}\right) m_1(R_1) + \left(\frac{1}{\omega_2}\right) m_2(R_2) + \cdots + \left(\frac{1}{\omega_n}\right) m_n(R_n).$$

Thus, at least one of these transfers must result in an increase in the sum. Assume that $i, j = 1, 2, \ldots, n$ are distinct, $A \subseteq P_i$, and transferring A from Player i to Player j increases the sum. Then,

$$\left(\frac{1}{\omega_i}\right) m_i(P_i) + \left(\frac{1}{\omega_j}\right) m_j(P_j) < \left(\frac{1}{\omega_i}\right) m_i(P_i \backslash A) + \left(\frac{1}{\omega_j}\right) m_j(P_j \cup A).$$

It follows that

$$\left(\frac{1}{\omega_i}\right) m_i(P_i) + \left(\frac{1}{\omega_j}\right) m_j(P_j) < \left(\frac{1}{\omega_i}\right) m_i(P_i) - \left(\frac{1}{\omega_i}\right) m_i(A) + \left(\frac{1}{\omega_j}\right) m_j(P_j) + \left(\frac{1}{\omega_j}\right) m_j(A)$$

and thus

$$\left(\frac{1}{\omega_i}\right) m_i(A) < \left(\frac{1}{\omega_j}\right) m_j(A).$$

This implies that $\frac{m_i(A)}{m_j(A)} < \frac{\omega_i}{\omega_j}$, and thus $\{a \in A : \frac{f_i(a)}{f_j(a)} < \frac{\omega_i}{\omega_j}\}$ has positive measure. Then, since $A \subseteq P_i$, it is not the case that $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every $a \in P_i$. This tells us that P is not w-associated with ω.

For the reverse direction, we suppose that P is not w-associated with ω. Then, for some distinct $i, j = 1, 2, \ldots, n$, the set $\{a \in P_i : \frac{f_i(a)}{f_j(a)} < \frac{\omega_i}{\omega_j}\}$ has positive measure. Call this set A. Since $\omega = \mathrm{RD}(\alpha)$, it follows that

$$A = \left\{ a \in P_i : \frac{f_i(a)}{f_j(a)} < \frac{\left(\frac{\frac{1}{\alpha_i}}{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_n}}\right)}{\left(\frac{\frac{1}{\alpha_j}}{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_n}}\right)} \right\} = \left\{ a \in P_i : \frac{f_i(a)}{f_j(a)} < \frac{\alpha_j}{\alpha_i} \right\}.$$

Then $\frac{m_i(A)}{m_j(A)} < \frac{\alpha_j}{\alpha_i}$ and hence $\alpha_i m_i(A) < \alpha_j m_j(A)$. Define a partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ as follows: for each $k = 1, 2, \ldots, n$,

$$Q_k = \begin{cases} P_k \backslash A & \text{if } k = i \\ P_k \cup A & \text{if } k = j \\ \mathrm{P}_k & \text{if } k \neq i, k \neq j \end{cases}$$

Partition Q is the partition obtained from partition P by transferring A from Player i to Player j.

It follows that

$$\begin{aligned} &\alpha_i m_i(P_i) + \alpha_j m_j(P_j) \\ &\quad = \alpha_i m_i(P_i \backslash A) + \alpha_i m_i(A) + \alpha_j m_j(P_j) \\ &\quad < \alpha_i m_i(P_i \backslash A) + \alpha_j m_j(A) + \alpha_j m_j(P_j) \\ &\quad = \alpha_i m_i(P_i \backslash A) + \alpha_j m_j(P_j \cup A) = \alpha_i m_i(Q_i) + \alpha_j m_j(Q_j). \end{aligned}$$

Hence, since $Q_k = P_k$ for every $k \neq i, k \neq j$,

$$\begin{aligned} &\alpha_1 m_1(P_1) + \alpha_2 m_2(P_2) + \cdots + \alpha_n m_n(P_n) \\ &\quad < \alpha_1 m_1(Q_1) + \alpha_2 m_2(Q_2) + \cdots + \alpha_n m_n(Q_n). \end{aligned}$$

Thus, P does not maximize the convex combination of measures corresponding to α. This completes the proof of the theorem. □

Suppose that partitions P and Q are p-equivalent. Then certainly P and Q maximize the same convex combinations of measures. By the theorem, this implies that P and Q are w-associated with the same points of S^+. In other words, w-association respects p-equivalence, and so we may say that a certain p-class of partitions is w-associated with a point in S^+. This observation and the theorem immediately yield Corollary 10.7. We shall use and extend this idea in Chapter 12.

We shall generally adhere to the notational convention used in the theorem: ω denotes a point to be used as in w-association and α denotes a point to be used as in the maximization of a convex combination of measures.

Corollary 10.7

a. *A p-class of partitions maximizes more than one convex combination of measures corresponding to points of S^+ if and only if it is w-associated with more than one point of S^+.*
b. *Fix $\alpha \in S^+$. More than one p-class of partitions maximizes the convex combination of measures corresponding to α if and only if more than one p-class of partitions is w-associated with* $\mathrm{RD}(\alpha)$.

We are almost ready to state our characterization of Pareto maximality using the notion of w-associated. We first discuss an additional assumption. In our characterization, we shall assume that all partitions to be considered give a piece of cake of positive measure to each player. Let Part^+ denote the set of all such partitions. Thus, $\mathrm{Part}^+ = \{P = \langle P_1, P_2, \ldots, P_n \rangle \in \mathrm{Part} : \text{for each } i, P_i$ has positive measure$\}$. Or, equivalently, $\mathrm{Part}^+ = \{P \in \mathrm{Part} : m(P) \in S^+\}$. We will discuss the reasons for this assumption later in this section. For now, we wish to show that this assumption should be viewed as nothing more than a minor annoyance. This is a consequence of the following.

Lemma 10.8 *Let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition and set $\delta_P = \{i \le n : P_i$ has positive measure$\}$. Then P is Pareto maximal if and only if the partition $\langle P_i : i \in \delta_P \rangle$ is a Pareto maximal partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P.*

Proof: Let P and δ_P be as in the statement of the lemma.

For the forward direction, we simply note that any partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P that is Pareto bigger than $\langle P_i : i \in \delta_P \rangle$ would immediately yield a partition of C among all players that is Pareto bigger than P.

For the reverse direction, suppose that P is not Pareto maximal. We must show that $\langle P_i : i \in \delta_P \rangle$ is not a Pareto maximal partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P.

For $i \notin \delta_P$, P_i has measure zero and hence, by redistributing P_i among the players named by δ_P, if necessary, we may assume that $P_i = \emptyset$. Then, $\bigcup_{i \in \delta_P} P_i = C$, and thus P is a partition of all of C among the players named by δ_P.

Since P is not Pareto maximal we may let $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ be a partition that is Pareto bigger than P. Set $\delta_Q = \{i \leq n : Q_i \text{ has positive measure}\}$. Then, $\delta_P \subseteq \delta_Q$. Arguing as before, we may assume that, for any $i \notin \delta_Q$, $Q_i = \emptyset$ and therefore $\bigcup_{i \in \delta_Q} Q_i = C$. We consider two cases.

Case 1: $\delta_P = \delta_Q$. Then $\langle Q_i : i \in \delta_P \rangle$ is a partition of C among the players named by δ_P and is Pareto bigger than $\langle P_i : i \in \delta_P \rangle$. Hence, $\langle P_i : i \in \delta_P \rangle$ is not a Pareto maximal partition of C among the players named by δ_P.

Case 2: $\delta_P \neq \delta_Q$. Fix any $j \in \delta_P$ and define a partition $\langle R_i : i \in \delta_P \rangle$ of C among the players named by δ_P as follows: for each $i \in \delta_P$,

$$R_i = \begin{cases} Q_i & \text{if } i \neq j \\ Q_j \cup \left(\bigcup_{k \notin \delta_P} Q_k \right) & \text{if } i = j \end{cases}$$

We may view partition R as arising from partition Q by having all players not named by δ_P give all of their cake to Player j and considering R as a partition among only the players named by δ_P. We claim that $\langle R_i : i \in \delta_P \rangle$ is Pareto bigger than $\langle P_i : i \in \delta_P \rangle$.

We recall that Q is Pareto bigger than P. Hence, for every $i = 1, 2, \ldots, n$, and, in particular, for every $i \in \delta_P, m_i(Q_i) \geq m_i(P_i)$. Then, for every $i \in \delta_P$ with $i \neq j$, we have $m_i(R_i) = m_i(Q_i) \geq m_i(P_i)$. And, $m_j(R_j) = m_j(Q_j \cup (\bigcup_{k \notin \delta_P} Q_k)) = m_j(Q_j) + m_j(\bigcup_{k \notin \delta_P} Q_k) > m_j(Q_j)$. The last inequality follows from the fact that $\delta_Q \backslash \delta_P \neq \emptyset$ and, for any $k \in \delta_Q \backslash \delta_P$, Q_k has positive measure.

This establishes that $\langle R_i : i \in \delta_P \rangle$ is a Pareto bigger partition of C among the players named by δ_P than is $\langle P_i : i \in \delta_P \rangle$, and hence $\langle P_i : i \in \delta_P \rangle$ is not a Pareto maximal partition of C among the players named by δ_P.

This completes the proof of the lemma. □

The lemma tells us that in considering the possible Pareto maximality of some partition we may simply ignore players that receive no cake and consider the partition to be a partition of only the players who receive a piece of cake of positive measure.

Our characterization of Pareto maximality is the following.

Theorem 10.9 *Fix a partition* $P \in \text{Part}^+$. *P is Pareto maximal if and only if P is w-associated with ω for some $\omega \in S^+$.*

The reverse direction of the theorem (without the assumption of absolute continuity) was proved by D. Weller [43].

Proof of Theorem 10.9: Suppose $P \in \text{Part}^+$. For the forward direction, we assume that P is Pareto maximal. By Theorem 7.4, P maximizes the convex combination of measures corresponding to some $\alpha \in S$. We claim that $\alpha \in S^+$. Suppose, by way of contradiction, that the ith component of α is zero and fix some j so that the jth component of α is not zero. (There must be at least one non-zero component, since the components are all non-negative and sum to one.) Since $P \in \text{Part}^+$, we know that Player i has a piece of cake of positive measure. Let R be the partition obtained from P by transferring Player i's piece to Player j. The convex combination of measures corresponding to α produces a larger sum when applied to R than when applied to P. This contradicts the fact that P maximizes the convex combination of measures corresponding to α. Thus, $\alpha \in S^+$. Let $\omega = \text{RD}(\alpha)$. Then $\omega \in S^+$ and, by Theorem 10.6, P is w-associated with ω.

For the reverse direction, we assume that P is w-associated with some $\omega \in S^+$. By Theorem 10.6, P maximizes the convex combination of measures corresponding to $\text{RD}(\omega)$. Theorem 7.4 implies that P is Pareto maximal. □

Theorem 10.9 can give us additional perspective on two ideas having to do with the relationship between the Pareto maximality of a partition and the Pareto maximality of "subparts" of the partition. We examine these ideas in the following two examples. As we shall see, in Example 10.10, "subparts" means subcollections of the players, as in the notion of proper subpartition Pareto maximal (see Definition 6.1). In Example 10.11, "subparts" means subsets of the pieces of cake that make up the partition. Informally stated, the question we investigate in these examples is whether or not the union of Pareto maximal partitions is Pareto maximal, where "union" means something very different in each case.

Example 10.10 Recall Theorem 6.2, which states that if a partition P is a Pareto maximal then it is proper subpartition Pareto maximal. Example 6.3 established that the converse of this result is false. In Chapter 8, we were able to gain some perspective on both the truth of Theorem 6.2 and the failure of its converse. (See Example 8.10 and the discussion following this example.) Theorem 10.9 provides us with a geometric perspective on these issues. Suppose

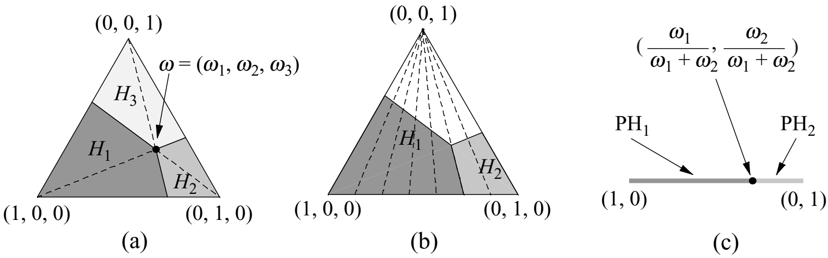


Figure 10.4

$P = \langle P_1, P_2, P_3 \rangle$ is a partition of C among Player 1, Player 2, and Player 3, and that $P \in \text{Part}^+$.

Concerning the truth of Theorem 6.2, suppose that P is Pareto maximal. Then, by Theorem 10.9, P is w-associated with some $\omega \in S^+$. Consider Figure 10.4. Figure 10.4a is the same as Figure 10.3. As we described previously, the point $\omega = (\omega_1, \omega_2, \omega_3)$ determines the three closed regions H_1, H_2, and H_3 of the simplex. We assume that each point of P_1 is associated with a point in H_1, each point of P_2 is associated with a point in H_2, and each point of P_3 is associated with a point in H_3. Then P is w-associated with ω.

To see that P is proper subpartition Pareto maximal, let us see why, for example, the partition $\langle P_1, P_2 \rangle$ is a Pareto maximal partition of $P_1 \cup P_2$ between Player 1 and Player 2. We wish to consider the associated RNS, which lies on the one-simplex consisting of the line segment between Player 1's vertex, (1, 0, 0), and Player 2's vertex, (0, 1, 0), but which we shall now think of as the line segment in $\mathbf{R}^2$ between Player 1's vertex, (1, 0), and Player 2's vertex, (0, 1). We can view this RNS as arising by taking each point in the original RNS that is associated with a point in $P_1 \cup P_2$ (i.e., each point of the RNS that is in $H_1 \cup H_2$), forgetting about the third coordinate, and dividing each of the two remaining coordinates by their sum. Then the ratio of these coordinates remains unchanged but their sum becomes one; hence, the resulting ordered pair is in the one-simplex. Geometrically, this corresponds to simply projecting along a line segment from the point (0, 0, 1), through the given point, to a point on the line segment between (1, 0, 0) and (0, 1, 0). This is illustrated in Figure 10.4b. Each point along any dashed line is projected to the point of intersection of that dashed line with the line segment between (1, 0, 0) and (0, 1, 0).

The result of this projecting is shown is Figure 10.4c, where PH_1 denotes the projection of H_1 and PH_2 denotes the projection of H_2. Note that the projection of the point $\omega = (\omega_1, \omega_2, \omega_3)$ is the point $(\frac{\omega_1}{\omega_1+\omega_2}, \frac{\omega_2}{\omega_1+\omega_2})$. It is clear that every point in PH_1 is on the line segment between (1, 0) and $(\frac{\omega_1}{\omega_1+\omega_2}, \frac{\omega_2}{\omega_1+\omega_2})$, and every point in PH_2 is on the line segment between $(\frac{\omega_1}{\omega_1+\omega_2}, \frac{\omega_2}{\omega_1+\omega_2})$ and (0, 1). Hence,

$\langle P_1, P_2\rangle$ is w-associated with $(\frac{\omega_1}{\omega_1+\omega_2}, \frac{\omega_2}{\omega_1+\omega_2})$ and therefore is a Pareto maximal partition of $P_1 \cup P_2$ between Player 1 and Player 2.

Concerning the failure of the converse of Theorem 6.2, we return to Example 6.3. For convenience, we repeat that example here. The cake C is the interval $[0, 3)$ on the real number line. There are three players, Player 1, Player 2, and Player 3, and we define their measures m_1, m_2, and m_3, respectively, as follows, where m_L denotes Lebesgue measure on C: for any $A \subseteq C$,

$$\begin{aligned} m_1(A) &= .3m_L(A \cap [0, 1)) + .1m_L(A \cap [1, 2)) + .6m_L(A \cap [2, 3)) \\ m_2(A) &= .6m_L(A \cap [0, 1)) + .3m_L(A \cap [1, 2)) + .1m_L(A \cap [2, 3)) \\ m_3(A) &= .1m_L(A \cap [0, 1)) + .6m_L(A \cap [1, 2)) + .3m_L(A \cap [2, 3)) \end{aligned}$$

Let $P = \langle [0, 1), [1, 2), [2, 3)\rangle$. In Example 6.3, we showed that P is not Pareto maximal but is proper subpartition Pareto maximal. We shall illustrate the idea here using the notion of w-associated. Let us compute the RNS associated with the cake and measures of this example. We recall that, by definition, $\mu = m_1 + m_2 + m_3$.

For any $B \subseteq [0, 1)$,

$$\begin{aligned} \mu(B) &= m_1(B) + m_2(B) + m_3(B) = .3m_L(B) + .6m_L(B) + .1m_L(B) \\ &= m_L(B). \end{aligned}$$

Hence, for any such B, $m_1(B) = .3\mu(B)$, $m_2(B) = .6\mu(B)$, and $m_3(B) = .1\mu(B)$. This implies that, for almost every $a \in [0, 1)$, $f_1(a) = .3$, $f_2(a) = .6$, and $f_3(a) = .1$. We may assume (as discussed in the concluding paragraph of Chapter 9) that this is true for *every* $a \in [0, 1)$. Thus, for every such a, $f(a) = (.3, .6, .1)$. Similarly, for every $a \in [1, 2)$, $f(a) = (.1, .3, .6)$ and, for every $a \in [2, 3)$, $f(a) = (.6, .1, .3)$. Hence, the RNS for this example consists of the three-point set $\{(.3, .6, .1), (.1, .3, .6), (.6, .1, .3)\}$. It is pictured in Figure 10.5.

Recall that $P = \langle [0, 1), [1, 2), [2, 3)\rangle$. Then the cake given to Player 1, Player 2, and Player 3 corresponds to the points $\{(.3, .6, .1), (.1, .3, .6)$, and $(.6, .1, .3)\}$, respectively, and we have indicated these points in Figures 10.5a and 10.5b by ω^1, ω^2, and ω^3, respectively. Using Figure 10.5a and the notion of w-associated, it is not hard to see that P is proper subpartition Pareto maximal. For example, to see that $\langle [1, 2), [2, 3)\rangle$ is a Pareto maximal partition of $[1, 3)$ between Player 2 and Player 3, we consider the relevant simplex, which is the two-simplex consisting of the line segment between Player 2's vertex, $(0, 1, 0)$, and Player 3's vertex, $(0, 0, 1)$. The associated RNS is obtained, as described earlier, by projecting all points of the original RNS that correspond to points in $[1, 3)$, to this simplex, along the line segment from $(1, 0, 0)$. The

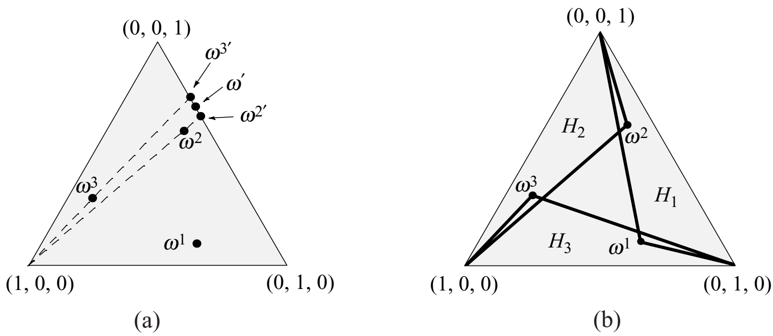


Figure 10.5

points $\omega^2 = (.1, .3, .6)$ and $\omega^3 = (.6, .1, .3)$ are the only such points (i.e., ω^1 does not correspond to a point in this set). The projections of these points are $\omega^{2'} = (0, \frac{.3}{.3+.6}, \frac{.6}{.3+.6}) = (0, \frac{1}{3}, \frac{2}{3})$ and $\omega^{3'} = (0, \frac{.1}{.1+.3}, \frac{.3}{.1+.3}) = (0, \frac{1}{4}, \frac{3}{4})$, as indicated in the figure. Since $\frac{1}{3} > \frac{1}{4}$ and $\frac{2}{3} < \frac{3}{4}$, it follows that $\omega^{2'}$ is closer to Player 2's vertex that is $\omega^{3'}$, and $\omega^{3'}$ is closer to Player 3's vertex than is $\omega^{2'}$. Therefore, with ω' any point between $\omega^{2'}$ and $\omega^{3'}$, as shown, the partition $\langle [1, 2), [2, 3) \rangle$ is w-associated with ω. Hence, by Theorem 10.9, $\langle [1, 2), [2, 3) \rangle$ is a Pareto maximal partition of $[1, 3)$ between Player 2 and Player 3. Similar arguments show that $\langle [0, 1), [2, 3) \rangle$ is a Pareto maximal partition of $[0, 1) \cup [2, 3)$ between Player 1 and Player 3, and that $\langle [0, 1), [1, 2) \rangle$ is a Pareto maximal partition of $[0, 2)$ between Player 1 and Player 2.

Next, we wish to use Figure 10.5b to illustrate the fact that P is not Pareto maximal. Suppose, by way of contradiction, that P is Pareto maximal. Then, by Theorem 10.9, P is w-associated with some ω in S^+. Consider the triangular regions H_1, H_2, and H_3 in the figure. Since the cake associated with ω^1 is to go to Player 1, it is not hard to see that ω must be in H_1. Similarly, since the cake associated with ω^2 is to go to Player 2 and the cake associated with ω^3 is to go to Player 3, ω must also be in H_2 and H_3. But $H_1 \cap H_2 \cap H_3 = \emptyset$. Hence, there is no such point ω and it follows that P is not Pareto maximal.

Example 10.11 We wish to consider the relationship between the following two statements, where P is a partition that gives a piece of cake of positive measure to each player:

a. P is Pareto maximal.
b. $P = \langle Q_1 \cup R_1, Q_2 \cup R_2, \ldots, Q_n \cup R_n \rangle$ where $C = A \cup B$, $A \cap B = \emptyset$, $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ is a Pareto maximal partition of A, and $R = \langle R_1, R_2, \ldots, R_n \rangle$ is a Pareto maximal partition of B.

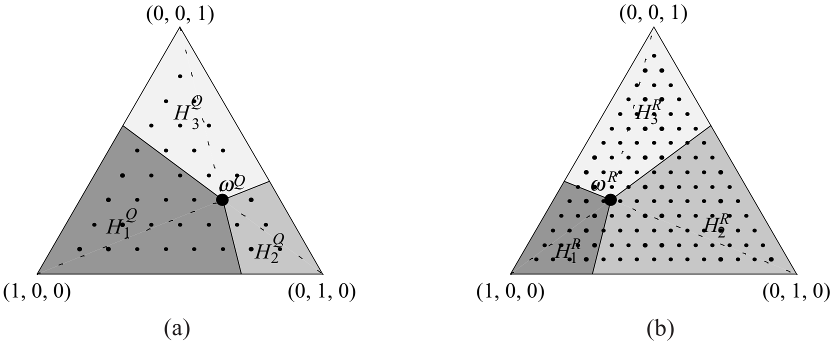


Figure 10.6

It is not hard to see (and we shall discuss this in light of Theorem 10.9 shortly) that statement a implies statement b. Whether statement b implies statement a is not so clear. We investigate this question in a slightly informal manner.

Suppose that there are three players, Player 1, Player 2, and Player 3, with measures m_1, m_2, and m_3, respectively. Assume that $C = A \cup B$, $A \cap B = \emptyset$, $Q = \langle Q_1, Q_2, Q_3 \rangle$ is a Pareto maximal partition of A in which every player receives a piece of cake of positive measure, and $R = \langle R_1, R_2, R_3 \rangle$ is a Pareto maximal partition of B in which every player receives a piece of cake of positive measure. Let us suppose that the three players' measures vary in very different ways from each other on A and also on B, and so the RNS associated with each of these sets contains many points throughout the simplex. This is illustrated in Figure 10.6. The displayed points in Figure 10.6a are the points of the RNS that correspond to A and the displayed points in Figure 10.6b are the points of the RNS that correspond to B.

By Theorem 10.9, there are points $\omega^Q, \omega^R \in S^+$ such that Q is w-associated with ω^Q and R is w-associated with ω^R. These points are shown in the figures. The regions H_1^Q, H_2^Q, and H_3^Q and the regions H_1^R, H_2^R, and H_3^R are the regions of the simplex determined by ω^Q and ω^R, respectively (as described previously and illustrated in Figure 10.3). Then, $f(Q_1) \subseteq H_1^Q$, $f(Q_2) \subseteq H_2^Q$, $f(Q_3) \subseteq H_3^Q$, $f(R_1) \subseteq H_1^R$, $f(R_2) \subseteq H_2^R$, and $f(R_3) \subseteq H_3^R$. Let $P = \langle Q_1 \cup R_1, Q_2 \cup R_2, Q_3 \cup R_3 \rangle$. Then P is a partition of C. We claim that P is not Pareto maximal.

If P were Pareto maximal, that P would be w-associated with some $\omega \in S^+$. Also, if P is w-associated with ω, then Q and R must each be w-associated with ω. However, given the RNSs for A and B as shown in Figure 10.6, it is clear that any point that is not close to ω^Q will not yield the partition Q of A, and any point that is not close to ω^R will not yield the partition R of B. Hence,

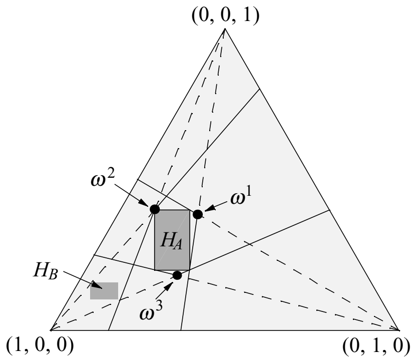


Figure 10.7

there is no such point ω with which P is w-associated and hence, by Theorem 10.9, P is not Pareto maximal. This establishes that statement b does not imply statement a.

It is easy to see that statement a implies statement b. If P is Pareto maximal, then P is w-associated with some $\omega \in S^+$. If A, B, Q, and R are as in statement b then, as noted in the preceding paragraph, the partition Q of A and the partition R of B are both w-associated with ω. Hence, Q is a Pareto maximal partition of A, and R is a Pareto maximal partition of B.

Suppose that $A \subseteq C$ has positive measure. Can we give A to any player we wish and still obtain a Pareto maximal partition? In other words, for each $i = 1, 2, \ldots, n$, does there exist a Pareto maximal partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ such that $A \subseteq P_i$? The answer is "obviously, yes!" since we can simply give all of C to Player i. Can we do better? Somewhat informally stated, the question is this: for any $i = 1, 2, \ldots, n$, how can we give A, and as little additional cake as possible, to Player i, and obtain a Pareto maximal partition? We illustrate the answer informally in the following example.

Example 10.12 Suppose there are three players, Player 1, Player 2, and Player 3. In addition, assume that $A \subseteq C$ has positive measure and $f(A) = H_A$ is as shown in Figure 10.7. Consider the three points ω^1, ω^2, and ω^3 shown in the figure. By Theorem 10.9, any partition that is w-associated with any of these points is Pareto maximal. It is clear from the figure that

- there are partitions w-associated with ω^1 that give all of A to Player 1.
- there are partitions w-associated with ω^2 that give all of A to Player 2.
- there are partitions w-associated with ω^3 that give all of A to Player 3.

(It may not be true that *all* partitions that are w-associated with ω^1 give all of A to Player 1. If one of the points of intersection of H_A with one of the relevant lines in the figure corresponds to a piece of cake of positive measure, then there exist partitions that are w-associated with ω^1 and do not give all of A to Player 1. A similar statement holds for ω^2 and ω^3.)

It is not hard to see that in each case there may be some cake in addition to A that must be presented to the given player in order to create a Pareto maximal partition. (This will depend on the location of the part of the RNS that we have not included in the picture, i.e., $f(C\backslash A)$.) For example, if $B \subseteq C$ has positive measure and $f(B) = H_B$ is as shown in the figure, then any Pareto maximal partition that gives all of A to Player 1 must also give all of B to Player 1. On the other hand, it is clear from the figure that any partition that gives all of A to Player 1, is Pareto maximal, and gives Player 1 a minimal piece subject to these two conditions, must be w-associated with ω^1. A similar statement holds for Player 2 and Player 3, using ω^2 and ω^3, respectively.

Why did we insist throughout this section that the point ω be chosen from the interior of the simplex and, in Theorem 10.9, that the partition P give a piece of cake of positive measure to each player? We now consider these two questions. Later in this section (see Theorem 10.15), we shall present a characterization of Pareto maximality that removes this restriction.

One answer to our first question appeals to our use of the function RD, given in Definition 10.5 and applied in the proof of Theorem 10.6. If $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is on the boundary of S, then at least one of its coordinates is zero. How do we then define RD(ω)? If exactly one of ω's coordinates is zero, then there is a reasonable way to define RD(ω). Suppose that, for some $i = 1, 2, \ldots, n$, $\omega_i = 0$ and, for all $j = 1, 2, \ldots, n$ with $j \neq i$, $\omega_j > 0$. We can define RD(ω) to be the element of S with a 1 in the ith position and 0s elsewhere. However, what if two coordinates of ω are 0? Let us suppose that, for some distinct $i, j = 1, 2, \ldots, n$, $\omega_i = \omega_j = 0$ and, for any $k = 1, 2, \ldots, n$ with $k \neq i$ and $k \neq j$, $\omega_k > 0$. Then certainly for any such k, we must define the kth coordinate of RD(ω) to be zero, and the sum of the ith and the jth coordinates must be one. But precisely what these two coordinates should be is unclear.

We next give a different perspective on our first question. Suppose $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is a point in the simplex. If ω is on the boundary, then at least one of the coordinates of ω is zero. Let us suppose that $\omega_i = 0$ and, for all $j = 1, 2, \ldots, n$ with $j \neq i$, $\omega_j > 0$. For any such j, $\frac{\omega_i}{\omega_j} = 0$ and we may set $\frac{\omega_j}{\omega_i} = \infty$. Part b of Lemma 9.1 implies that $0 < \frac{f_i(a)}{f_j(a)} < \infty$ for every $a \in C$. It follows that if $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition w-associated with ω then, except possibly for a set of measure zero, $P_i = C$ and, for every $j \neq i$, $P_j = \emptyset$. Thus, we see that if ω has coordinate i equal to zero and all other coordinates not equal to zero, then any partition w-associated with ω gives almost all of C to Player i. This situation seems to be perfectly acceptable. If, for example, there are three players, Player 1, Player 2, and Player 3, then choosing ω to be any point on the open line segment between (1, 0, 0) and (0, 1, 0) (i.e., between Player 1's and Player 2's vertices) results in a partition that gives all of the cake to Player 3. This is also consistent with the geometric perspective given in Figure 10.3.

However, as in our previous explanation, problems arise if we allow ω to have more than one coordinate equal to zero. If, for distinct $i, j = 1, 2, \ldots, n$, ω is such that $\omega_i = 0$ and $\omega_j = 0$, then $\frac{\omega_i}{\omega_j} = \frac{0}{0}$, and it is not clear how to evaluate the truth or falsity of an expression of the form "$\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$" in Definition 10.4. Let us again suppose that there are three players, Player 1, Player 2, and Player 3. Choosing ω to be (0, 0, 1), for example, which is Player 3's vertex, tells us to give no cake to Player 3 and to split the cake between Player 1 and Player 2. But, it does not tell us *how* to split the cake between Player 1 and Player 2. Thus we see, just as in our first explanation, that having one coordinate of ω equal to zero is fine, but having two coordinates equal to zero is problematic.

Next, we discuss our second question: why did we insist that partitions give each player a piece of cake of positive measure? As before, we give two answers. The first will again appeal to problems in Definition 10.5 and its use in Theorem 10.6. Consider first the two-player context and the two IPSs in Figure 10.8. (Figures 10.8a and 10.8b are repeats of Figures 2.1c and 2.1b,

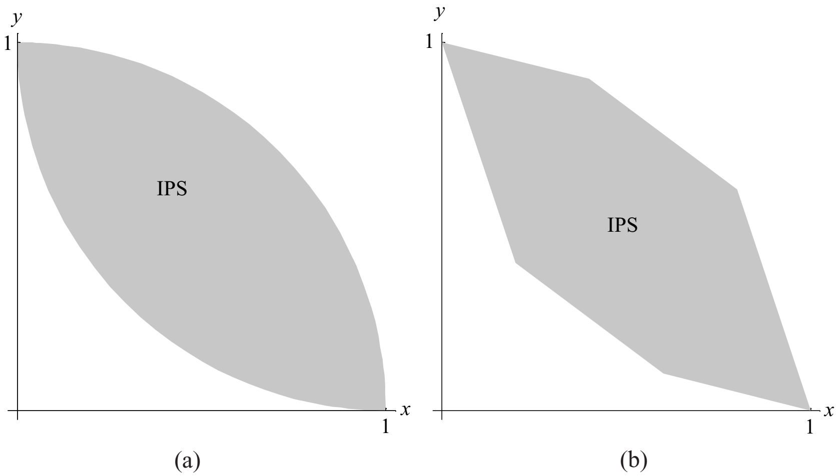


Figure 10.8

respectively.) When there are two players, the point (0, 1) is, of course, in the IPS, and is a point of first contact of the horizontal family of parallel lines with the IPS. The IPS in Figure 10.8a has a horizontal tangent at this point. This implies that the horizontal family of parallel lines is the *only* family of parallel non-negative lines that makes first contact with this IPS at (0, 1). (For the definition of non-negative line, see Definition 7.1.) On the other hand, the IPS in Figure 10.8b does not have a unique tangent line at (0, 1) and, hence, there are many families of parallel non-negative lines in addition to the horizontal family of parallel lines that make first contact with this IPS at (0, 1). Since the partition that gives all of the cake to Player 2 (i.e., the partition $\langle \phi, C \rangle$) corresponds to the point (0, 1), these observations tell us that this partition maximizes only the convex combination of measures corresponding to $\alpha = (0, 1)$ for the IPS in Figure 10.8a, but maximizes many convex combinations of measures besides the one corresponding to $\alpha = (0, 1)$ for the IPS in Figure 10.8b. In applying Definition 10.5 to α (as we did in the proof of Theorem 10.6), we have the same issue discussed earlier. One of the coordinates of α is equal to zero. As we have seen, there is a perfectly reasonable way to define RD(α) in this case.

However, consider the case of three players. The partition $\langle C, \phi, \phi \rangle$ maximizes the convex combination of measures corresponding to $\alpha = (1, 0, 0)$ and, as in the case of two players, it is not hard to see on geometric grounds that this may be the only convex combination of measures maximized by $\langle C, \phi, \phi \rangle$. (As in the two-player context, this will depend on the exact shape of the IPS. We shall have much more to say about this in Chapter 12.) Then, in applying Definition 10.5 to α, we have an apparently irresolvable problem (as previously discussed), since two of α's coordinates are equal to zero.

We next give a different reason to only consider partitions that give a piece of cake of positive measure to each player. This reason involves looking carefully at the RNS. We will illustrate by considering the contrast between the following two examples.

Example 10.13 Let C be the interval $[0, 2)$ on the real number line, let m_L be Lebesgue measure on this interval, and define m_1 and m_2 on C as follows: for any $A \subseteq C$,

$$m_1(A) = \frac{1}{3} m_L\left(A \cap [0, 1)\right) + \frac{2}{3} m_L\left(A \cap [1, 2)\right)$$

and

$$m_2(A) = \frac{2}{3} m_L\left(A \cap [0, 1)\right) + \frac{1}{3} m_L\left(A \cap [1, 2)\right)$$

Then, for any $A \subseteq [0, 1)$, $m_2(A) = 2m_1(A)$ and, for any $A \subseteq [1, 2)$, $m_1(A) = 2m_2(A)$. This implies that, for almost every $a \in [0, 1)$, $f_1(a) = \frac{1}{3}$ and $f_2(a) = \frac{2}{3}$,

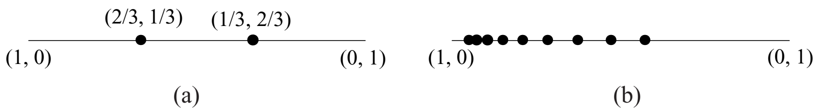


Figure 10.9

and, for almost every $a \in [1, 2)$, $f_1(a) = \frac{2}{3}$ and $f_2(a) = \frac{1}{3}$. Hence, for almost every $a \in [0, 1)$, $f(a) = (\frac{1}{3}, \frac{2}{3})$, and, for almost every $a \in [1, 2)$, $f(a) = (\frac{2}{3}, \frac{1}{3})$. We may assume (as discussed in the concluding paragraph of Chapter 9) that for *every* $a \in [0, 1)$, $f(a) = (\frac{1}{3}, \frac{2}{3})$ and for *every* $a \in [1, 2)$, $f(a) = (\frac{2}{3}, \frac{1}{3})$. It follows that the RNS consists of two points, $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$, as shown in Figure 10.9a.

Example 10.14 Let C be the non-negative real numbers, let m_L denote Lebesgue measure on this set, and define m_1 and m_2 on C as follows: for any $A \subseteq C$,

$$m_1(A) = \left(\frac{1}{2}\right) m_L\,(A \cap [0, 1)) + \left(\frac{1}{4}\right) m_L\,(A \cap [1, 2)) + \left(\frac{1}{8}\right) m_L\,(A \cap [2, 3)) + \cdots + \left(\frac{1}{2^k}\right) m_L\,(A \cap [k-1, k)) + \cdots$$

and

$$m_2(A) = \left(\frac{2}{3}\right) m_L\,(A \cap [0, 1)) + \left(\frac{2}{9}\right) m_L\,(A \cap [1, 2)) + \left(\frac{2}{27}\right) m_L\,(A \cap [2, 3)) + \cdots + \left(\frac{2}{3^k}\right) m_L\,(A \cap [k-1, k)) + \cdots$$

It is straightforward to show that m_1 and m_2 are countably additive and non-atomic, since m_L has these properties. We must show that $m_1(C) = m_2(C) = 1$. We do this as follows:

$$\begin{aligned} m_1(C) &= \left(\frac{1}{2}\right) m_L\,([0, 1)) + \left(\frac{1}{4}\right) m_L\,([1, 2)) + \left(\frac{1}{8}\right) m_L\,([2, 3)) + \cdots \\ &\quad + \left(\frac{1}{2^k}\right) m_L\,([k-1, k)) + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \cdots \\ &= \frac{\frac{1}{2}}{1 - \left(\frac{1}{2}\right)} = 1 \end{aligned}$$

and

$$m_2(C) = \left(\frac{2}{3}\right) m_L([0,1)) + \left(\frac{2}{9}\right) m_L([1,2)) + \left(\frac{2}{27}\right) m_L([2,3)) + \cdots$$
$$+ \left(\frac{2}{3^k}\right) m_L([k-1,k)) + \cdots$$
$$= \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^k} + \cdots$$
$$= \frac{\frac{2}{3}}{1-\left(\frac{1}{3}\right)} = 1$$

The third equality in each string of equalities uses the standard formula for summing a geometric series. Thus, m_1 and m_2 are measures. We wish to examine the RNS in this situation.

For any positive integer k, $m_1([k-1,k)) = (\frac{1}{2^k})m_L([k-1,k))$ and $m_2([k-1,k)) = (\frac{2}{3^k})m_L([k-1,k))$. Hence,

$$\frac{m_1([k-1,k))}{m_2([k-1,k))} = \frac{\left(\frac{1}{2^k}\right) m_L([k,k-1))}{\left(\frac{2}{3^k}\right) m_L([k,k-1))} = \frac{\left(\frac{1}{2^k}\right)}{\left(\frac{2}{3^k}\right)} = \frac{3^k}{2^{k+1}},$$

and thus $m_1([k,k-1)) = (\frac{3^k}{2^{k+1}})m_2([k,k-1))$. This implies that, for almost every $a \in [k-1,k)$, $f_1(a) = (\frac{3^k}{2^{k+1}})f_2(a)$. Then, since $f_1(a) + f_2(a) = 1$, it follows that

$$f_1(a) = \left(\frac{3^k}{2^{k+1}}\right) f_2(a) = \left(\frac{3^k}{2^{k+1}}\right)(1 - f_1(a)) = \left(\frac{3^k}{2^{k+1}}\right) - \left(\frac{3^k}{2^{k+1}}\right) f_1(a).$$

Therefore, $f_1(a)(1 + \frac{3^k}{2^{k+1}}) = \frac{3^k}{2^{k+1}}$ and, hence,

$$f_1(a) = \frac{\frac{3^k}{2^{k+1}}}{1 + \frac{3^k}{2^{k+1}}} = \frac{3^k}{2^{k+1} + 3^k}.$$

Also,

$$f_2(a) = 1 - f_1(a) = 1 - \frac{3^k}{2^{k+1} + 3^k} = \frac{2^{k+1}}{2^{k+1} + 3^k}$$

and so

$$f(a) = \left(\frac{3^k}{2^{k+1} + 3^k}, \frac{2^{k+1}}{2^{k+1} + 3^k}\right).$$

As usual, we may assume that, for *every* $a \in [k-1,k)$,

$$f(a) = \left(\frac{3^k}{2^{k+1} + 3^k}, \frac{2^{k+1}}{2^{k+1} + 3^k}\right).$$

This tells us that for each positive integer k the interval $[k-1, k)$ corresponds to the point $(\frac{3^k}{2^{k+1}+3^k}, \frac{2^{k+1}}{2^{k+1}+3^k})$ in the RNS. Hence, the RNS consists of all points of the form $(\frac{3^k}{2^{k+1}+3^k}, \frac{2^{k+1}}{2^{k+1}+3^k})$ for $k = 1, 2, \ldots$. Notice that

$$\lim_{k\to\infty}\left(\frac{3^k}{2^{k+1}+3^k}\right) = \lim_{k\to\infty}\left(\frac{1}{\left(\frac{2^{k+1}}{3^k}\right)+1}\right) = \lim_{k\to\infty}\left(\frac{1}{(2)\left(\frac{2}{3}\right)^k+1}\right) = 1$$

and

$$\lim_{k\to\infty}\left(\frac{2^{k+1}}{2^{k+1}+3^k}\right) = \lim_{k\to\infty}\left(1-\frac{3^k}{2^{k+1}+3^k}\right) = 1-1 = 0.$$

Hence, the sequence of points $(\frac{3^k}{2^{k+1}+3^k}, \frac{2^{k+1}}{2^{k+1}+3^k})$ for $k = 1, 2, \ldots$ has limit $(1, 0)$. This is illustrated in Figure 10.9b. (In the figure, we have shown only nine of the infinitely many points of the RNS.)

Let us consider the contrast between the previous two examples. Suppose that we wish to give all of the cake to Player 2. The RNSs for each of these situations, as shown in Figures 10.9a and 10.9b, are very different. For the cake and measures described in Example 10.13 and pictured in Figure 10.9a, if we pick any point ω in the open interval between $(1, 0)$ and $(\frac{2}{3}, \frac{1}{3})$, then the only partition that is w-associated with ω is the partition $\langle\phi, C\rangle$, i.e., the partition that gives all of C to Player 2. (Notice that if $\omega = (\frac{2}{3}, \frac{1}{3})$ then there are many partitions that are w-associated with ω, one of which is $\langle\phi, C\rangle$.) The key point here is that there is a gap in the simplex between $(1, 0)$ and the first point of the RNS associated with a piece of cake of positive measure.

In contrast with this situation, we now consider the cake and measures described in Example 10.14 and pictured in Figure 10.9b. If ω were a point of the simplex with which the partition $\langle\phi, C\rangle$ is w-associated then, as in the preceding paragraph, ω would have to be to the left of any point of the RNS associated with a piece of cake of positive measure. But in this example, the point $(1, 0)$ is a limit of such points and therefore, in contrast with the previous example, there is no gap in which to place ω. Hence, $\langle\phi, C\rangle$ is a partition that is not w-associated with any ω in S^+. This example illustrates that our insistence on considering only points ω in the interior of the simplex in Theorem 10.9 necessitates that we consider only partitions that give each player a piece of cake of positive measure.

Lemma 10.8 told us that our restriction to partitions that give a piece of cake of positive measure to each player is not a major restriction. As we discussed following the proof, this lemma tells us that we can simply ignore players that receive no cake. We can now be more specific by combining this idea with Theorem 10.9. As in Lemma 10.8, for any partition $P = \langle P_1, P_2, \ldots, P_n\rangle$, let

$\delta_P = \{i \leq n : P_i$ has positive measure$\}$. In addition, let S_{δ_P} be the $(|\delta_P| - 1)$-simplex. We shall identify players named by δ_P with the $|\delta_P|$ vertices of the $(|\delta_P| - 1)$-simplex in the natural order-preserving way. For $\omega \in S^+_{\delta_P}$ (i.e., for ω in the interior of S_{δ_P}), we shall say that the partition $\langle P_i : i \in \delta_P \rangle$ of $\bigcup_{i \in \delta_P} P_i$ is w-associated with ω if and only if Definition 10.4 holds for the partition $\langle P_i : i \in \delta_P \rangle$ and the point ω, with this identification.

Theorem 10.15 *Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$. P is Pareto maximal if and only if the partition $\langle P_i : i \in \delta_P \rangle$ of $\bigcup_{i \in \delta_P} P_i$ is w-associated with ω for some $\omega \in S^+_{\delta_P}$.*

Proof: Fix $P = \langle P_1, P_2, \ldots, P_n \rangle$. Then $\langle P_i : i \in \delta_P \rangle$ is a partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P and each of these players receives a piece of cake of positive measure.

By Lemma 10.8, P is Pareto maximal if and only if $\langle P_i : i \in \delta_P \rangle$ is a Pareto maximal partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P. By Theorem 10.9, $\langle P_i : i \in \delta_P \rangle$ is a Pareto maximal partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P if and only if it is w-associated with ω for some $\omega \in S^+_{\delta_P}$. This establishes the theorem. □

We conclude this section by considering the chores versions of the results of this section. All of the results adjust in a natural way, once we make the appropriate adjustment in the notion of w-associated, as given in Definition 10.4. This adjustment involves simply reversing the relevant inequality.

Definition 10.16 Suppose that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition and $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. We shall say that P is *chores w-associated with ω* if and only if the following holds for all distinct $i, j = 1, 2, \ldots, n$:

$$\frac{f_i(a)}{f_j(a)} \leq \frac{\omega_i}{\omega_j} \text{ for almost every } a \in P_i$$

We used Figure 10.3 to give a geometric perspective on what it means for a partition P to be w-associated with a point $\omega \in S^+$. The analogous perspective for chores w-associated is given in Figure 10.10. In the standard context, as illustrated in Figure 10.3, we think of the point ω as determining regions H_1, H_2, and H_3, that are *close to* Player 1, Player 2, and Player 3, respectively (since bits of cake associated with closer points in the RNS are more desirable). Now, in the chores context, as illustrated in Figure 10.10, we think of the point ω as determining regions H_1, H_2, and H_3 that are *far from* Player 1, Player 2, and Player 3, respectively (since bits of cake associated with farther away points in the RNS are more desirable). For ω as in the figure, a partition is chores w-associated with ω if and only if, for each $i = 1, 2, 3$, the points in

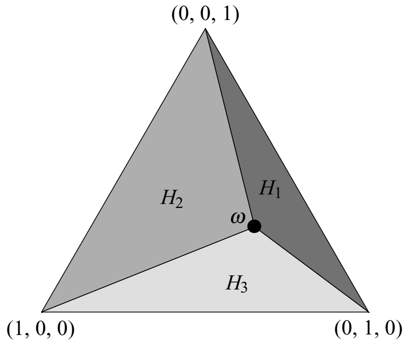


Figure 10.10

C corresponding to interior points of an H_i go to Player i, and points of C corresponding to boundary points of some H_i go to any of the players associated with this boundary.

The adjustments to Theorem 10.6, Corollary 10.7, Lemma 10.8, and Theorems 10.9 and 10.15 are straightforward and are presented as Theorem 10.17, Corollary 10.18, Lemma 10.19, and Theorems 10.20 and 10.21, respectively. The proofs of Theorem 10.17, Corollary 10.18, and Theorems 10.20 and 10.21 are trivial adjustments of the proofs of Theorem 10.6, Corollary 10.7, and Theorems 10.9 and 10.15, respectively, and we omit them. The proof of Lemma 10.19 is easier than was the proof of Lemma 10.8, and we include it in the following.

Theorem 10.17 *Fix partition $P = \langle P_1, P_2, \ldots, P_n \rangle$, $\omega \in S^+$, and $\alpha \in S^+$, with $\alpha = \mathrm{RD}(\omega)$. P is chores w-associated with ω if and only if P minimizes the convex combination of measures corresponding to α.*

Corollary 10.18

a. *A p-class of partitions minimizes more than one convex combination of measures corresponding to points of S^+ if and only if it is chores w-associated with more than one point of S^+.*
b. *Fix $\alpha \in S^+$. More than one p-class of partitions minimizes the convex combination of measures corresponding to α if and only if more than one p-class of partitions is chores w-associated with* $\mathrm{RD}(\alpha)$.

Lemma 10.19 *Let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition and set $\delta_P = \{i \le n :$ P_i has positive measure$\}$. Then P is Pareto minimal if and only if the partition $\langle P_i : i \in \delta_P \rangle$ is a Pareto minimal partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P.*

The difference between the proofs of Lemmas 10.8 and 10.19 is in the reverse direction. For the proof of the reverse direction of Lemma 10.8, we assumed, by way of contradiction, that partition Q was Pareto bigger than P. We then considered two cases, depending on whether or not Q gave a piece of cake of positive measure to some player that received a piece of cake of measure zero in partition P. For the present result, this distinction does not arise. If Q is Pareto smaller than P, then it is certainly not possible that some player who receives a piece of cake of measure zero in partition P receives a piece of cake of positive measure in partition Q.

Proof of Lemma 10.19: Let P and δ_P be as in the statement of the lemma.

As was the case for Lemma 10.8, the forward direction is trivial, since any partition $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P that is Pareto smaller than $\langle P_i : i \in \delta_P \rangle$ would immediately yield a partition of C among all players that is Pareto smaller than P.

For the reverse direction, suppose that P is not Pareto minimal and that Q is Pareto smaller than P. Then, $\{i \leq n : Q_i \text{ has positive measure}\} \subseteq \delta_P$. By redefining Q on a set of measure zero, if necessary, we may assume that $\langle Q_i : i \in \delta_P \rangle$ is a partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P. This implies that the partition $\langle Q_i : i \in \delta_P \rangle$ is a Pareto smaller partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P than is $\langle P_i : i \in \delta_P \rangle$. Hence, $\langle P_i : i \in \delta_P \rangle$ is not a Pareto minimal partition of $\bigcup_{i \in \delta_P} P_i$ among the players named by δ_P. □

Theorem 10.20 *Fix a partition $P \in \text{Part}^+$. P is Pareto minimal if and only if P is chores w-associated with ω for some $\omega \in S^+$.*

Theorem 10.21 *Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$. P is Pareto minimal if and only if the partition $\langle P_i : i \in \delta_P \rangle$ is chores w-associated with ω for some $\omega \in S_{\delta_P}$.*

10C. The Situation Without Absolute Continuity

In this section we make no general assumptions about absolute continuity. Our goal is to characterize Pareto maximality in this context. We present two approaches (Theorems 10.23 and 10.28.) We shall not need to assume (as we did for most of the previous section) that each player receives a piece of cake that he or she believes to be of positive measure.

As we did in the previous chapter, we adopt the convention that expressions such as "almost every" or "positive measure" refer to the measure $\mu = m_1 + m_2 + \cdots + m_n$ unless otherwise stated.

By Corollary 9.11, the failure of absolute continuity implies that the boundary of the RNS is associated with a piece of cake of positive measure. In other words, $\mu(\{a \in C : f(a) \text{ is on the boundary of the simplex}\}) > 0$.

Fix some $A \subseteq C$. In the presence of absolute continuity, there exist Pareto maximal partitions that give all of A to any player we wish. (See Example 10.12 and the paragraph preceding this example.) This is not the case if absolute continuity fails. If $m_i(A) = 0$ and $m_j(A) > 0$, then no Pareto maximal partition can given A to Player i, since if P is a partition that gives A to Player i, and Q is the partition that results from P by transferring A from Player i to Player j, then Q is Pareto bigger than P and, hence, P is not Pareto maximal. Thus, Pareto maximality demands that any piece of cake that has positive measure to at least one player cannot be given to a player for whom that piece has measure zero. We wish to consider this idea in terms of the RNS. This requires that we refer to "faces of the simplex."

Although the notion of "a face of the simplex" is fairly intuitive (and we have already used this notion in Chapter 9), it will be useful to give a precise definition. For any non-empty $\delta \subseteq \{1, 2, \ldots, n\}$, we define the *face of the simplex corresponding to* δ to be $\{(p_1, p_2, \ldots, p_n) \in S : p_i = 0 \text{ for every } i \notin \delta\}$. Equivalently, the face of the simplex corresponding to δ is the convex hull of the set of vertices of players named by δ. Notice that we do not insist that δ be proper. Hence, we consider the whole simplex to be a face of itself. If we wish to exclude this possibility by requiring that δ be a proper subset of $\{1, 2, \ldots, n\}$, we shall use the term *proper face*.

For any $p = (p_1, p_2, \ldots, p_n)$ in the simplex, let $\delta_p^0 = \{i \leq n : p_i = 0\}$ and let $\delta_p^+ = \{i \leq n : p_i > 0\} = \{1, 2, \ldots, n\} \backslash \delta_p^0$. Note that any point p of the simplex is on the face corresponding to δ_p^+. Suppose that $a \in C$ and that $f(a)$ is on the boundary of the simplex. Then $\delta_{f(a)}^0$ is non-empty and $f(a)$ is on the face of the simplex corresponding to $\delta_{f(a)}^+$. For each $i \in \delta_{f(a)}^0$, Player i gives value zero to point a. Hence, in any Pareto maximal partition, point a cannot go to any player named by $\delta_{f(a)}^0$ but must be given to a player named by $\delta_{f(a)}^+$. Thus, cake corresponding to a point p of the RNS that is on the boundary of the simplex must be distributed among the players named by δ_p^+. (Of course, there can be a measure-zero set of exceptions.) In other words, for any $p \in$ RNS, if p is on a face of the simplex, then, in any Pareto maximal partition, all cake corresponding to p must be distributed among those players whose vertices determine this face. This is precisely the same as our previous conclusion that, for any $A \subseteq C$ and $i, j = 1, 2, \ldots, n$, if $m_i(A) = 0$ and $m_j(A) > 0$ then no Pareto maximal partition can give A to Player i.

Let us consider the definition of "the partition P is w-associated with $\omega \in S^+$" given by Definition 10.4. "The partition P is w-associated with $\omega \in S^+$" makes sense in the absence of absolute continuity as long as we adopt some arithmetic conventions. Consider the term "$\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$" in Definition 10.4.

Since $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$, $\frac{\omega_i}{\omega_j}$ is a positive number divided by a positive number and, hence, is a positive number. On the other hand, as discussed in the preceding paragraph, $f(a)$ may be on the boundary of the IPS, and thus one or both of $f_i(a)$ and $f_j(a)$ could be zero. Of course, an expression of the form "$\frac{0}{\text{positive number}} \geq$ positive number" is always false. We also adopt the natural convention that an expression of the form "$\frac{\text{positive number}}{0} \geq$ positive number" is always true.

How about expressions of the form "$\frac{0}{0} \geq$ positive number?" Suppose that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition, $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$, and we wish to consider what it means for P to be w-associated with ω. Fix some $a \in C$ and assume that, for some $i = 1, 2, \ldots, n$, $f_i(a) = 0$. Then we should not have $a \in P_i$. We ensure that this is the case by declaring any expression of the form "$\frac{0}{0} \geq$ positive number" to be false. (The alert reader may notice that, in our present context, it really does not matter whether we consider such an expression to be true or false. In applying Definition 10.4, it may be that, for some $j = 1, 2, \ldots, n$ with $j \neq i$, $\frac{f_i(a)}{f_j(a)} = \frac{0}{0}$ and, hence, an expression of the form "$\frac{0}{0} \geq$ positive number" does arise. However, since $f_1(a) + f_2(a) + \cdots + f_n(a) = 1$, we know that, for at least one $k = 1, 2, \ldots, n$, $f_k(a) > 0$. Then certainly the expression "$\frac{f_i(a)}{f_k(a)} \geq$ positive number" is false, and this guarantees that $a \notin P_i$, regardless of whether we consider "$\frac{0}{0} \geq$ positive number" to be true or false. However, it will be convenient for our work later in this section to adopt the convention that any such expression is false.)

In searching for the correct adjustment of Theorem 10.9 when absolute continuity fails, we note that the proof of Theorem 10.9 used Theorem 10.6. We claim that Theorem 10.6 holds in our present context, where we no longer assume that absolute continuity holds. While the proof of this result did not rely on absolute continuity, it did involve terms of the form $\frac{f_i(a)}{f_j(a)}$, and such terms may now have a zero in the numerator and/or the denominator. However, with our new arithmetic rules and our convention that "positive measure" refers to the measure $\mu = m_1 + m_2 + \cdots + m_n$, the given proof is still correct for our present setting. We also note that the proof of the reverse direction of the Theorem 10.9 did not require absolute continuity and, therefore, holds in our present setting. However, the forward direction of Theorem 10.9 might not hold if the measures are not absolutely continuous with respect to each other. This is established by the following.

Example 10.22 This example is similar to Example 10.14, but has a small addition. Let $C' = [-1/2, 0)$ and let C'' be the non-negative real numbers. We define the cake C to be $C' \cup C''$. We wish to define measures m_1 and m_2 on C. It suffices to define m_1 and m_2 on C' and on C'' since, for any $A \subseteq C$, $m_1(A) = m_1(A \cap C') + m_1(A \cap C'')$ and $m_2(A) = m_2(A \cap C') + m_2(A \cap C'')$. Let m_L denote Lebesgue measure on the real number line.

We first define m_1 and m_2 on C''. We do this as in Example 10.14, with a change in the constants for m_1. Fix $A \subseteq C''$. We define

$$m_1(A) = \left(\frac{1}{4}\right) m_L\left(A \cap [0, 1)\right) + \left(\frac{1}{8}\right) m_L\left(A \cap [1, 2)\right) + \left(\frac{1}{16}\right) m_L\left(A \cap [2, 3)\right) + \cdots + \left(\frac{1}{2^{k+1}}\right) m_L\left(A \cap [k-1, k)\right) + \cdots$$

and

$$m_2(A) = \left(\frac{2}{3}\right) m_L\left(A \cap [0, 1)\right) + \left(\frac{2}{9}\right) m_L\left(A \cap [1, 2)\right) + \left(\frac{2}{27}\right) m_L\left(A \cap [2, 3)\right) + \cdots + \left(\frac{2}{3^k}\right) m_L\left(A \cap [k-1, k)\right) + \cdots.$$

To define m_1 and m_2 on C', we let $m_1(A) = m_L(A)$ and $m_2(A) = 0$ for any $A \subseteq C'$.

We must show that $m_1(C) = m_2(C) = 1$. We do this as follows:

$$\begin{aligned} m_1(C) &= m_1(C') + m_1(C'') \\ &= \frac{1}{2} + \left(\frac{1}{4}\right) m_L\left([0, 1)\right) + \left(\frac{1}{8}\right) m_L\left([1, 2)\right) \\ &\quad + \left(\frac{1}{16}\right) m_L\left([2, 3)\right) + \cdots + \left(\frac{1}{2^{k+1}}\right) m_L\left([k-1, k)\right) + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k+1}} + \cdots \\ &= \frac{\frac{1}{2}}{1 - \left(\frac{1}{2}\right)} = 1 \end{aligned}$$

and

$$\begin{aligned} m_2(C) &= m_2(C') + m_2(C'') \\ &= 0 + \left(\frac{2}{3}\right) m_L\left([0, 1)\right) + \left(\frac{2}{9}\right) m_L\left([1, 2)\right) + \left(\frac{2}{27}\right) m_L\left([2, 3)\right) + \cdots \\ &\quad + \left(\frac{2}{3^k}\right) m_L\left([k-1, k)\right) + \cdots \\ &= \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^k} + \cdots \\ &= \frac{\frac{2}{3}}{1 - \left(\frac{1}{3}\right)} = 1 \end{aligned}$$

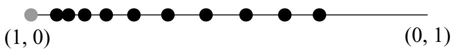


Figure 10.11

Thus, m_1 and m_2 are (countably additive, non-atomic, probability) measures. What is the corresponding RNS?

We first consider $f(a)$ for $a \in C'$. We have $m_1(C') = \frac{1}{2}$ and $m_2(C') = 0$. It follows that, for almost every $a \in C'$, $f_1(a) = 1$ and $f_2(a) = 0$. Hence, $f(a) = (1, 0)$. We may assume that, for *every* $a \in C'$, $f(a) = (1, 0)$.

Next, we consider $f(a)$ for $a \in C''$. Arguing precisely as in Example 10.14, we find that, for any positive integer k and almost every $a \in [k-1, k)$, $f_1(a) = \frac{3^k}{2^{k+2}+3^k}$ and $f_2(a) = \frac{2^{k+2}}{2^{k+2}+3^k}$, and therefore $f(a) = (\frac{3^k}{2^{k+2}+3^k}, \frac{2^{k+2}}{2^{k+2}+3^k})$. We may assume that, for *every* $a \in [k-1, k)$, $f(a) = (\frac{3^k}{2^{k+2}+3^k}, \frac{2^{k+2}}{2^{k+2}+3^k})$.

It follows that the RNS consists of the point $(1, 0)$ and all points of the form $(\frac{3^k}{2^{k+2}+3^k}, \frac{2^{k+2}}{2^{k+2}+3^k})$ for $k = 1, 2, \ldots$. This is illustrated in Figure 10.11. (We have lightened the point $(1, 0)$ to distinguish it from the other points in the RNS.) The important fact in what follows is that no point of the form $(\frac{3^k}{2^{k+2}+3^k}, \frac{2^{k+2}}{2^{k+2}+3^k})$ is equal to $(1, 0)$, but $\mathrm{Lim}_{k\to\infty}(\frac{3^k}{2^{k+2}+3^k}) = 1$ and $\mathrm{Lim}_{k\to\infty}(\frac{2^{k+2}}{2^{k+2}+3^k}) = 0$ and, therefore, the sequence of points $(\frac{3^k}{2^{k+2}+3^k}, \frac{2^{k+2}}{2^{k+2}+3^k})$ for $k = 1, 2, \ldots$ has limit $(1, 0)$. (These limits can be computed as in Example 10.14. The reader may notice a slight peculiarity in the figure. The points in the sequence go right to left, but get slightly farther apart early in the sequence before they start getting closer together.)

Since $m_1(C') > 0$ and $m_2(C') = 0$, we know that any Pareto maximal partition must give all of C' to Player 1. (To use the perspective developed earlier in this section, all of C' must go to Player 1 since it corresponds to the point $(1, 0)$ in the RNS and this point is on the face of the RNS determined by $(1, 0)$, which is Player 1's vertex.) Consider the partition $P = \langle C', C'' \rangle$. We claim that P is Pareto maximal but is not w-associated with any $\omega \in S^+$. It is not hard to see that this partition is Pareto maximal, since no transfer of cake from Player 1 to Player 2 will help Player 2 (since $m_2(C') = 0$), but any transfer from Player 2 to Player 1 that helps Player 1 will hurt Player 2 (since, for any $A \subseteq C''$, if $m_1(A) > 0$, then $m_2(A) > 0$). However, as we saw in Example 10.14, any $\omega \in S^+$ (i.e., any ω strictly between the points $(1, 0)$ and $(0, 1)$) with which the partition $\langle C', C'' \rangle$ is w-associated would have to be to the left of all of the points of the RNS associated with C''. These are the points $(\frac{3^k}{2^{k+2}+3^k}, \frac{2^{k+2}}{2^{k+2}+3^k})$ for $k = 1, 2, \ldots$, and these points have limit $(1, 0)$. Hence, there is no such point ω.

This example tells us that Theorem 10.9 does not hold if absolute continuity fails, and it also strongly suggests that the difficulty in finding an appropriate adjustment of Theorem 10.9 to our present setting lies in our restriction that ω lie in S^+, the interior of the simplex. We now wish to remove this restriction and make sense of "partition P is w-associated with point $\omega \in S$" when ω is on the boundary of S. To do this, we need to decide on the truth of the inequality $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ when ω_i or ω_j (or both) are equal to zero.

The four terms making up the two fractions in the expression "$\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$" can each be positive or zero. Thus, each fraction is either positive, zero, is of the form "$\frac{\text{positive number}}{0}$," which we shall refer to as "∞," or else is $\frac{0}{0}$. The truth of any such inequality that involves only zero and/or a positive number or numbers is trivial to evaluate, as usual. The inequalities involving ∞ but not $\frac{0}{0}$ are also easy to evaluate:

- "$\infty \geq 0$" is always true.
- "$0 \geq \infty$" is always false.
- "$\infty \geq$ positive number" is always true.
- "positive number $\geq \infty$" is always false.
- "$\infty \geq \infty$" is always true.

What about inequalities involving $\frac{0}{0}$? We previously adopted and justified the convention that inequalities of the form "$\frac{0}{0} \geq$ positive number" are always false. We now extend this to say that the inequalities $\frac{0}{0} \geq 0$, $\frac{0}{0} \geq \infty$, $\frac{0}{0} \geq \frac{0}{0}$, and $0 \geq \frac{0}{0}$ are all always false. Our reason is precisely as it was before, namely, to rule out any player receiving a point of cake to which he or she assigns value zero.

We now declare that the inequality $\infty \geq \frac{0}{0}$ and inequalities of the form "positive number $\geq \frac{0}{0}$" are always true. We justify this as follows. Suppose that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition, $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S$, and $\omega_i = \omega_j = 0$, where $i \neq j$. Then $\frac{\omega_i}{\omega_j} = \frac{0}{0}$. In Definition 10.4, the result of declaring that the inequality $\infty \geq \frac{0}{0}$ and inequalities of the form "positive number $\geq \frac{0}{0}$" are always true is to not exclude a point from being in P_i based only on the fact that $\omega_i = \omega_j = 0$.

Our definition of "partition P is w-associated with point $\omega \in S$" is precisely as in Definition 10.4, with our convention that "almost every $a \in P_i$" refers to the measure $\mu = m_1 + m_2 + \cdots + m_n$ and the addition of the arithmetic rules given previously.

Our first approach to characterizing Pareto maximality using w-association is Theorem 10.23. It is an attempt to import Theorem 10.9 as directly as possible. As we have seen, Theorem 10.9 is not true in our present setting, and the adjustment we make results in a theorem that is not an "if and only if" statement and, hence, is not a characterization of Pareto maximality. Our second approach (Theorem 10.28) provides a complete characterization.

Theorem 10.23 *Let P be a partition of C.*

a. If P is Pareto maximal then P is w-associated with ω for some $\omega \in S$.
b. If P is w-associated with ω for some $\omega \in S^+$, then P is Pareto maximal.

Proof: Fix partition $P = \langle P_1, P_2, \dots, P_n \rangle$ of C.

For part a, we assume that P is Pareto maximal. Then, for every $i, j = 1, 2, \dots, n$, if $A \subseteq P_i$ and $m_i(A) = 0$, we must have $m_j(A) = 0$. (In other words, P is non-wasteful. See Definition 6.5.) This implies that, for each $i = 1, 2, \dots, n$, $f_i(a) > 0$ for almost every $a \in P_i$.

As we have previously discussed, given any face of the simplex, Pareto maximality implies that P gives almost all of the cake associated with that face to the players whose vertices determine that face. Now we consider the following related question: is there a face of the simplex such that all of the cake given out to the players whose vertices determine that face is associated with that face? If we do not insist that the face be proper, then there certainly is such a face, since we consider the simplex to be a face of itself. We allow this possibility here, and we consider the smallest such face.

Choose $\delta \subseteq \{1, 2, \dots, n\}$ to be of minimal (non-zero) size such that, for every $j \notin \delta$ and $k \in \delta$, $m_j(P_k) = 0$. (The face of the simplex corresponding to δ is as described in the previous paragraph.) By Theorem 6.2, P is proper subpartition Pareto maximal and, hence, the partition $\langle P_i : i \in \delta \rangle$ is a Pareto maximal partition of $\bigcup_{i \in \delta} P_i$ among the players named by δ. Therefore, by part a of Theorem 7.10, $\langle P_i : i \in \delta \rangle$ maximizes the convex combination of the measures $\langle m_i : i \in \delta \rangle$ corresponding to some $(\alpha_i : i \in \delta)$ in the $(|\delta| - 1)$-simplex. We assume here, as we did earlier in this section (see the paragraph preceding the statement of Theorem 10.15), that we have made the natural (order-preserving) identification between the players named by δ and the $|\delta|$ vertices of the $(|\delta| - 1)$-simplex.

We claim that, for each $i \in \delta$, $\alpha_i > 0$. Suppose that this is not the case and let $\delta' = \{i \in \delta : \alpha_i = 0\}$. Then $\delta' \neq \emptyset$ and, since $\sum_{i \in \delta} \alpha_i = 1$, it follows that δ' is a proper subset of δ. For any $j \in \delta \backslash \delta'$ and $k \in \delta'$, $m_j(P_k) = 0$ since, if not, then a transfer of P_k from Player k to Player j would result in a larger sum in the convex combination of the measures $\langle m_i : i \in \delta \rangle$ corresponding to $(\alpha_i : i \in \delta)$. Since $\delta' \subseteq \delta$, we also know that, for any $j \notin \delta$ and $k \in \delta'$, $m_j(P_k) = 0$. Thus, for *any* $j \notin \delta'$ and $k \in \delta'$, $m_j(P_k) = 0$. Since $\delta' \neq \emptyset$ and δ' is a proper subset of δ, this contradicts our minimality assumption on δ. Hence, $\alpha_i > 0$ for each $i \in \delta$.

Recall that Theorem 10.6 holds in the present context, in which absolute continuity is not assumed. Since $\alpha_i > 0$ for each $i \in \delta$, this result implies that the partition $\langle P_i : i \in \delta \rangle$ is w-associated with $\text{RD}(\alpha_i : i \in \delta)$. Set $\text{RD}(\alpha_i : i \in \delta) = (\omega'_i : i \in \delta) = \omega'$. Then, for each $i \in \delta$, $\omega'_i > 0$. Define $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S$ as follows: for each $i = 1, 2, \ldots, n$,

$$\omega_i = \begin{cases} \omega'_i & \text{if } i \in \delta \\ 0 & \text{if } i \notin \delta \end{cases}$$

Then, $\omega \in S$. Notice that, for each $i = 1, 2, \ldots, n$, $\omega_i = 0$ if and only if $i \notin \delta$.

We claim that P is w-associated with ω. Fix distinct $i, j = 1, 2, \ldots, n$. We must show that $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every $a \in P_i$. We consider four cases.

Case 1: $i \in \delta$, $j \in \delta$. Then $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every $a \in P_i$ since $\langle P_i : i \in \delta \rangle$ is w-associated with ω' and $\frac{\omega_i}{\omega_j} = \frac{\omega'_i}{\omega'_j}$.

Case 2: $i \in \delta$, $j \notin \delta$. Then $\omega_i = \omega'_i > 0$ and $\omega_j = 0$. As noted at the beginning of this proof, for almost every $a \in P_i$, $f_i(a) > 0$. Since $j \notin \delta$, $m_j(P_i) = 0$. Hence, $f_j(a) = 0$ for almost every $a \in P_i$. Therefore, for almost every such a, $\frac{f_i(a)}{f_j(a)} = \frac{\text{positive number}}{0} = \infty$. And, $\frac{\omega_i}{\omega_j} = \frac{\text{positive number}}{0} = \infty$. Hence, for almost every $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$.

Case 3: $i \notin \delta$, $j \in \delta$. Then $\omega_i = 0$ and $\omega_j = \omega'_j > 0$. Since $f_i(a) > 0$ for almost every $a \in P_i$, we know that, for almost every such a, either $\frac{f_i(a)}{f_j(a)} = \frac{\text{positive number}}{\text{positive number}} = \text{positive number}$ or $\frac{f_i(a)}{f_j(a)} = \frac{\text{positive number}}{0} = \infty$. Since $\frac{\omega_i}{\omega_j} = \frac{0}{\text{positive number}} = 0$, it follows in either case that, for almost every $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$.

Case 4: $i \notin \delta$, $j \notin \delta$. Then $\omega_i = \omega_j = 0$. As in Case 3, we know that, for almost every $a \in P_i$, either $\frac{f_i(a)}{f_j(a)} = \text{positive number}$ or $\frac{f_i(a)}{f_j(a)} = \infty$. And, since $\frac{\omega_i}{\omega_j} = \frac{0}{0}$, our arithmetic rules imply that, for almost every $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$.

This establishes part a.

For part b, we assume that $\omega \in S^+$ and P is w-associated with ω. By Theorem 10.6, P maximizes the convex combination of measures corresponding to $\text{RD}(\omega)$. Then, since $\text{RD}(\omega) \in S^+$, it follows from part b of Theorem 7.10 that P is Pareto maximal. This completes the proof of the theorem. □

We note that the relationship between Theorems 10.9 and 10.23 is analogous to the relationship between Theorems 7.4 and 7.10. We saw in Chapter 7 that the converses to parts a and b of Theorem 7.10 are each false.

Example 10.22 tells us that the converse to part b of Theorem 10.23 is false. We next give an example to show that the converse to part a of this result is also false.

Example 10.24 Let C' be any cake and let m_1' and m_2' be distinct measures on C' that are absolutely continuous with respect to each other. Let $\langle P_1, P_2 \rangle$ be a partition of C' that is not Pareto maximal. Let C'' be any cake disjoint from C', let m_3'' be any measure on C'', and set $P_3 = C''$. Define a new cake $C = C' \cup C''$ and define measures m_1, m_2, and m_3 on C as follows: for any $A \subseteq C$,

$$m_1(A) = m_1'(A \cap C')$$
$$m_2(A) = m_2'(A \cap C')$$
$$m_3(A) = m_3''(A \cap C'')$$

Let $P = \langle P_1, P_2, P_3 \rangle$. Then P is a partition of C. Since the partition $\langle P_1, P_2 \rangle$ of C' is not Pareto maximal, P is not proper subpartition Pareto maximal and hence, by Theorem 6.2, it follows that P is not a Pareto maximal partition of C. However, we claim that P is w-associated with (0, 0, 1). We must show that, for distinct $i, j = 1, 2, 3$, and almost every $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$, where we have set $(\omega_1, \omega_2, \omega_3) = (0, 0, 1)$. We verify this as follows.

For almost every $a \in P_1$, $\frac{f_1(a)}{f_2(a)} = \frac{\text{positive number}}{\text{positive number}} = \text{positive number} \geq \frac{0}{0} = \frac{\omega_1}{\omega_2}$. ("$\frac{f_1(a)}{f_2(a)} = \frac{\text{positive number}}{\text{positive number}}$" uses the fact that m_1' and m_2' are absolutely continuous with respect to each other.) Similarly, for almost every $a \in P_2$, $\frac{f_2(a)}{f_1(a)} \geq \frac{\omega_2}{\omega_1}$. For almost every $a \in P_1$, $\frac{f_1(a)}{f_3(a)} = \frac{\text{positive number}}{0} = \infty \geq \frac{0}{1} = \frac{\omega_1}{\omega_3}$. Similarly, for almost every $a \in P_2$, $\frac{f_2(a)}{f_3(a)} \geq \frac{\omega_2}{\omega_3}$. For almost every $a \in P_3$, $\frac{f_3(a)}{f_1(a)} = \frac{\text{positive number}}{0} = \infty \geq \frac{1}{0} = \frac{\omega_3}{\omega_1}$. Similarly, for almost every $a \in P_3$, $\frac{f_3(a)}{f_2(a)} \geq \frac{\omega_3}{\omega_2}$. We have established that P is not Pareto maximal but is w-associated with (0, 0, 1). This establishes that the converse to part a of Theorem 10.23 is false.

Simply put, the point (0, 0, 1) can make no distinctions between the points of C that go to Player 1 and those that go to Player 2. Thus, in general, there is no reason to expect that a partition w-associated with (0, 0, 1) is Pareto maximal.

By way of motivation for our characterization of Pareto maximality (Theorem 10.28), consider the following example.

Example 10.25 Suppose that there are three players, Player 1, Player 2, and Player 3, with measures m_1, m_2, and m_3, respectively, such that the corresponding RNS is as in Figure 10.12. First, we clarify what is happening in the figure, and then we informally describe how one could construct a cake and measures

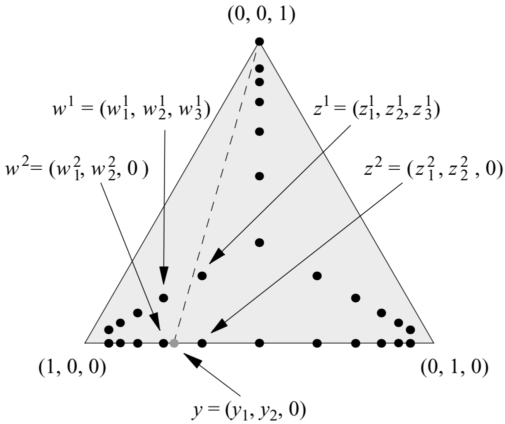


Figure 10.12

that yield this RNS. We can describe the RNS in the figure as consisting of five parts. It contains

a. an infinite collection of points between (1, 0, 0) and (0, 1, 0), with (1, 0, 0) and (0, 1, 0) both being limit points of this collection;
b. an infinite sequence of points beginning at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and continuing toward the point (1, 0, 0), with (1, 0, 0) being the limit of these points;
c. an infinite sequence of points beginning at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and continuing toward the point (0, 1, 0), with (0, 1, 0) being the limit of these points;
d. an infinite sequence of points beginning at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and continuing toward the point (0, 0, 1), with (0, 0, 1) being the limit of these points; and
e. the point (0, 0, 1).

In the figure, we have shown six of the points in each of the sequences described in conditions b, c, and d (the first point in each of these sequences is the same) and have shown eleven of the points in the sequence described in condition a.

It would be cumbersome, but not difficult, to explicitly define a cake C and corresponding measures m_1, m_2, and m_3, having the RNS described in the preceding paragraph and pictured in Figure 10.12. We will informally describe a part of this construction. Consider, for example, conditions d and e. Let the cake be as in Example 10.22, and let measures m_1 and m_2 from that example be the present measures m_3 and m_2, respectively, and let $m_1 = m_2$. This provides us with the point of condition e and the sequence of condition d, except that the starting point of this sequence, and the spacing between the points of the sequence, is not quite right. These can easily be made right by slightly altering the definitions of the measures. Cakes and measures that yield the sequences in the RNS described by conditions a, b, and c can be defined in a similar manner. These various cakes and measures can be combined in a natural way to make one cake and three measures whose RNS satisfies the five preceding conditions.

Define A_1, A_2, and A_3 as follows:

A_1 = all cake associated with points on the line segment between $(1, 0, 0)$ and $(0, 1, 0)$
A_2 = all cake associated with points in the interior of the simplex
A_3 = all cake associated with the point $(0, 0, 1)$

In other words, for any $a \in C$:

$a \in A_1$ if and only if $f(a) = (p_1, p_2, 0)$ for some p_1 and p_2
$a \in A_2$ if and only if $f(a) \in (p_1, p_2, p_3)$, for some $p_1, p_2, p_3 > 0$
$a \in A_3$ if and only if $f(a) = (0, 0, 1)$

Thus, A_1 consists of all cake associated with points in the RNS given by condition a, A_2 consists of all cake associated with points in the RNS given by conditions b, c, and d, and A_3 consists of all cake associated with the point in the RNS given by condition e. Then $C = A_1 \cup A_2 \cup A_3$ and A_1, A_2, and A_3 are pairwise disjoint.

Our previous discussion tells us that, for any p in the RNS, if p is on a face of the simplex, then in any Pareto maximal partition, all cake corresponding to p must be distributed among those players whose vertices determine that face. Thus, any such partition must give none of A_1 to Player 3, since $f(A_1)$ is a subset of the line segment between $(1, 0, 0)$ and $(0, 1, 0)$, and must give all of A_3 to Player 3, since $f(A_3) = \{(0, 0, 1)\}$. A_2 can be distributed in many different ways among the three players, using the notion of w-associated, as in part b of Theorem 10.23. Similarly, A_1 can be distributed in many different ways between Player 1 and Player 2, using the notion of w-associated, as applied to the RNS that corresponds to just Player 1 and Player 2 and piece A_1. We wish to consider partitions that are not describable by choosing some point ω in S^+ and applying part b of Theorem 10.23. (These are partitions of the sort considered in Example 10.22 for the case of two players.) In particular, we shall study the two extreme cases: giving all of A_2 to Player 3 and giving none of A_2 to Player 3.

First, we need to consider the points w^1, w^2, z^1, z^2, and y, and the dashed line segment, in the figure:

- w^1 is the point in the sequence of points given by condition b that is to the left of the dashed line and is closest to this line.

- w^2 is the point in the sequence of points given by condition a that is to the left of the dashed line and is closest to this line.
- z^1 is the point in the sequence of points given by condition b that is to the right of the dashed line and is closest to this line.
- z^2 is the point in the sequence of points given by condition a that is to the right of the dashed line and is closest to this line.
- y is the point of intersection of the dashed line with the line connecting the points (1, 0, 0) and (0, 1, 0).

Notice that neither the point y, nor any point on the dashed line except for the point (0, 0, 1), is in the RNS and, hence, there is no cake of positive measure associated with either of these objects. (We have drawn the point y lighter to distinguish it from the points of the RNS.)

We shall need the following inequalities, each of which follows easily from the figure:

- $\frac{w_2^1}{w_1^1} < \frac{y_2}{y_1}$
- $\frac{w_2^2}{w_1^2} < \frac{y_2}{y_1}$
- $\frac{z_1^1}{z_2^1} < \frac{y_1}{y_2}$
- $\frac{z_1^2}{z_2^2} < \frac{y_1}{y_2}$

Consider the following two partitions:

$P^1 = \langle P_1^1, P_2^1, P_3^1 \rangle$, where
  $P_1^1 =$ all cake in A_1 associated with points in the RNS to the left of y,
  $P_2^1 =$ all cake in A_1 associated with points in the RNS to the right of y, and
  $P_3^1 = A_2 \cup A_3$.
$P^2 = \langle P_1^2, P_2^2, P_3^2 \rangle$, where
  $P_1^2 =$ all cake in $A_1 \cup A_2$ associated with points in the RNS to the left of the dashed line,
  $P_2^2 =$ all cake in $A_1 \cup A_2$ associated with points in the RNS to the right of the dashed line, and
  $P_3^2 = A_3$.

We claim that both P^1 and P^2 are Pareto maximal partitions. We first note that both partitions give all of A_3 to Player 3 and split A_1 between Player 1 and Player 2, as our previous discussion shows must be so for every Pareto maximal partition. We shall show that each of these partitions is Pareto maximal by using partition ratios. We recall that, for distinct $i, j = 1, 2, 3$, $\text{pr}_{ij} = \sup\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i$ and either $m_i(A) \neq 0$ or $m_j(A) \neq 0\}$.

First, we consider P^1. We claim that the partition ratios pr_{31} and pr_{32} each equal ∞^*. (For the meaning of ∞^*, see Notation 8.21. For the relevant multiplication rules involving ∞^*, see Definition 8.22.) To see that $\text{pr}_{31} = \infty^*$, we first note that no point in P_3^1 is associated with a point of the RNS that has positive first coordinate and third coordinate zero. Hence, $\text{pr}_{31} \neq \infty^{**}$. Clearly the ratio of the first to the third coordinate of points in the sequence given by condition b goes to infinity. Since each point in this sequence corresponds to a positive-measure piece of cake in P_3^1, this implies that $\text{pr}_{31} = \infty^*$. Similarly, $\text{pr}_{32} = \infty^*$.

It is easy to see that $\text{pr}_{13} = \text{pr}_{23} = 0$ and that $\text{pr}_{12} = \frac{w_2^2}{w_1^2}$ and $\text{pr}_{21} = \frac{z_1^2}{z_2^2}$. Then, computing all cyclic products, we have:

$$\begin{aligned}
\text{pr}_{12}\text{pr}_{21} &= \left(\frac{w_2^2}{w_1^2}\right)\left(\frac{z_1^2}{z_2^2}\right) < \left(\frac{y_2}{y_1}\right)\left(\frac{y_1}{y_2}\right) = 1\\
\text{pr}_{13}\text{pr}_{31} &= (0)(\infty^*) = 0\\
\text{pr}_{23}\text{pr}_{32} &= (0)(\infty^*) = 0\\
\text{pr}_{12}\text{pr}_{23}\text{pr}_{31} &= \left(\frac{w_2^2}{w_1^2}\right)(0)(\infty^*) = 0\\
\text{pr}_{32}\text{pr}_{21}\text{pr}_{13} &= (\infty^*)\left(\frac{z_1^2}{z_2^2}\right)(0) = 0
\end{aligned}$$

Since all cyclic products are less than or equal to one, Theorem 8.24 implies that P^1 is Pareto maximal.

Next, we consider P^2. It is easy to see that $\text{pr}_{31} = \text{pr}_{32} = 0$.

We note that $\text{pr}_{13} = \frac{w_3^1}{w_1^1}$ and $\text{pr}_{23} = \infty^*$. (The argument that $\text{pr}_{23} = \infty^*$ is similar to the argument given for P^1 that $\text{pr}_{31} = \infty^*$. Choosing to have the dashed line pass to the left of the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ was arbitrary. If we had chosen the dashed line so that it passed to the right of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, then the roles of pr_{13} and pr_{23} would have been reversed.)

Finally, we see that $\text{pr}_{12} = \sup\{\frac{w_2^1}{w_1^1}, \frac{w_2^2}{w_1^2}\}$ and $\text{pr}_{21} = \sup\{\frac{z_1^1}{z_2^1}, \frac{z_1^2}{z_2^2}\}$. Then, computing all cyclic products, we have:

$$\begin{aligned}
\text{pr}_{12}\text{pr}_{21} &= \left(\sup\left\{\frac{w_2^1}{w_1^1}, \frac{w_2^2}{w_1^2}\right\}\right)\left(\sup\left\{\frac{z_1^1}{z_2^1}, \frac{z_1^2}{z_2^2}\right\}\right) < \left(\frac{y_2}{y_1}\right)\left(\frac{y_1}{y2}\right) = 1\\
\text{pr}_{13}\text{pr}_{31} &= \left(\frac{w_3^1}{w_1^1}\right)(0) = 0\\
\text{pr}_{23}\text{pr}_{32} &= (\infty^*)(0) = 0\\
\text{pr}_{12}\text{pr}_{23}\text{pr}_{31} &= \left(\sup\left\{\frac{w_2^1}{w_1^1}, \frac{w_2^2}{w_1^2}\right\}\right)(\infty^*)(0) = 0\\
\text{pr}_{32}\text{pr}_{21}\text{pr}_{13} &= (0)\left(\sup\left\{\frac{z_1^1}{z_2^1}, \frac{z_1^2}{z_2^2}\right\}\right)\left(\frac{w_3^1}{w_1^1}\right) = 0
\end{aligned}$$

Since all cyclic products are less than or equal to one, Theorem 8.24 implies that P_2 is Pareto maximal.

It is not hard to see that neither partition P^1 nor partition P^2 is w-associated with any point in S^+. However, each of these partitions is w-associated with a point on the boundary of S. Partition P^1 is w-associated with the point y, and partition P^2 is w-associated with the point $(0, 0, 1)$. There is an important difference between these two situations. Whereas P^1 is the only partition that is w-associated with y, there are many partitions that are w-associated with $(0, 0, 1)$, and many of these partitions are not Pareto maximal. We discussed the reasons for each of these situations earlier in this chapter. If we are only interested in Pareto maximal partitions, using "w-associated with y" does work, because the point y has only one coordinate that is equal to zero. Using "w-associated with $(0, 0, 1)$" does not work, because both the first and second coordinates of this point are equal to zero, and thus this point can make no distinctions between which points of C go to Player 1 and which go to Player 2. In particular, as long as A_3 goes to Player 3, the cake associated with points in A_1 and A_2 can be given out arbitrarily between Player 1 and Player 2 and the resulting partition will be w-associated with the point $(0, 0, 1)$. Some such partitions will be Pareto maximal and some will not. Thus, this approach does not provide a characterization.

We will show how to describe each of these partitions by an iterative procedure. This procedure, when properly generalized, will give us our desired characterization, which is Theorem 10.28. For P^1 and P^2, the procedure will have two stages. In stage 1, we identify one face of the simplex and we partition the cake associated with this face among the players whose vertices determine this face, using part b of Theorem 10.23. Next, in stage 2, we partition the remainder of the cake among the remaining player(s), again using part b of Theorem 10.23.

For P^1, we focus first on the face consisting of the line segment between $(1, 0, 0)$ and $(0, 1, 0)$. Notice that this line segment is a face of the simplex, since it is the set of all points (p_1, p_2, p_3) of the simplex for which $p_3 = 0$. For stage 1, we apply part b of Theorem 10.23, thinking of A_1 as the cake and Player 1 and Player 2 as the players, using the point y in place of ω. Note that y is an interior point of the relevant simplex, which is the one-simplex. Next, we focus on the remaining cake, which is $A_2 \cup A_3$, and the remaining player, Player 3. The relevant simplex is the trivial zero-simplex associated with Player 3. For stage 2, we apply part b of Theorem 10.23, thinking of $A_2 \cup A_3$ as the cake and using the single point of the zero-simplex, $(0, 0, 1)$, in place of ω. (We shall consider the single point that makes up the zero-simplex to be an interior point of the simplex, since its coordinate in $\mathbf{R}^1$, i.e., on the real number line, is one, a positive number.) This gives all of $A_2 \cup A_3$ to Player 3. This two-stage approach yields P^1, as desired.

For P^2, we focus first on the face consisting of just the point (0, 0, 1). We note that this point is a face of the simplex, since it is the set of all points (p_1, p_2, p_3) of S for which $p_1 = 0$ and $p_2 = 0$. We also note that there is a piece of cake, namely A_3, associated with this point. For stage 1, we apply part b of Theorem 10.23, thinking of A_3 as the cake and Player 3 as the only player, using the single point of the associated zero-simplex, (0, 0, 1), in place of ω. This gives all of A_3 to Player 3. Next, we focus on the remaining cake, which is $A_1 \cup A_2$, and the remaining players, Player 1 and Player 2. The relevant simplex is the one-simplex associated with Player 1 and Player 2. We may identify this one-simplex with the line segment between (1, 0, 0) and (0, 1, 0). To make this identification, we proceed as we did earlier in this chapter (see Example 10.10 and Figure 10.4). We ignore the third coordinate of each point along this line segment and identify points of the RNS other than (0, 0, 1) with their projections to this line segment. We make this identification simply by changing the third coordinates of all points to zero, and changing the first and second coordinates so that their sum is one but so that their ratio remains unchanged. In other words, each point $(p_1, p_2, p_3) \neq (0, 0, 1)$ of the RNS is identified with the point $(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}, 0)$. Geometrically, this means simply projecting along the line segment from the point (0, 0, 1), through the given point, to a point on the line segment between (1, 0, 0) and (0, 1, 0). This is illustrated in Figure 10.13. In the figure, any point along one of the dashed lines is projected to the point of intersection of that dashed line with the line between (1, 0, 0) and (0, 1, 0). Clearly, this procedure projects points that are to the left of the dashed line in Figure 10.12 to points that are to the left of y and projects points that are to the right of this dashed line to points that are to the right of y. For stage 2, we apply part b of Theorem 10.23, thinking of $A_1 \cup A_2$ as the cake, and using the

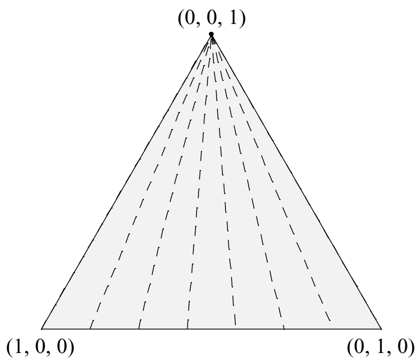


Figure 10.13

point y in place of ω, once we have made the identification just described. This two-stage approach yields P^2, as desired.

In moving toward making this procedure general and precise, we note the following two facts that held in the previous example, and will hold in the general case:

- A given player is eligible to receive cake at only one stage.
- Any piece of cake given out at any stage before a given player is eligible to receive cake has measure zero to that player.

We shall use the notion of "partition sequence pair," which was given by Definition 7.11. In what follows, we shall refer to the sequence $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, rather than to the sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, as in the statement of Definition 7.11. This is in keeping with our convention of using ω in the context of w-association and using α when referring to coefficients in convex combinations of measures.

Suppose that $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$ is a partition of $\{1, 2, \ldots, n\}$ (as in Definition 7.11). For each $k = 1, 2, \ldots, t$, set $\mu^{\gamma_k} = \sum_{i \in \gamma_k} m_i$.

Definition 10.26 Let (ω, γ) be a partition sequence pair with $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, and let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition. P is *w-associated with* (ω, γ) if and only if the following two conditions hold:

a. For every $k = 1, 2, \ldots, t$ and distinct $i, j \in \gamma_k$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every (with respect to μ^{γ_k}) $a \in P_i$.
b. Either
   i. for every $k = 1, 2, \ldots, t$, and almost every $a \in C$, $a \in \bigcup_{i \in \gamma_k} P_i$ if and only if $a \notin \bigcup_{k' < k} \bigcup_{i \in \gamma_{k'}} P_i$, $f_j(a) > 0$ for some $j \in \gamma_k$, and $f_{j'}(a) = 0$ for all $j' \in \gamma_{k'}$ with $k' = k + 1, k + 2, \ldots, t$, or
   ii. for every $k = 1, 2, \ldots, t$ and almost every $a \in C$, $a \in \bigcup_{i \in \gamma_k} P_i$ if and only if $a \notin \bigcup_{k' > k} \bigcup_{i \in \gamma_{k'}} P_i$, $f_j(a) > 0$ for some $j \in \gamma_k$, and $f_{j'}(a) = 0$ for all $j' \in \gamma_{k'}$ with $k' = 1, 2, \ldots, k - 1$.

Notice that if (ω, γ) is a partition sequence pair with $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\gamma = \langle \{1, 2, \ldots, n\} \rangle$ (i.e., γ is the trivial partition of $\{1, 2, \ldots, n\}$ into one piece), then $\omega \in S^+$ and condition a says that, for distinct $i, j = 1, 2, \ldots, n$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every $a \in P_i$, and condition b is trivially true. In this case, Definition 10.26 reduces to Definition 10.4. Thus, we may view Definition 10.4 as a special case of Definition 10.26.

In condition a of Definition 10.26, once we restrict our attention to γ_k, the relevant measures are the measures m_i for $i \in \gamma_k$ and, hence, the sum of these measures, μ^{γ_k}, plays the role usually played by $\mu = m_1 + m_2 + \cdots + m_n$. Thus we see that condition a is equivalent to the following:

> for every $k = 1, 2, \ldots, t$, the partition $\langle P_i : i \in \gamma_k \rangle$ of $\bigcup_{i \in \gamma_k} P_i$ is w-associated with $(\omega_i : i \in \gamma_k)$.

Concerning condition a, we also note what may appear to be a subtle error. Once we have restricted our attention to only the players named by γ_k and we use the measure μ^{γ_k} in place of the measure μ, we are really working with a new RNS, and this new RNS will have density functions that need not be the same as the original density functions. (To see this, recall that, for any $a \in C$, $f_1(a) + f_2(a) + \cdots + f_n(a) = 1$. Thus, for any $a \in C$, if $f_i(a) > 0$ for some $i \notin \gamma_k$, then $\sum_{i \in \gamma_k} f_i(a) < 0$. Hence, these original density functions are not legitimate density functions for the new restricted RNS.) However, we are justified in ignoring this issue since we are only concerned with ratios of the form $\frac{f_i(a)}{f_j(a)}$, and it is not hard to see that such ratios are the same for the original density functions as for the new density functions.

Condition b of Definition 10.26 gives us our iterated perspective. Consider condition bi. The idea here (as was the case for a-maximization in Chapter 7 and for Example 10.25) is that we may imagine t stages, one for each of the pieces of γ. At each such stage k, cake is given out to the players named by γ_k. Point $a \in C$ is given to some player at this stage if and only if

- it has not previously been given out (i.e., $a \notin \bigcup_{k' < k} \bigcup_{i \in \gamma_{k'}} P_i$),
- it has value to some player named by γ_k (i.e., $f_j(a) > 0$ for some $j \in \gamma_k$), and
- it has no value to any player considered later in the procedure (i.e., $f_{j'}(a) = 0$ for all $j' \in \gamma_{k'}$ with $k' = k + 1, k + 2, \ldots, t$).

The discussion would be analogous if condition bii is satisfied instead of condition bi.

Before stating our characterization of Pareto maximality, we need to define a new function that generalizes the function $\text{RD} : S^+ \to S^+$. We defined this function in the previous section (Definition 10.5) and used it in the proof of Theorem 10.6. Recall that this function associates a point $\alpha \in S^+$ that provides coefficients for a convex combination of measures, with a point $\omega \in S^+$ that is to be used in the context of w-association. We now use this same idea to associate each partition sequence pair that is to be thought of in the context of a-maximization (as in Definition 7.12) with a partition sequence pair that is to be thought of in the context of w-association (as in Definition 10.26). We do this by applying the function RD to each of the lower-dimensional simplices corresponding to the pieces that make up the partition γ of $\{1, 2, \ldots, n\}$. Let Ψ denote the set of all partition sequence pairs.

Definition 10.27 We define $\mathrm{RD}_\Psi : \Psi \to \Psi$ as follows. Fix $(\alpha, \gamma) \in \Psi$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$. For each $k = 1, 2, \ldots, t$, define $(\omega_i : i \in \gamma_k) = \mathrm{RD}(\alpha_i : i \in \gamma_k)$, and set $\mathrm{RD}_\Psi((\alpha, \gamma)) = (\omega, \gamma)$, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$.

Since $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$ is a partition of $\{1, 2, \ldots, n\}$, each ω_i for $i = 1, 2, \ldots, n$ has been defined and, hence, the definition of RD_Ψ makes sense.

As we discussed when we defined the RD function, simplices of different sizes have different RD functions, but we use this notation for all of the RD functions. Thus, in general, the RD functions used in the definition of RD_Ψ will be different functions, depending on the size of the relevant simplex. We also note that $\mathrm{RD}_\Psi : \Psi \to \Psi$ is a bijection and that, for any $(\alpha, \gamma) \in \Psi$, $\mathrm{RD}_\Psi(\mathrm{RD}_\Psi(\alpha, \gamma)) = (\alpha, \gamma)$.

Our characterization of Pareto maximality using the notion of w-association is the following.

Theorem 10.28 *A partition P is Pareto maximal if and only if it is w-associated with some partition sequence pair.*

The proof of Theorem 10.28 uses Theorem 7.13, which characterized Pareto maximality using the notion of a-maximization of a partition sequence pair. The best informal perspective on the theorem is an iterative one, as was the case for our characterizations of Pareto maximality using a-maximization and b-maximization in Chapter 7. Our proof does not clearly convey this. After presenting this proof, we shall give an alternate proof of the forward direction that will give us this perspective, and then we will discuss how the reverse direction of the theorem provides us with an iterative approach to constructing Pareto maximal partitions.

Theorem 10.28 follows easily from the next lemma.

Lemma 10.29 *Fix a partition sequence pair* (α, γ). *P a-maximizes* (α, γ) *and is non-wasteful if and only if P is w-associated with* $\mathrm{RD}_\Psi((\alpha, \gamma))$. *(For the definition of a-maximization, see Definition 7.12. For the definition of non-wasteful, see Definition 6.5.)*

Proof: Fix a partition sequence pair (α, γ), where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, and let $\mathrm{RD}_\Psi((\alpha, \gamma)) = (\omega, \gamma)$, where $\omega = (\omega_1, \omega_2, \ldots,$

ω_n). We must show that P a-maximizes (α, γ) and is non-wasteful if and only if P is w-associated with (ω, γ). We begin by establishing a claim.

Claim Condition a of Definition 7.12 holds if and only if condition a of Definition 10.26 holds.

Proof of Claim: We must show that the following two statements are equivalent:

a$'$. For every $k = 1, 2, \ldots, t$, partition $\langle P_i : i \in \gamma_k \rangle$ of $\bigcup_{i \in \gamma_k} P_i$ maximizes the convex combination of the measures $\langle m_i : i \in \gamma_k \rangle$ corresponding to $(\alpha_i : i \in \gamma_k)$.

a$''$. For every $k = 1, 2, \ldots, t$ and distinct $i, j \in \gamma_k$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every (with respect to μ^{γ_k}) $a \in P_i$.

As we discussed previously, Theorem 10.6 holds without the assumption of absolute continuity and hence, since each α_i is positive, we may apply this result to statement a', with $\langle P_i : i \in \gamma_k \rangle$ playing the role of the partition P in the theorem. Hence, statement a' holds if and only if,

c. for every $k = 1, 2, \ldots, t$, partition $\langle P_i : i \in \gamma_k \rangle$ of $\bigcup_{i \in \gamma_k} P_i$ is w-associated with

$$\mathrm{RD}(\alpha_i : i \in \gamma_k) = \left(\frac{\frac{1}{\alpha_i}}{\sum_{i' \in \gamma_k} \frac{1}{\alpha_{i'}}} : i \in \gamma_k \right).$$

By the definition of w-associated, statement c holds if and only if,

d. for every $k = 1, 2, \ldots, t$ and distinct $i, j \in \gamma_k$,

$$\frac{f_i(a)}{f_j(a)} \geq \frac{\left(\frac{\frac{1}{\alpha_i}}{\sum_{i' \in \gamma_k} \frac{1}{\alpha_{i'}}} \right)}{\left(\frac{\frac{1}{\alpha_j}}{\sum_{i' \in \gamma_k} \frac{1}{\alpha_{i'}}} \right)} = \frac{\alpha_j}{\alpha_i}$$

for almost every (with respect to μ^{γ_k}) $a \in P_i$.

For each such i, j, and k, we know that

$$\omega_i = \frac{\frac{1}{\alpha_i}}{\sum_{i' \in \gamma_k} \frac{1}{\alpha_{i'}}}$$

and

$$\omega_j = \frac{\frac{1}{\alpha_j}}{\sum_{i' \in \gamma_k} \frac{1}{\alpha_{i'}}}$$

Hence, $\frac{\alpha_j}{\alpha_i} = \frac{\omega_i}{\omega_j}$. Thus, statement d holds if and only if statement a'' holds. This establishes that statements a' and a'' are equivalent and, hence, completes the proof of the claim.

We return now to the proof of the lemma. We wish to show that P a-maximizes the partition sequence pair (α, γ) and is non-wasteful if and only if P is w-associated with the partition sequence pair (ω, γ).

For the forward direction, we assume that P a-maximizes the partition sequence pair (α, γ) and is non-wasteful. By the claim, condition a of Definition 10.26 is satisfied. We shall assume that P and (α, γ) satisfy condition bi of Definition 7.12. The proof is similar if instead they satisfy condition bii of this definition.

We claim that P and (ω, γ) satisfy condition bi of Definition 10.26. Suppose, by way of contradiction, that this is not so. Then, for some $k = 1, 2, \ldots, t$, it is not true that, for almost every $a \in C$, $a \in \bigcup_{i \in \gamma_k} P_i$ if and only if $a \notin \bigcup_{k' < k} \bigcup_{i \in \gamma_{k'}} P_i$, $f_j(a) > 0$ for some $j \in \gamma_k$, and $f_{j'}(a) = 0$ for all $j' \in \gamma_{k'}$ with $k' = k+1, k+2, \ldots, t$. Fix such a k and let

$A = \{a \in C : a \in \bigcup_{i \in \gamma_k} P_i$ and either $a \in \bigcup_{k' < k} \bigcup_{i \in \gamma_{k'}} P_i$, or $f_j(a) = 0$ for all $j \in \gamma_k$, or $f_{j'}(a) > 0$ for some $j' \in \gamma_{k'}$ with $k' = k+1, k+2, \ldots, t\}$ and

$B = \{a \in C : a \notin \bigcup_{i \in \gamma_k} P_i, a \notin \bigcup_{k' < k} \bigcup_{i \in \gamma_{k'}} P_i, f_j(a) > 0$ for some $j \in \gamma_k$, and $f_{j'}(a) = 0$ for all $j' \in \gamma_{k'}$ with $k' = k+1, k+2, \ldots, t\}$.

Then either $\mu(A) > 0$ or $\mu(B) > 0$. We consider each of these two cases.

Case 1: $\mu(A) > 0$. Since P is a partition of C, it follows that, for any $a \in A$, since $a \in \bigcup_{i \in \gamma_k} P_i$, we must have $a \notin \bigcup_{k' < k} \bigcup_{i \in \gamma_{k'}} P_i$. Hence, if we let

$A_1 = \{a \in C : a \in \bigcup_{i \in \gamma_k} P_i$ and $f_j(a) = 0$ for all $j \in \gamma_k\}$ and

$A_2 = \{a \in C : a \in \bigcup_{i \in \gamma_k} P_i$ and $f_{j'}(a) > 0$ for some $j' \in \gamma_{k'}$ with $k' = k+1, k+2, \ldots, t\}$,

then either $\mu(A_1) > 0$ or $\mu(A_2) > 0$.

If $\mu(A_1) > 0$, then for each $j \in \gamma_k$, $m_j(A_1) = 0$. This implies that, for some $j' \notin \gamma_k$, $m_{j'}(A_1) > 0$. But, since $A_1 \subseteq \bigcup_{i \in \gamma_k} P_i$, this contradicts our assumption that P is non-wasteful.

If $\mu(A_2) > 0$, then for some $A_3 \subseteq A_2$, $j \in \gamma_k$, $k' = k+1, k+2, \ldots, t$, and $j' \in \gamma_{k'}$, we have $A_3 \subseteq P_j$ and $m_{j'}(A_3) > 0$. But then $m_{j'}(P_j) > 0$. This contradicts condition bi of Definition 7.12.

Case 2: $\mu(B) > 0$. Notice that, for every $a \in B$, $a \in \bigcup_{k' > k} \bigcup_{i \in \gamma_{k'}} P_i$. It follows that, for some $B_1 \subseteq B$, $k' = k+1, k+2, \ldots, t$, and $j' \in \gamma_{k'}$, we have $B_1 \subseteq P_{j'}$ and $\mu(B_1) > 0$. But, for every $a \in B$, $f_{j'}(a) = 0$, and hence $m_{j'}(B_1) = 0$. This implies that, for some $j = 1, 2, \ldots, n$ with $j \neq j'$, $m_j(B_1) > 0$, which contradicts our assumption that P is non-wasteful. This establishes the forward direction of the lemma.

For the reverse direction, we assume that P is w-associated with the partition sequence pair (ω, γ). We must show that P a-maximizes (α, γ) and is non-wasteful. We first show that P a-maximizes (α, γ). By the claim, condition a of Definition 7.12 is satisfied.

Assume that P and (ω, γ) satisfy condition bi of Definition 10.26. The proof is similar if instead they satisfy condition bii of this definition.

Fix any $k, k' = 1, 2, \ldots, t$ with $k < k'$, $j \in \gamma_k$, and $j' \in \gamma_{k'}$. Condition bi of Definition 10.26 implies that $f_{j'}(a) = 0$ for almost every $a \in P_j$ and, therefore, $m_{j'}(P_j) = 0$. This establishes that condition bi of Definition 7.12 holds; hence, P a-maximizes the partition sequence pair (α, γ).

It remains for us to show that P is non-wasteful. Fix any $k = 1, 2, \ldots, t$, $j \in \gamma_k$, and $A \subseteq P_j$ with $m_j(A) = 0$. We must show that, for every $j' = 1, 2, \ldots, n$, $m_{j'}(A) = 0$. Suppose, by way of contradiction, that this is not so, and let k' be maximal so that $m_{j'}(A) > 0$ for some $j' \in \gamma_{k'}$. We consider three cases.

Case 1 $k' < k$. Let $A_1 = \{a \in A : f_{j'}(a) > 0\}$. Since $m_{j'}(A) > 0$, it follows that $m_{j'}(A_1) > 0$. We claim that the following three conditions are satisfied for almost every $a \in A_1$:

i. $a \notin \bigcup_{k'' < k'} \bigcup_{i \in \gamma_{k''}} P_i$.
ii. $f_{j''}(a) > 0$ for some $j'' \in \gamma_{k'}$.
iii. $f_{j''}(a) = 0$ for all $j'' \in \gamma_{k''}$ with $k'' = k' + 1, k' + 2, \ldots, t$.

Condition i is obvious, since $A_1 \subseteq A \subseteq P_j$, $j \in \gamma_k$, and $k' < k$. Condition ii follows from our definition of A_1, since $j' \in \gamma_{k'}$. Condition iii follows from our maximality assumption on k'. Then, since P and (ω, γ) satisfy condition bi of Definition 10.26, it follows from this definition (with k' in place of k) that, for almost every $a \in A_1$, $a \in \bigcup_{i \in \gamma'_k} P_i$. But $A_1 \subseteq P_j$ and, since $j \in \gamma_k$, $j \notin \gamma_{k'}$. This is a contradiction.

Case 2: $k' = k$. Let $A_1 = \{a \in A : f_j(a) = 0$ and $f_{j'}(a) > 0\}$. Since $m_j(A) = 0$ and $m_{j'}(A) > 0$, it follows that $\mu(A_1) > 0$. Since $A_1 \subseteq A \subseteq P_j$, condition a of Definition 10.26 implies that $\frac{f_j(a)}{f_{j'}(a)} \geq \frac{\omega_j}{\omega_{j'}}$ for almost every (with respect to μ^{γ_k}) $a \in A_1$. But for each such a, $f_j(a) = 0$, and thus the inequality $\frac{f_j(a)}{f_{j'}(a)} \geq \frac{\omega_j}{\omega_{j'}}$ is false, since $\frac{\omega_j}{\omega_{j'}}$ is a positive number. This is a contradiction.

Case 3: $k' > k$. By condition bi of Definition 10.26, for almost every $a \in A$, $f_{j'}(a) = 0$. This implies that $m_{j'}(A) = 0$, which contradicts our assumption that $m_{j'}(A) > 0$.

This completes the proof of the lemma. □

Proof of Theorem 10.28: The theorem follows immediately from the Lemma 10.29 and Theorem 7.13. □

We may view Theorem 10.28 as a general result that holds whether or not the measures are absolutely continuous with respect to each other, and Theorem 10.9 as a special case that holds if $P \in \text{Part}^+$ and absolute continuity holds. This involves viewing Definition 10.4 as a special case of Definition 10.26, as discussed previously (see the discussion following Definition 10.26).

As discussed earlier, it is best to view the theorem in iterative terms. To provide this perspective, we shall consider the forward and reverse directions separately. For the forward direction, we give an alternate proof. This second approach is analogous to the approach used in proving the forward direction of Theorem 7.13. ("If a partition P is Pareto maximal, then it a-maximizes some partition sequence pair and is non-wasteful.") That proof involved repeated use of part a of Theorem 7.10. In a similar manner, we will repeatedly use part a of Theorem 10.23 to establish the forward direction of Theorem 10.28.

Alternative Proof of the Forward Direction of Theorem 10.28: Suppose that partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is Pareto maximal. We must show that P is w-associated with some partition sequence pair.

By part a of Theorem 10.23, P is w-associated with some $\omega^1 = (\omega_1^1, \omega_2^1, \ldots, \omega_n^1) \in S$. Let $\gamma_1 = \{i \leq n : \omega_i^1 > 0\}$ and, for each $i \in \gamma_1$, let $\omega_i = \omega_i^1$. If $\gamma_1 = \{1, 2, \ldots, n\}$, set $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and we are done, since then P is w-associated with the partition sequence pair $\langle \omega, \gamma \rangle$, where γ is the trivial partition of $\{1, 2, \ldots, n\}$ into one piece.

If $\gamma_1 \neq \{1, 2, \ldots, n\}$, then the partition $\langle P_i : i \in \gamma_1 \rangle$ of $\bigcup_{i \in \gamma_1} P_i$ is w-associated with $(\omega_i : i \in \gamma_1)$, which is a point in the interior of the $(|\gamma_1| - 1)$-simplex. (As we have done previously in this section, we identify the players named by γ_1 with the $|\gamma_1|$ vertices of the $(|\gamma_1| - 1)$-simplex in the natural order-preserving way.)

Fix any j and k with $j \in \gamma_1$ and $k \notin \gamma_1$. We claim that $m_k(P_j) = 0$. Since P is w-associated with ω, it follows that, for almost every $a \in P_j$, $\frac{f_j(a)}{f_k(a)} \geq \frac{\omega_j}{\omega_k}$. But, since $j \in \gamma_1$ and $k \notin \gamma_1$, we know that $\omega_j > 0$ and $\omega_k = 0$. Hence, for almost every $a \in P_j$, $\frac{f_j(a)}{f_k(a)} \geq \frac{\omega_j}{0} = \infty$. This implies that for almost every $a \in P_j$, $f_k(a) = 0$. Hence, $m_k(P_j) = 0$.

Next, consider the set $C_2 = C \backslash (\bigcup_{i \in \gamma_1} P_i)$. By the above, for any $k \notin \gamma_1$, $m_k(\bigcup_{i \in \gamma_1} P_i) = 0$. Then,

$$m_k(C_2) = m_k\left(C \Big\backslash \left(\bigcup_{i \in \gamma_1} P_i\right)\right) = m_k(C) - m_k\left(\bigcup_{i \in \gamma_1} P_i\right) = 1 - 0 = 1.$$

We now take the view that the cake is C_2, with measures m_i for $i \notin \gamma_1$. We apply part a of Theorem 10.23, as we did earlier, except that now we apply it to C_2. Doing so, we obtain $\gamma_2 \subseteq \{1, 2, \ldots, n\} \backslash \gamma_1$ and $(\omega_i : i \in \gamma_2)$ so that the partition $\langle P_i : i \in \gamma_2 \rangle$ of $\bigcup_{i \in \gamma_2} P_i$ is w-associated with $(\omega_i : i \in \gamma_2)$, which is a point in the interior of the $(|\gamma_2| - 1)$-simplex. We continue in this manner. Since each $\gamma_j \neq \emptyset$, we must arrive at some $\gamma_t \subseteq \{1, 2, \ldots, n\} \backslash (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{t-1})$ and $(\omega_i : i \in \gamma_t)$ so that the partition $\langle P_i : i \in \gamma_t \rangle$ of $\bigcup_{i \in \gamma_t} P_i$ is w-associated with $(\omega_i : i \in \gamma_t)$, which is a point in the interior of the $(|\gamma_t| - 1)$-simplex, and $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t = \{1, 2, \ldots, n\}$. At this point, our construction is complete. It is straightforward to show that conditions a and bi of Definition 10.26 are satisfied with $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$. Hence, P is w-associated with $\langle \omega, \gamma \rangle$. This establishes the forward direction of the theorem. □

We now consider an iterative perspective on the reverse direction of Theorem 10.28. The reverse direction of the theorem provides us with a method of constructing Pareto maximal partitions. Pick any face of S. This face is determined by the vertices of some collection of players. Let $\gamma_1 \subseteq \{1, 2, \ldots, n\}$ be the set of names of these players. Pick any $\omega^1 = (\omega_i : i \in \gamma_1)$ with all positive coefficients (i.e., choose any ω^1 that is an interior point of the $(|\gamma_1| - 1)$-simplex associated with the players named by γ_1). Consider the piece of cake associated with this face, and let $\langle P_i : i \in \gamma_1 \rangle$ be any partition of this piece among the players named by γ_1 that is w-associated with ω^1.

Next, consider the new cake $C_2 = C \backslash \bigcup_{i \in \gamma_1} P_i$ and its corresponding simplex, whose vertices now correspond to the players named by $\{1, 2, \ldots, n\} \backslash \gamma_1$, and continue precisely as before. Choose any face of this simplex, let $\gamma_2 \subseteq \{1, 2, \ldots, n\} \backslash \gamma_1$ be the set of names of the players whose vertices determine this face, pick any $\omega^2 = (\omega_i : i \in \gamma_2)$ with all positive coefficients, and choose any partition $\langle P_i : i \in \gamma_2 \rangle$ of the piece of cake corresponding to this face, among the players named by γ_2, that is w-associated with ω^2.

We continue in this manner where, for each $i = 2, 3, \ldots,$ $\gamma_i \subseteq \{1, 2, \ldots, n\} \backslash (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{i-1})$. Since each $\gamma_j \neq \emptyset$, we eventually arrive at some γ_t where $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t = \{1, 2, \ldots, n\}$. Set $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, and $P = \langle P_1, P_2, \ldots, P_n \rangle$. Then P is w-associated with the partition sequence pair $\langle \omega, \gamma \rangle$.

The important point here, in terms of satisfying condition bi of Definition 10.26, is that if $A \subseteq C$ corresponds to points of the RNS that lie on some face of a simplex, then A has value zero to any player whose vertex is not one of the vertices that determine this face. This tells us that any cake given out at any stage of this procedure has value zero to any player not yet considered. It follows from Theorem 10.28 that any partition produced by this iterative procedure is Pareto maximal.

On the other hand, if P is Pareto maximal, then the forward direction of Theorem 10.28 provides us with a partition sequence pair with which P is w-associated. This partition sequence pair provides exact instructions for constructing P, using the iterative procedure just described. Hence, we see that a partition is Pareto maximal if and only if it can be produced by this iterative procedure.

In Example 10.25, we showed that the partitions P^1 and P^2 are both Pareto maximal. After we established this, we showed that we can view each of these partitions as arising from a certain iterative construction. We now see that this construction is exactly the construction described in the preceding few paragraphs, which comes from the reverse direction of Theorem 10.28.

In Chapter 7, we discussed a social hierarchy among the players. (See the paragraph following the proof of Theorem 7.13.) In Example 10.25, we would say that in the process of creating partition P^1, Player 3 has higher social status than Player 1 or Player 2 since, in Player 3's view, the cake given to Player 1 and to Player 2 in the first stage has no value and the entire cake is given to him or her in the second stage. On the other hand, in the process of creating partition P^2, Player 1 and Player 2 have higher social status since, in their view, the cake given to Player 3 in the first stage has no value and the entire cake is given out to them in the second stage.

Notice that, in our iterated perspective on the partitions of Example 10.25, players considered later in the procedure have higher social status. We could have altered the iterated procedure of this example to focus on condition bii rather than condition bi of Definition 10.26. If we had done so, then players considered earlier in the procedure would have higher social status.

We close this section by considering the chores versions of our characterization theorem of this section. We saw in Chapter 7 that, in contrast with the fairness context, and the efficiency context when absolute continuity holds, the chores versions of efficiency theorems when absolute continuity fails can be very different from the standard versions. We will see the same situation here.

Recall Definition 7.21: a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is c-non-wasteful if and only if, for any $i = 1, 2, \ldots, n$ and $A \subseteq P_i$ with $m_i(A) > 0$, we have $m_j(A) > 0$ for every $j = 1, 2, \ldots, n$. Clearly every Pareto minimal partition is c-non-wasteful. This says something very simple about the RNS picture. It says that any piece of cake associated with the interior of any proper face of the simplex must be given to players who are not among those whose vertices determine that face (with, as usual, a measure-zero set of possible exceptions). Precisely which of these players receives how much of this piece of cake is irrelevant to everyone. Thus, we can think of the cake as being partitioned at the start into two pieces. The first piece consists of all bits of cake associated with points on any proper face of the simplex (i.e., with points on the boundary of the simplex) and the other piece consists of all bits of cake associated with points that are not on a proper face of the simplex (i.e., with interior points of the simplex). The first collection of points should be given out among the players as we have just described. The measures are absolutely continuous with respect to each other on the second piece, and thus we simply use Theorem 10.21 to partition this set in a Pareto minimal way using the notion of chores w-associated.

For partition $P = \langle P_1, P_2, \ldots, P_n \rangle$, we let $\delta_P = \{i \leq n : P_i$ has positive measure$\}$, and we let S_{δ_P} be the $(|\delta_P| - 1)$-simplex where, as usual, we identify players named by δ_P with the $|\delta_P|$ vertices of the $(|\delta_P| - 1)$-simplex in the natural order-preserving way. By redefining P on a set of measure zero, if necessary, we may assume that $\bigcup_{i \in \delta_P} P_i = C$.

Theorem 10.30 *Let $A = \{a \in C : f(a)$ is on the boundary of the simplex$\}$, let $B = C \backslash A$, and let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be a partition. P is Pareto minimal if and only if the following two conditions are satisfied:*

a. For any $i = 1, 2, \ldots, n$ and almost every $a \in P_i \cap A$, $f_i(a) = 0$ (or, equivalently, for almost every $a \in A$, a is not given to any of the players whose vertices determine the face of the simplex of which $f(a)$ is an interior point).
b. For some $\omega \in S^+_{\delta_P}$, the partition $\langle P_i \cap B : i \in \delta_P \rangle$ of B is chores w-associated with ω.

By Theorem 10.20, condition b is equivalent to the assertion that $\langle P_1 \cap B, P_2 \cap B, \ldots, P_n \cap B \rangle$ is a Pareto minimal partition of B.

The proof of Theorem 10.30 follows easily from our preceding discussion and we omit it. Clearly, this result is simpler than Theorem 10.28 (which involved the notion of partition sequence pair and was best perceived in iterative terms). This is similar to what we saw in Chapter 7 when we studied maximization and minimization of convex combinations of measures (compare Theorems 7.13 and 7.18 with Theorems 7.23 and 7.24, respectively).

11
The Shape of the IPS

In this chapter, we investigate the possible shapes of the IPS. In Section 11A, we consider the two-player context, where we shall provide a complete answer. In Section 11B, we consider the n-player context for $n > 2$, where we are able to provide only a partial answer. For any cake C and corresponding measures $m_1, m_2, \ldots, m_n$, let $\text{IPS}(C; m_1, m_2, \ldots, m_n)$ denote the IPS corresponding to cake C and measures $m_1, m_2, \ldots, m_n$ on C. We make no general assumptions about absolute continuity in this chapter.

11A. The Two-Player Context

In Chapter 2, we considered various properties of the IPS for the case of two players. In particular, we established Theorem 2.4, which told us that the IPS

a. is a subset of $[0, 1]^2$,
b. contains the points $(1, 0)$ and $(0, 1)$,
c. is closed,
d. is convex, and
e. is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$.

In this section, we show that these five properties completely characterize the possible shapes of the IPS. In other words, for any $G \subseteq \mathbf{R}^2$ that satisfies these five conditions, there is a cake C and measures m_1 and m_2 on C so that $G = \text{IPS}(C; m_1, m_2)$. Thus, once we have established this, we shall know, for example, that each of the objects in Figure 2.1 are IPSs for some cake C and measures m_1 and m_2. In the next section, we shall see that the situation is quite different when there are more than two players. Our main result in this section is the following.

Theorem 11.1 *Let G be a subset of $\mathbf{R}^2$. There exists a cake C and measures m_1 and m_2 on C such that $G = \text{IPS}(C; m_1, m_2)$ if and only if G*

a. is a subset of $[0, 1]^2$,
b. contains the points $(1, 0)$ and $(0, 1)$,
c. is closed,
d. is convex, and
e. is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$.

We wish to prove two simple lemmas that will be useful in the proof of the theorem. For convenience, let IPS^{out} denote the outer boundary of the IPS. (For the definition of "outer boundary" in the two-player context, see Definition 3.7.)

Suppose that G is any set that satisfies the five conditions of the theorem. Conditions a and d tell us that the boundary of G is a connected and closed curve. Conditions a and b imply that this curve contains the points $(1, 0)$ and $(0, 1)$. We can define the outer boundary of G, denoted by G^{out}, in precisely the same way that we defined the outer boundary of the IPS in Chapter 3. Thus, G^{out} consists of all points (x, y) on the boundary of G for which $x + y \geq 1$. Equivalently, G^{out} is the subset of the boundary of G that includes the points $(1, 0)$ and $(0, 1)$ and all other points of the boundary that are not on the same side of the segment connecting $(1, 0)$ and $(0, 1)$ as is the origin.

Lemma 11.2 *Suppose that G and H each satisfy the five conditions of the theorem. If $G^{\text{out}} \subseteq H^{\text{out}}$, then $G^{\text{out}} = H^{\text{out}}$.*

Proof: Assume that G and H each satisfy the five conditions of the lemma. By condition d, G^{out} and H^{out} are each connected curves. By conditions a and b, each has endpoints $(1, 0)$ and $(0, 1)$. Then, the only way we could have $G^{\text{out}} \subseteq H^{\text{out}}$ and $G^{\text{out}} \neq H^{\text{out}}$ is if H^{out} is self-intersecting, and this is certainly not possible, given that H^{out} is a subset of the boundary of a convex set. □

A more explicit proof of Lemma 11.2 could be constructed by using the fact that G^{out} and H^{out} can each be viewed as the range of a one-to-one function of the angle between the positive x axis and the line segments connecting the origin to points on the curve. Both functions have the same domain ($[0, 90°]$). It is then straightforward to show that $G^{\text{out}} \subseteq H^{\text{out}}$ implies $G^{\text{out}} = H^{\text{out}}$.

Lemma 11.3 *Any set satisfying the five conditions of the theorem is uniquely determined by its outer boundary. In other words, if G and H each satisfy the five conditions of the theorem and $G^{\text{out}} = H^{\text{out}}$, then $G = H$.*

Proof: Let G be any set satisfying the five conditions of the theorem. By condition e, the outer boundary of G uniquely determines the inner boundary of G. Conditions a and b imply that the outer boundary has endpoints $(1, 0)$, $(0, 1)$. Hence, the inner boundary has endpoints $(1, 0)$ and $(0, 1)$. It follows that the outer and inner boundaries together make a closed curve and, by conditions c and d, G consists of this curve together with the region enclosed by this curve. This determination of G from its outer boundary is clearly unique. □

Proof of Theorem 11.1: The forward direction is Theorem 2.4.

For the reverse direction, suppose that G is a subset of $\mathbf{R}^2$ that satisfies the five given conditions. We must find a cake C and measures m_1 and m_2 on C such that $G = \text{IPS}(C; m_1, m_2)$.

We define the cake C to be G^{out}. For any $A \subseteq C$, let A_1 be the projection of A onto the x axis and let A_2 be the projection of A onto the y axis. Let m_L be Lebesgue measure on the real number line and define m_1 and m_2 on C as follows: for any such $A \subseteq C$, $m_1(A) = m_L(A_1)$ and $m_2(A) = m_L(A_2)$.

It is easy to see that m_1 and m_2 are (countably additive, non-atomic, probability) measures on C. We shall say that a piece of cake has positive measure if and only if it has positive measure with respect to the measure $\mu = m_1 + m_2$.

We claim that $G = \text{IPS}(C; m_1, m_2)$. For simplicity, let IPS denote $\text{IPS}(C; m_1, m_2)$ for the remainder of the proof.

Since we already know that the forward direction of the theorem is true, we know that IPS satisfies the five conditions of the theorem. Thus, since G and IPS each satisfy these conditions, it follows from Lemmas 11.2 and 11.3 that in order to show $G = \text{IPS}$ it suffices to show $G^{\text{out}} \subseteq \text{IPS}^{\text{out}}$.

We need to develop some notation. For any $(p, q) \in C$, let $UL(p, q)$ be that portion of the curve C that is between $(0, 1)$ and (p, q), including $(0, 1)$ but not (p, q), and let $LR(p, q)$ be that portion of the curve C that is between (p, q) and $(1, 0)$, including both (p, q) and $(1, 0)$. (Our specification of which set includes the point (p, q) is arbitrary.) The terms "UL" and "LR" are meant to denote "upper left" and "lower right," respectively.

Suppose that $(p, q) \in G^{\text{out}}$ and, hence, $(p, q) \in C$. We must show that $(p, q) \in \text{IPS}^{\text{out}}$. Consider the partition of C given by $\langle \text{UL}(p, q), \text{LR}(p, q) \rangle$. We have

$$\begin{aligned} m_1(\text{UL}(p, q)) &= m_L(\text{projection of UL}(p, q) \text{ onto the } x \text{ axis}) \\ &= m_L([0, p)) = p \end{aligned}$$

and

$$\begin{aligned} m_2(\text{LR}(p, q)) &= m_L(\text{projection of LR}(p, q) \text{ onto the } y \text{ axis}) \\ &= m_L([0, q]) = q. \end{aligned}$$

This tells us that $(p, q) \in \text{IPS}$. To show that $(p, q) \in \text{IPS}^{\text{out}}$, we consider three cases.

Case 1: $p = 1$. In this case, (p, q) is at or above the point $(1, 0)$. Since $\text{IPS} \subseteq [0, 1]^2$, it follows that $(p, q) \in \text{IPS}^{\text{out}}$.

Case 2: $q = 1$. In this case, (p, q) is at or to the right of the point $(0, 1)$. As in case 1, since $\text{IPS} \subseteq [0, 1]^2$, it follows that $(p, q) \in \text{IPS}^{\text{out}}$.

Case 3: $p \neq 1$ and $q \neq 1$. Since $(p, q) \in G^{\text{out}}$, we know that $p \neq 0$ and $q \neq 0$, since any point on G^{out} with one coordinate equal to zero must have the other coordinate equal to one. We establish two claims from which the desired result will easily follow.

Claim 1 If $A \subseteq \text{UL}(p, q)$ and $B \subseteq \text{LR}(p, q)$ each have positive measure, then $\frac{m_2(A)}{m_1(A)} \leq \frac{m_2(B)}{m_1(B)}$.

Proof of Claim: For any line ℓ, let $s(\ell)$ be the slope of ℓ. We begin by noting that, given two secant lines ℓ_1 and ℓ_2 to C, if ℓ_1 connects two points of $\text{UL}(p, q)$, and ℓ_2 connects two points of $\text{LR}(p, q)$, then $|s(\ell_1)| \leq |s(\ell_2)|$. (It is possible that ℓ_1 is a horizontal line, in which case $s(\ell_1) = 0$, or that ℓ_2 is a vertical line, in which case $s(\ell_2) = \infty$. In either of these cases, the given inequality is certainly true. Note that if ℓ_1 is not horizontal then $s(\ell_1)$ is negative, and if ℓ_2 is not vertical then $s(\ell_2)$ is negative.)

Let D be any non-empty connected subset of C. In particular, let us assume that D runs between the two points (a, b) and (c, d) of C. Then, $m_1(D) = |c - a|$ and $m_2(D) = |d - b|$, and so $\frac{m_2(D)}{m_1(D)} = \left|\frac{d-b}{c-a}\right|$, which is the absolute value of the slope of the secant line to C between (a, b) and (c, d).

Suppose that $A \subseteq \text{UL}(p, q)$ and $B \subseteq \text{LR}(p, q)$ are each connected and non-empty. The previous two paragraphs imply that $\frac{m_2(A)}{m_1(A)} \leq \frac{m_2(B)}{m_1(B)}$. This establishes the claim for the special case where A and B are each connected subsets of C. Using this, it is straightforward to verify that the claim holds if A and B are each (finite or infinite) unions of connected subsets of C.

We wish to define open subsets and open intervals of C. As we did before, for any $A \subseteq C$, let A_1 be the projection of A onto the x axis and let A_2 be the projection of A onto the y axis. We shall say that A is *open* if and only if $A_1 \backslash \{(1, 0), (0, 0)\}$ and $A_2 \backslash \{(0, 1), (0, 0)\}$ are each open, and that it is an *open interval* if and only if $A_1 \backslash \{(1, 0), (0, 0)\}$ and $A_2 \backslash \{(0, 1), (0, 0)\}$ are each open intervals. (We consider the empty set to be an open interval of the real number line.) Because every open set on the real number line is the union of open intervals, it is not hard to see that every open subset of C is the union of open intervals of C. Then, since every open interval of C is a connected subset of C, our preceding work tells us that the claim holds if $A \subseteq \mathrm{UL}(p, q)$ and $B \subseteq \mathrm{LR}(p, q)$ are each non-empty open subsets of C.

To establish the claim, we suppose, by way of contradiction, that there exist positive-measure sets $A \subseteq \mathrm{UL}(p, q)$ and $B \subseteq \mathrm{LR}(p, q)$ such that $\frac{m_2(A)}{m_1(A)} > \frac{m_2(B)}{m_1(B)}$. We can approximate each of these fractions as closely as we wish using open subsets of $\mathrm{UL}(p, q)$ and $\mathrm{LR}(p, q)$. Hence, there are non-empty open sets $A \subseteq \mathrm{UL}(p, q)$ and $B \subseteq \mathrm{LR}(p, q)$ with $\frac{m_2(A)}{m_1(A)} > \frac{m_2(B)}{m_1(B)}$. This contradicts our conclusion in the previous paragraph and, hence, establishes the claim.

Claim 2 $\langle \mathrm{UL}(p, q), \mathrm{LR}(p, q)\rangle$ is a Pareto maximal partition of C.

Proof of Claim: We use partition ratios. By Theorem 8.9, we must show that, for any cyclic sequence φ, $\mathrm{CP}(\varphi) \leq 1$. We consider the product of the two partition ratios that exist in this $n = 2$ case. We have

$$\begin{aligned}\mathrm{pr}_{12} &= \sup\left\{\frac{m_2(A)}{m_1(A)} : A \subseteq \mathrm{UL}(p, q) \text{ and either } m_1(A) > 0 \text{ or } m_2(A) > 0\right\}\\ &= \sup\left\{\frac{m_2(A)}{m_1(A)} : A \subseteq \mathrm{UL}(p, q) \text{ and } A \text{ has positive measure}\right\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{pr}_{21} &= \sup\left\{\frac{m_1(B)}{m_2(B)} : B \subseteq \mathrm{LR}(p, q) \text{ and either } m_1(B) > 0 \text{ or } m_2(B) > 0\right\}\\ &= \sup\left\{\frac{m_1(B)}{m_2(B)} : B \subseteq \mathrm{LR}(p, q) \text{ and B has positive measure}\right\}.\end{aligned}$$

Note that since we are in Case 3, $m_1(\mathrm{UL}(p, q)) > 0$ and $m_2(\mathrm{LR}(p, q)) > 0$. We wish to show that $\mathrm{pr}_{12}\mathrm{pr}_{21} \leq 1$

Suppose, by way of contradiction, that $\mathrm{pr}_{12}\mathrm{pr}_{21} > 1$. Then there exist $A \subseteq \mathrm{UL}(p, q)$ and $B \subseteq \mathrm{LR}(p, q)$, both of positive measure, such that $(\frac{m_2(A)}{m_1(A)})(\frac{m_1(B)}{m_2(B)}) > 1$. It follows that $\frac{m_2(A)}{m_1(A)} > \frac{m_2(B)}{m_1(B)}$. This contradicts Claim 1 and, hence, establishes Claim 2.

Returning to the proof of Case 3 of the theorem, it follows from Claim 2 that, since $m_1(\mathrm{UL}(p, q)) = p$ and $m_2(\mathrm{LR}(p, q)) = q$, the point (p, q) is on the outer Pareto boundary of the IPS. Depending on whether the measures are, or are not, absolutely continuous with respect to each other, it follows from Theorem 3.9, or Theorem 3.24, respectively, that $(p, q) \in \mathrm{IPS}^{\mathrm{out}}$. This establishes Case 3 and, hence, completes the proof of the theorem. □

11B. The Case of Three or More Players

The statement of Theorem 11.1 generalizes in a natural way to three players. Unfortunately, this generalization is false. In this section, we examine why this is so. This will involve continuing our study of notions of symmetry appropriate for the $n > 2$ context that we began in Section 4A. We will close this section with open questions regarding the possibility of generalizing Theorem 11.1.

Suppose that C is a cake and m_1, m_2, and m_3 are measures on C. Theorem 4.3 tells us that the first four conditions of Theorem 2.4 generalize in a natural way to the three-player context. In other words, IPS$(C; m_1, m_2, m_3)$

a. is a subset of $[0, 1]^3$,
b. contains the two-simplex,
c. is closed, and
d. is convex.

We also showed (see Theorem 4.2) that IPS$(C; m_1, m_2, m_3)$ consists precisely of the two-simplex if and only if the three measures are identical.

In Section 4A, we considered generalizations of condition e of Theorem 2.4, which told us that, in the two-player context, the IPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$. These generalizations are given by Theorem 4.8 and Corollary 4.9. These results suggest, but do not establish, that the most obvious generalization to the three-player context of symmetry about $(\frac{1}{2}, \frac{1}{2})$ (i.e., symmetry about $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$), is false. We now show that this is so and then connect this with our study of the possible shapes of the IPS. (We did not establish that this generalization is false in Chapter 4, since it will be convenient for us to use partition ratios to do so, and we did not introduce partition ratios until Chapter 8.) For convenience, we first state what Corollary 4.9 says about the three-player context.

Corollary 11.4 *Suppose that $p = (p_1, p_2, p_3) \in$ IPS. If q is the point such that p, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and q are collinear, with $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ between p and q, and the distance from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to q is one-half the distance from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to p, then $q = (\frac{1-p_1}{2}, \frac{1-p_2}{2}, \frac{1-p_3}{2})$ and $q \in$ IPS.*

Theorem 11.5 *Corollary 11.4 is the best possible result of this sort. In other words, if the "one-half" in the corollary is replaced by any larger number, the statement would not be true.*

Proof: It suffices to present an example of a cake C, measures m_1, m_2, and m_3 on C, and a point p on the outer boundary of IPS$(C; m_1, m_2, m_3)$, such that if q is related to p as in Corollary 11.4 then q is on the inner boundary of IPS$(C; m_1, m_2, m_3)$.

Let the cake C be the interval $[0, 3)$ and let m_L be Lebesgue measure on this interval. Define measures m_1, m_2, and m_3 on C as follows: for any $A \subseteq C$,

$$\begin{aligned} m_1(A) &= .6m_L(A \cap [0, 1)) + .2m_L(A \cap [1, 2)) + .2m_L(A \cap [2, 3)) \\ m_2(A) &= .2m_L(A \cap [0, 1)) + .6m_L(A \cap [1, 2)) + .2m_L(A \cap [2, 3)) \\ m_3(A) &= .2m_L(A \cap [0, 1)) + .2m_L(A \cap [1, 2)) + .6m_L(A \cap [2, 3)) \end{aligned}$$

It is easy to verify that m_1, m_2, and m_3 are (countably additive, non-atomic, probability) measures on C and that they are absolutely continuous with respect to each other.

Consider the partition $P = \langle [0, 1), [1, 2), [2, 3) \rangle$ of C. We claim that this partition is Pareto maximal. We establish this by computing the partition ratios and applying Theorem 8.9.

To compute pr_{12}, we note that, for any positive-measure $B \subseteq [0, 1)$, $\frac{m_2(B)}{m_1(B)} = \frac{.2m_L(B)}{.6m_L(B)} = \frac{1}{3}$. Hence, $\mathrm{pr}_{12} = \frac{1}{3}$. The other partition ratios are computed similarly and we find that all partition ratios equal $\frac{1}{3}$. It is then easy to check all cyclic products:

$$\mathrm{pr}_{12}\mathrm{pr}_{21} = \mathrm{pr}_{13}\mathrm{pr}_{31} = \mathrm{pr}_{23}\mathrm{pr}_{32} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\mathrm{pr}_{12}\mathrm{pr}_{23}\mathrm{pr}_{31} = \mathrm{pr}_{32}\mathrm{pr}_{21}\mathrm{pr}_{13} = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

Since all cyclic products are less than one, it follows from Theorem 8.9 that P is Pareto maximal. This tells us that $m(P)$ is on the outer Pareto boundary of IPS$(C; m_1, m_2, m_3)$ and, thus, is on the outer boundary of IPS$(C; m_1, m_2, m_3)$. We compute the coordinates of $m(P)$ as follows:

$$\begin{aligned} m(P) &= (m_1([0, 1)), m_2([1, 2)), m_3([2, 3))) \\ &= (.6m_L([0, 1)), .6m_L([1, 2)), .6m_L([2, 3))) = (.6, .6, .6) \end{aligned}$$

We know that $(\frac{1-.6}{2}, \frac{1-.6}{2}, \frac{1-.6}{2}) = (.2, .2, .2)$ is related to $(.6, .6, .6)$ as q is related to p in Corollary 11.4, and this result tells us that $(.2, .2, .2) \in$ IPS$(C; m_1, m_2, m_3)$. To prove the theorem, it suffices to show that $(.2, .2, .2)$ is on the inner boundary of the IPS.

We first show that $(.2, .2, .2)$ is a Pareto minimal point. We wish to find a partition that corresponds to the point $(.2, .2, .2)$ and to show that this partition is Pareto minimal. The key to finding such a partition lies in our alternate proof of Corollary 4.9. Each player's piece (according to partition

$P = \langle[0, 1), [1, 2), [2, 3)\rangle$ of C, as before) can be divided into what each of the three players believes are two equal pieces. (In general, this sort of step requires Lyapounov's theorem, Theorem 1.3. However, in this case, Lyapounov's theorem is not needed, since all players always agree about what is a division of $[0, 1)$ or $[1, 2)$ or $[2, 3)$ into two equal pieces.) Each player then gives one of these two pieces to each of the other two players.

Proceeding with this idea, we consider the partition $Q = \langle[1.5, 2.5), [0, .5) \cup [2.5, 3), [.5, 1.5)\rangle$. We compute the coordinates of the corresponding point in the IPS, $m(Q)$, as follows:

$$\begin{aligned} m(Q) &= (m_1([1.5, 2.5)), m_2([0, .5) \cup [2.5, 3)), m_3([.5, 1.5))) \\ &= (.2m_L([1.5, 2)) + .2m_L([2, 2.5)), .2m_L([0, .5)) \\ &\quad + .2m_L([2.5, 3)), .2m_L([.5, 1)) + .2m_L([1, 1.5))) = (.2, .2, .2) \end{aligned}$$

We must show that Q is Pareto minimal and, hence, that $(.2, .2, .2)$ is a Pareto minimal point of $\text{IPS}(C; m_1, m_2, m_3)$. To do so, we compute the chores partition ratios (see Definition 8.12).

To compute qr_{12}, suppose that $B \subseteq [1.5, 2.5)$ and B has positive measure. Then,

$$\begin{aligned} \frac{m_2(B)}{m_1(B)} &= \frac{m_2(B \cap [1.5, 2)) + m_2(B \cap [2, 2.5))}{m_1(B \cap [1.5, 2)) + m_1(B \cap [2, 2.5))} \\ &= \frac{.6m_L(B \cap [1.5, 2)) + .2m_L(B \cap [2, 2.5))}{.2m_L(B \cap [1.5, 2)) + .2m_L(B \cap [2, 2.5))}. \end{aligned}$$

This tells us that

- if $B \subseteq [1.5, 2)$, then $\frac{m_2(B)}{m_1(B)} = \frac{.6m_L(B\cap[1.5,2))}{.2m_L(B\cap[1.5,2))} = 3$.
- if $B \subseteq [2, 2.5)$, then $\frac{m_2(B)}{m_1(B)} = \frac{.2m_L(B\cap[2,2.5))}{.2m_L(B\cap[2,2.5))} = 1$.
- if $B \not\subseteq [1.5, 2)$ and $B \not\subseteq [2, 2.5)$, then $1 \le \frac{m_2(B)}{m_1(B)} \le 3$.

Thus, $\text{qr}_{12} = \inf\{\frac{m_2(B)}{m_1(B)} : B \subseteq [1.5, 2.5) \text{ and B has positive measure}\} = 1$. The other chores partition ratios are computed similarly, and we find that all chores partition ratios equal one. This implies that all chores cyclic products are equal to one and so, by Theorem 8.14, Q is a Pareto minimal partition of C. Thus, $(.2, .2, .2)$ is a Pareto minimal point.

It follows that $(.2, .2, .2)$ is on the inner Pareto boundary of $\text{IPS}(C; m_1, m_2, m_3)$. This implies that $(.2, .2, .2)$ is on the inner boundary of $\text{IPS}(C; m_1, m_2, m_3)$ and, hence, establishes the theorem. □

In the preceding proof, the measures are all absolutely continuous with respect to each other. We presented such an example to show that Corollary 11.4 is the best possible result of this sort, even if the measures are absolutely continuous with respect to each other. There is an easier example that involves the most extreme failure of absolute continuity. Suppose that m_1, m_2, and m_3 concentrate on disjoint sets. (See Definition 5.39 and the discussion following the definition.) Then $(0, 0, 0) \in \text{IPS}(C; m_1, m_2, m_3)$ and $(1, 1, 1) \in \text{IPS}(C; m_1, m_2, m_3)$. Since $\text{IPS}(C; m_1, m_2, m_3)$ is a subset of $[0, 1]^3$, it follows that $(0, 0, 0)$ is on the inner boundary and that $(1, 1, 1)$ is on the outer boundary of $\text{IPS}(C; m_1, m_2, m_3)$. Note that $(1, 1, 1)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and $(0, 0, 0)$ are collinear, with $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ between $(1, 1, 1)$ and $(0, 0, 0)$, and the distance from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to $(0, 0, 0)$ is one-half the distance from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to $(1, 1, 1)$. Then, with $(1, 1, 1)$ and $(0, 0, 0)$ playing the roles of p and q, respectively, in Corollary 11.4, it is clear that this corollary is the best possible. A point q corresponding to a number greater than the "one-half" in the corollary would have all negative coordinates, and this is certainly not possible for a point in the IPS.

It is not hard to construct examples as in the proof of Theorem 11.5, or as in the preceding paragraph, when there are more than three players.

Theorem 11.5 tells us that the natural generalization of Theorem 11.1 to the three-player context is false since, in general, we do not have the symmetry about $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ that we may have expected, given the symmetry about $(\frac{1}{2}, \frac{1}{2})$ that holds in the two-player context. However, Corollary 11.4 (or, more generally, Corollary 4.9) does tell us that the outer boundary of the IPS and the inner boundary of the IPS have some connection. We cannot, for example, have the outer boundary be far away from the simplex and have the inner boundary be very close to the simplex. Can we say more when there are three or more players? In other words:

Open Question 11.6 Is there anything specific we can say about any connection between the shape of the outer boundary and the inner boundary of the IPS for $n = 3$, or for $n > 3$?

It appears that a generalization of Theorem 11.1 to the context of three or more players would necessitate a clear affirmative answer to this question. In the absence of such an answer, a natural move is to disconnect the issues of the shape of the outer boundary and the shape of the inner boundary. In other words, we can ask the following.

Open Question 11.7 What are the possible shapes of the "outer IPS," i.e., the subset of the IPS consisting of all points of the IPS that are not on the same side of the simplex as is the origin?

Clearly, this is the same as asking:

What are the possible shapes of the outer boundary of the IPS?

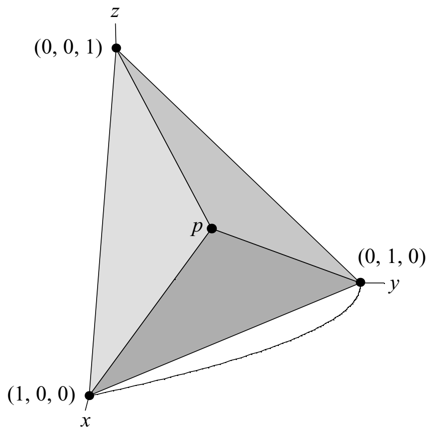


Figure 11.1

The same can be asked about the "inner IPS" or the inner boundary of the IPS. Then, one might ask whether the answer to each of these questions is provided by something like Theorem 11.1, with the obvious variation of conditions a and b ("G is a subset of $\{(p_1, p_2, \ldots, p_n) \in [0, 1]^n : p_1 + p_2 + \cdots + p_n \geq 1\}$" and "$G$ contains the points $(1, 0, 0, \ldots, 0, 0)$, $(0, 1, 0, \ldots, 0, 0), \ldots, (0, 0, 0, \ldots, 0, 1)$") and with conditions c and d. Such a theorem is certainly true for the separate "outer" and "inner" questions for the two-player context. The following observation establishes that this is not so in the three-player context.

Observation 11.8 Pick any point $p = (p_1, p_2, p_3)$ with $0 \leq p_1 \leq 1, 0 \leq p_2 \leq 1, 0 \leq p_3 \leq 1$, and $p_1 + p_2 + p_3 > 1$. If the "outer boundary" version of Theorem 11.1 for the three-player context were true, then there would exist a cake and corresponding measures such that the associated outer IPS is $\mathrm{CH}(\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (p_1, p_2, p_3)\})$. (Recall that "CH" denotes "convex hull.") Equivalently, there would exist such a cake and corresponding measures so that the outer boundary of the associated IPS consists of the three triangles determined by the point (p_1, p_2, p_3) and each of the three pairs of points from $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. We claim that this is not possible. Let $G = \mathrm{CH}(\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (p_1, p_2, p_3)\})$ and let G^{out} denote its outer boundary. This situation is illustrated in Figure 11.1. The three triangles making up G^{out} are shaded.

Suppose, by way of contradiction, that cake C and measures m_1, m_2, and m_3 on C are such that the outer boundary of $\mathrm{IPS}(C; m_1, m_2, m_3)$ is G^{out}. The fact that $\mathrm{IPS}(C; m_1, m_2, m_3)$ consists of more that just the simplex tells us that the measures m_1, m_2, and m_3 are not identical. Suppose, without loss of generality, that $m_1 \neq m_2$, and consider the collection of Pareto maximal partitions of C in which Player 3 gets no cake. The corresponding subset of the outer boundary of the IPS is a curve in the $z = 0$ plane. Since $m_1 \neq m_2$, this curve is not the line segment from $(1, 0, 0)$ to $(0, 1, 0)$. One possibility for this curve is shown in the figure. It is clearly not a subset of G^{out}, since G^{out}'s intersection with the $z = 0$ plane consists of the line segment from $(1, 0, 0)$ to $(0, 1, 0)$. Hence, there is no such cake C and measures m_1, m_2, and m_3 on C such that the outer boundary of $\text{IPS}(C; m_1, m_2, m_3)$ is G^{out}. (Of course, there are many different possibilities for this curve. Our conclusion does not depend on the particular curve that we have drawn.)

Observation 11.8 connects with ideas discussed in Chapters 3 and 5. Theorem 3.9 told us that in the two-player context if absolute continuity holds then the outer Pareto boundary and the inner Pareto boundary of the IPS are equal to the outer boundary and the inner boundary, respectively, of the IPS. Theorem 3.22 told us that this correspondence fails in the absence of absolute continuity. We saw in Theorem 5.13 that this correspondence also fails, even if the measures are all absolutely continuous with respect to each other, if there are more than two players and the measures are not all equal. Observation 11.8 provides some additional perspective. In this observation, the region between the curve in the $z = 0$ plane discussed in the previous paragraph and the line segment from $(1, 0, 0)$ to $(0, 1, 0)$, lies on the outer boundary of the IPS but not on the outer Pareto boundary.

We note that Observation 11.8 is valid with or without any absolute continuity assumptions.

We close this section by asking whether Theorem 11.1 can be generalized to the three-player context if we focus separately on the outer and inner IPS, drop condition e, and add a new condition to require that the appropriate curve in the $z = 0$ plane, and the analogous curves in the $x = 0$ and the $y = 0$ planes, be included whenever a point (p_1, p_2, p_3), as in Observation 11.8, is included. More specifically, we ask the following.

Open Question 11.9 Is there a function h with domain the unit cube, whose range consists of subsets of $\mathbf{R}^3$, so that the following result is true:

Let G be a subset of $\mathbf{R}^3$. There exists a cake C and measures m_1, m_2, and m_3 on C such that G is the "outer $\text{IPS}(C; m_1, m_2, m_3)$" (i.e., $G = \{(p_1, p_2, p_3) \in \text{IPS}(C; m_1, m_2, m_3) : p_1 + p_2 + p_3 \geq 1\}$) if and only if G

a. is a subset of $\{(p_1, p_2, p_3) \in [0, 1]^3 : p_1 + p_2 + p_3 \geq 1\}$;
b. contains the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$;

c. is closed;
d. is convex; and
e. is closed under h (i.e., if $(p_1, p_2, p_3) \in G$, then $f(p_1, p_2, p_3) \subseteq G$).

The intuition here is that the function h is producing the necessary three curves in the three coordinate planes, as discussed before.

Of course, given an affirmative answer to Open Question 11.9, a natural next step is to pursue an analogous result for more than three players.

12

The Relationship Between the IPS and the RNS

In Chapter 7, we studied the relationship between the IPS and maximization of convex combinations of measures. In Chapter 10 (see Theorem 10.6) we used the notion of w-association to study the relationship between the RNS and maximization of convex combinations of measures. In this chapter, we put these ideas together to enable us to understand the relationship between the IPS and the RNS. In Section 12A, we introduce a relation that will be useful in Sections 12B, 12C, and 12D. In Section 12B, we examine this relation in the two-player context. In Section 12C, we consider the general n-player context. We assume in Sections 12A, 12B, and 12B that the measures are absolutely continuous with respect to each other. In Section 12D, we consider the situation without this assumption. In Section 12E, we also do not assume that the measures are absolutely continuous with respect to each other and we use the IPS and the RNS together to show that there exists a partition that is Pareto maximal and envy-free.

12A. Introduction

We recall that S^+ denotes the interior of the simplex S. For $\omega \in S^+$, we let ω^* denote the set of partitions that are w-associated with ω. Then, for every $\omega \in S^+$, $\omega^* \neq \emptyset$ and, by Theorem 10.9, each partition in ω^* is Pareto maximal. If P and Q are p-equivalent partitions, then $P \in \omega^*$ if and only if $Q \in \omega^*$. (See the paragraph following the proof of Theorem 10.6.) It follows that if $P \in \omega^*$ then every partition in $[P]_p$ is Pareto maximal and is in ω^*. In this case, we shall say that the p-class $[P]_p$ is in ω^*, or that $[P]_p$ is w-associated with ω. Theorem 10.9 also tells us that any Pareto maximal partition P that gives a positive-measure piece of cake to each player is in ω^* for some $\omega \in S^+$.

The following three ideas and related questions will be central to our approach in this chapter. The first involves the RNS, the second involves the IPS, and the third involves the relationship between the RNS and the IPS.

a. The first concerns the RNS. For any $\omega \in S^+$, there is at least one p-class of Pareto maximal partitions that is w-associated with ω. Is there more than one? On the other hand, for any Pareto maximal partition P that gives a positive-measure piece of cake to each player, there is at least one point $\omega \in S^+$ with which $[P]_p$ is w-associated. Is there more than one? More generally, when (if ever) is the relation between Pareto maximal p-classes and points in S^+, given by "w-association," one-one, one-many, many-one, or many-many?
b. The second concerns the IPS. For any point p on the outer boundary of the IPS, there is at least one $\alpha \in S$ such that the family of parallel hyperplanes with coefficients given by α makes first contact with the IPS at p. Is there more than one? On the other hand, for any $\alpha \in S$, there is at least one p on the outer boundary of the IPS that is the point of first contact with the IPS of the family of parallel hyperplanes with coefficients given by α. Is there more than one? More generally, when (if ever) is the relation between points on the outer boundary of the IPS and points in S, given by "point(s) of first contact with the IPS of families of parallel hyperplanes," one-one, one-many, many-one, or many-many?
c. Recall that
   - Theorem 10.6 establishes that the function RD (see Definition 10.5) provides a one-to-one correspondence between points in S^+ to be thought of as in w-association and points in S^+ to be thought of as in the maximization of convex combinations of measures.
   - in Chapter 7 (see the discussion following the proof of Theorem 7.4) we saw that for any $\alpha \in S$ a partition P maximizes the convex combination of measures associated with α if and only if $m(P)$ is a point of first contact with the IPS of the family of parallel hyperplanes with coefficients given by α.

   This yields a correspondence between points in S^+ to be thought of as in w-association and points on the outer Pareto boundary of the IPS. This correspondence will be central to our examination of the relationship between the IPS and the RNS.

Notice that there is a slight awkwardness in that some of these ideas involve points from S^+ and others involve points from S. This will not cause any difficulties.

Studying these ideas and answering the related questions will enable us to connect various geometric properties of the IPS and the RNS. We begin by introducing a relation.

Definition 12.1 We define the relation M between S^+ and the set of p-classes of Pareto maximal partitions as follows: for $\omega \in S^+$ and P, a Pareto maximal partition, $M(\omega, [P]_p)$ holds if and only if P is w-associated with ω.

As discussed earlier, if partitions P and Q are p-equivalent partitions and $\omega \in S^+$, then P is w-associated with ω if and only if Q is w-associated with ω. Hence, M is well defined.

By the one-to-one correspondence given to us by the function RD, we will allow the first coordinate of M to denote either a point in S^+ to be thought of as in w-association or else a point in S^+ to be thought of as in the maximization of a convex combination of measures. (It will be clear by context which is meant.)

Recall that the function m provides a one-to-one correspondence between p-classes of partitions and points in the IPS (see Definition 4.1 and the discussion following the definition), and notice that if $m([P]_p) = q$ then P is Pareto maximal if and only if q is on the outer Pareto boundary of the IPS. Hence, m provides a one-to-one correspondence between Pareto maximal p-classes of partitions and points on the outer Pareto boundary of the IPS. Using this correspondence, we now allow the second coordinate of M to denote either a p-class of Pareto maximal partitions or else a point on the outer Pareto boundary of the IPS. (As before, it will be clear by context which is meant.)

Fix $\omega, \alpha \in S^+$ with $\text{RD}(\omega) = \alpha$. (Recall the notational convention that we introduced in Chapter 10: ω denotes a point to be thought of as in w-association, and α denotes a point to be thought of as in the maximization of a convex combination of measures.) Also, fix a Pareto maximal partition P and a point q on the outer Pareto boundary of the IPS with $m(P) = q$. By Definition 12.1 and the preceding discussion, the following are equivalent:

- $M(\omega, [P]_p)$: Every partition in $[P]_p$ is w-associated with ω.
- $M(\omega, q)$: Every partition in $m^{-1}(q)$ is w-associated with ω.
- $M(\alpha, [P]_p)$: Every partition in $[P]_p$ maximizes the convex combination of measures corresponding to α.
- $M(\alpha, q)$: Every partition in $m^{-1}(q)$ maximizes the convex combination of measures corresponding to α.

Notice that by the correspondence between maximization of convex combinations of measures and points of first contact with the IPS of families of parallel hyperplanes we know that $M(\alpha, q)$ holds if and only if the family of parallel hyperplanes with coefficients given by α makes first contact with the IPS at the point q (and possibly at other points too).

We shall freely use M in any of these four ways in this chapter. The questions asked in ideas a and b can now be seen to be the same question: when, if ever, is

M one-one, many-one, one-many, or many-many? Answering these questions will be an important part of our exploration of the relationship between the IPS and the RNS.

We shall not examine the chores versions of the ideas in this section until we consider the general n-player context in Section 12C.

12B. Relating the IPS and the RNS in the Two-Player Context

In this section, we restrict our attention to the case of two players. Hence, the setting for the RNS is the one-simplex, i.e., the line segment from (1, 0) to (0, 1). As usual, we shall denote this set by S and its interior by S^+. We recall that in the two-player context, when the measures are absolutely continuous with respect to each other, the outer boundary of the IPS is the same as the outer Pareto boundary of the IPS. (See Theorem 3.9. This is not the case if either there are more than two players or if absolute continuity fails. See Theorems 3.22, 5.13, and 5.35.)

By our work in Chapter 10, we know that for any $\omega \in S^+$ if we give to Player 1 all bits of cake that correspond to points of the RNS between (1, 0) and ω, give to Player 2 all bits of cake that correspond to points of the RNS between ω and (0, 1), and distribute all bits of cake that correspond to ω arbitrarily, the resulting partition is Pareto maximal and is w-associated with ω. Every Pareto maximal partition that gives a positive-measure piece of cake to each player is obtained in this way, for some $\omega \in S^+$.

We wish to revisit four of the IPSs from Figure 2.1. We have copied Figures 2.1a, 2.1b, 2.1c, and 2.1d (which are the four figures in Figure 2.1 that give IPSs for situations where the measures are absolutely continuous with respect to each other) as Figures 12.1ai, 12.1bi, 12.1ci, and 12.1di, respectively. In each of these figures, we have darkened the outer boundary. By Theorem 11.1, we know that for each of the four regions pictured there is a cake C and measures m_1 and m_2 on C so that the given figure is the corresponding IPS.

Our goal is to understand the corresponding RNSs. We claim that Figures 12.1aii, 12.1bii, 12.1cii, and 12.1dii are the RNSs corresponding to the IPSs in Figures 12.1ai, 12.1bi, 12.1ci, and 12.1di, respectively, in the sense that whatever cake and measures produce the given IPS also produce the corresponding RNS. We discuss these four correspondences in Examples 12.2, 12.3, 12.6, and 12.8.

Example 12.2 The correspondence between the IPS of Figure 12.1ai and the RNS of Figure 12.1aii.

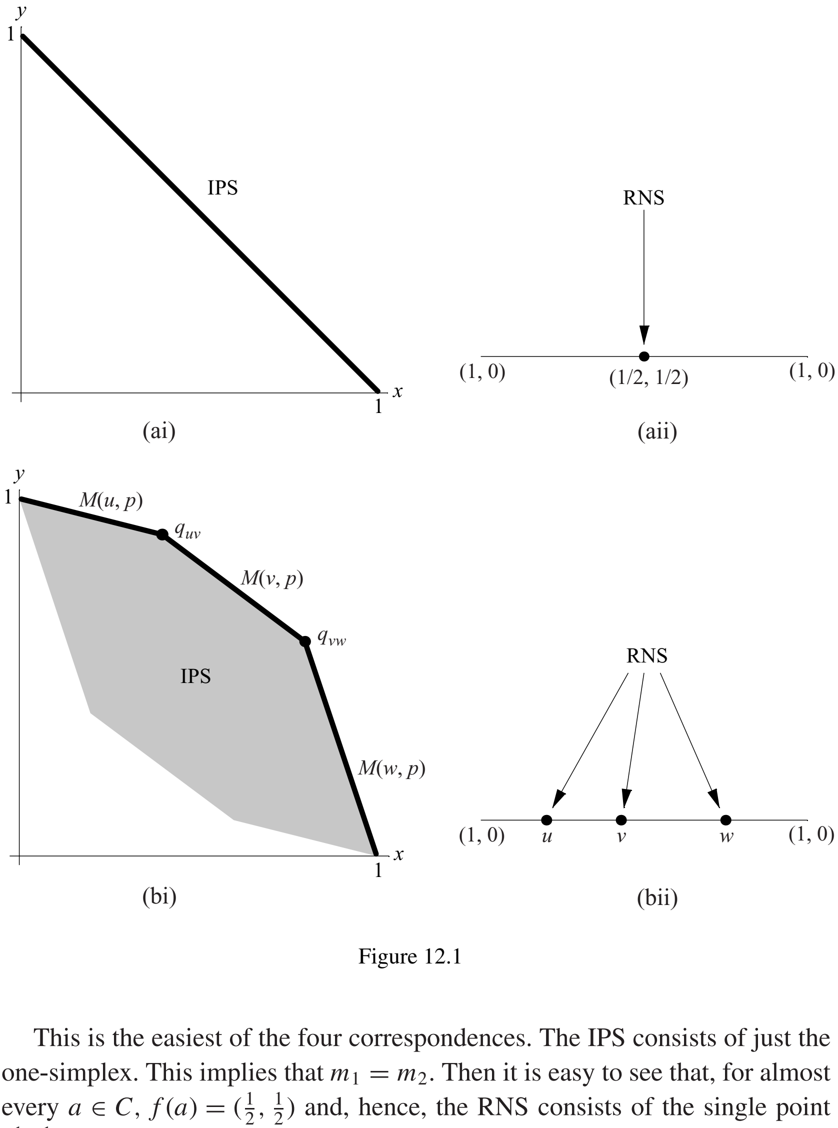


Figure 12.1

This is the easiest of the four correspondences. The IPS consists of just the one-simplex. This implies that $m_1 = m_2$. Then it is easy to see that, for almost every $a \in C$, $f(a) = (\frac{1}{2}, \frac{1}{2})$ and, hence, the RNS consists of the single point $(\frac{1}{2}, \frac{1}{2})$. We also note that

- for any ω between $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$, a partition is in ω^* if and only if it is p-equivalent to $\langle \emptyset, C \rangle$;
- for any ω between $(\frac{1}{2}, \frac{1}{2})$ and $(0, 1)$, a partition is in ω^* if and only if it is p-equivalent to $\langle C, \emptyset \rangle$; and
- since the entire cake is associated with the point $(\frac{1}{2}, \frac{1}{2})$, every partition is in $(\frac{1}{2}, \frac{1}{2})^*$.

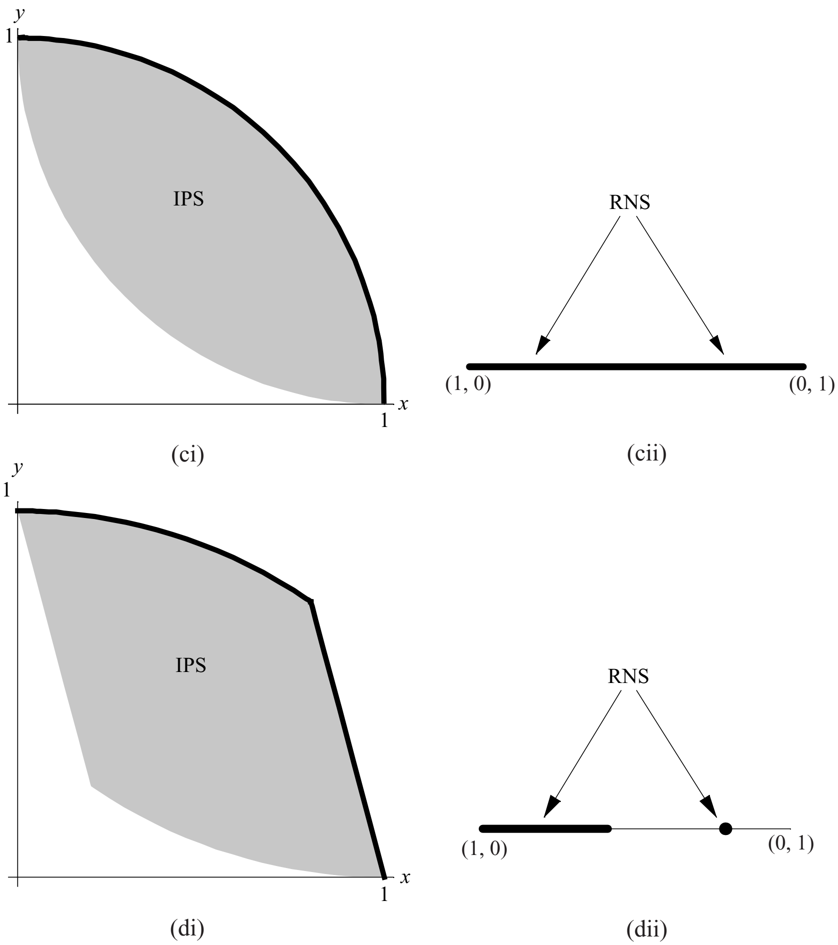


Figure 12.1 (*cont.*)

Hence,

- for any ω between $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$, and any p on the outer boundary of the IPS, $M(\omega, p)$ holds if and only if $p = (0, 1)$;
- for any ω between $(\frac{1}{2}, \frac{1}{2})$ and $(0, 1)$, and any p on the outer boundary of the IPS, $M(\omega, p)$ holds if and only if $p = (1, 0)$; and
- $M((\frac{1}{2}, \frac{1}{2}), p)$ holds for all p on the outer boundary of the IPS.

Notice that the first two of these situations illustrate that M can be many-one and the third illustrates that M can be one-many.

Example 12.3 The correspondence between the IPS of Figure 12.1bi and the RNS of Figure 12.1bii.

In the RNS picture, $u = (\frac{4}{5}, \frac{1}{5})$, $v = (\frac{4}{7}, \frac{3}{7})$, and $w = (\frac{1}{4}, \frac{3}{4})$. Each point of the cake C corresponds to one of these three points and each of these three points corresponds to a piece of cake of positive measure. Hence, the RNS consists precisely of these three points.

To see the correspondence between these two figures, we imagine a point ω in the RNS picture, beginning at $(1, 0)$ and moving to the right. For ω between $(1, 0)$ and u, ω^* consists of the single partition $\langle \emptyset, C\rangle$ and, hence, for any such ω, $M(\omega, p)$ holds if and only if $p = (0, 1)$.

For $\omega = u$, $\omega^* = u^*$ consists of the set of all partitions of the form $\langle P_1, P_2\rangle$, where P_1 contains only (but not necessarily all) points of C that correspond to the point u in the RNS. We wish to understand the shape of the corresponding part of the outer boundary of the IPS, i.e., the part that consists of all points p such that $M(u, p)$ holds.

Suppose that $P = \langle P_1, P_2\rangle$ and $Q = \langle Q_1, Q_2\rangle$ are distinct partitions in u^*. Set $A = P_1 \backslash Q_1$ and let $B = Q_1 \backslash P_1$. Then $Q_2 \backslash P_2 = A$ and $P_2 \backslash Q_2 = B$. We may view A as the piece of cake that Player 1 gives to Player 2, and B as the piece of cake that Player 2 gives to Player 1, in going from partition P to partition Q. Every point in A and every point in B is associated (via f) with the point $u = (\frac{4}{5}, \frac{1}{5})$ in the RNS.

Consider the quantity $\frac{m_2(Q_2)-m_2(P_2)}{m_1(Q_1)-m_1(P_1)}$:

$$\begin{aligned}\frac{m_2(Q_2)-m_2(P_2)}{m_1(Q_1)-m_1(P_1)} &= \frac{m_2((P_2\cap Q_2)\cup A)-m_2((P_2\cap Q_2)\cup B)}{m_1((P_1\cap Q_1)\cup B)-m_1((P_1\cap Q_1)\cup A)}\\ &= \frac{m_2(P_2\cap Q_2)+m_2(A)-m_2(P_2\cap Q_2)-m_2(B)}{m_1(P_1\cap Q_1)+m_1(B)-m_1(P_1\cap Q_1)-m_1(A)}\\ &= \frac{m_2(A)-m_2(B)}{m_1(B)-m_1(A)} = -\frac{m_2(B)-m_2(A)}{m_1(B)-m_1(A)}\end{aligned}$$

Since every point in A and every point in B is associated with $u = (\frac{4}{5}, \frac{1}{5})$, it follows that $\frac{m_2(A)}{m_1(A)} = \frac{m_2(B)}{m_1(B)} = \frac{1/5}{4/5} = \frac{1}{4}$. Hence, we have

$$\frac{m_2(Q_2)-m_2(P_2)}{m_1(Q_1)-m_1(P_1)} = -\frac{m_2(B)-m_2(A)}{m_1(B)-m_1(A)} = -\frac{m_2(B)-m_2(A)}{4m_2(B)-4m_2(A)} = -\frac{1}{4}.$$

But $\frac{m_2(Q_2)-m_2(P_2)}{m_1(Q_1)-m_1(P_1)}$ is simply the slope of the line segment between $(m_1(P_1), m_2(P_2))$ and $(m_1(Q_1), m_2(Q_2))$ in the IPS picture and, since both P and Q are in u^* and hence are Pareto maximal, we know that $(m_1(P_1), m_2(P_2))$ and $(m_1(Q_1), m_2(Q_2))$ are both on the outer boundary of the IPS. Since P and Q are arbitrarily chosen partitions in u^*, this tells us that the portion of the outer boundary of the IPS that is associated with u (via M) is a line segment with slope $-\frac{1}{4}$. It is the line segment labeled "$M(u, p)$" in Figure 12.1bi.

We continue to imagine the point ω moving from left to right in Figure 12.1bii. From $\omega = u$ until $\omega = v$, ω^* remains the same, since there are no points of the RNS in this open line segment. For such an ω, ω^* consists of the single partition $\langle P_1, P_2\rangle$, where P_1 is the piece of cake associated with u and P_2 is the piece of cake consisting of points associated with v or with w. This partition corresponds to the point q_{uv} in Figure 12.1bi. In other words, for such an ω, $M(\omega, p)$ holds if and only $p = q_{uv}$.

For $\omega = v$, an analysis similar to that used for $\omega = u$ tells us that the portion of the outer boundary of the IPS shown in Figure 12.1bi that is associated with v (via M) is a line segment with slope $-\frac{3}{4}$. It is labeled "$M(v, p)$" in Figure 12.1bi. The point q_{uv} is what we shall call a *corner point* on the outer boundary of the IPS. The slope of the outer boundary makes a discontinuous jump at such a point.

Continuing with this analysis, we find that, for any ω between v and w, $M(\omega, p)$ holds if and only if $p = q_{vw}$, and the portion of the outer boundary of the IPS that is associated with w (via M) is a line segment with slope –3, labeled "$M(w, p)$" in Figure 12.1bi. Finally, for ω between w and (0, 1), $M(\omega, p)$ holds if and only if $p = (1, 0)$.

Before continuing with our analysis of the other IPSs and RNSs in Figure 12.1, we make some observations. Observations 12.4 and 12.5 follow from the preceding analysis together with our work in Chapter 7 on relating the IPS and the maximization of convex combinations of measures. These observations are only meant (for the moment) to apply to the case of two players. They will help motivate our general results in the next section. We recall that if a partition P maximizes some convex combination of measures then every partition that is p-equivalent to P maximizes this same convex combination of measures, and so we may refer to the p-class of partitions that maximizes a convex combination of measures.

Observation 12.4 The following three items correspond to each other:

a. an individual point in the RNS that is associated with a positive-measure piece of cake,
b. a line segment on the outer boundary of the IPS, and
c. a situation where M is a one-many relation.

Observation 12.5 The following three items correspond to each other:

a. a gap in the RNS,
b. a corner point on the outer boundary of the IPS, and
c. a situation where M is a many-one relation.

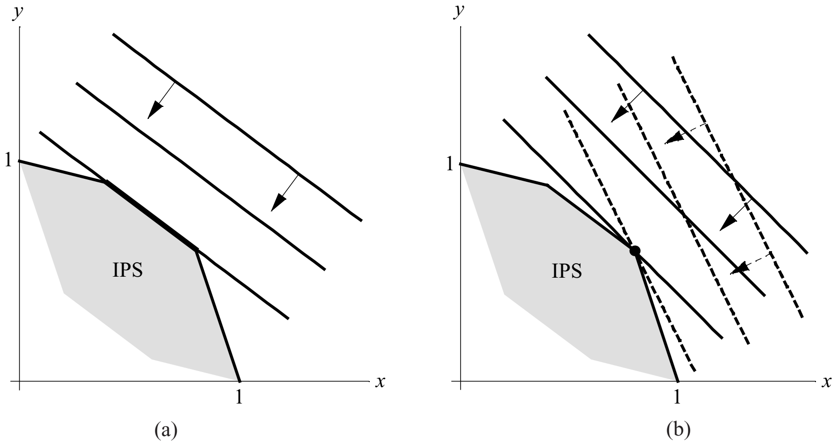


Figure 12.2

Given the various correspondences discussed in the previous section, we can provide another perspective on these observations using the notion of points of first contact of families of parallel non-negative lines with the IPS. The three items of Observation 12.4 correspond to there being a family of parallel non-negative lines that makes first contact with the IPS at many points. This is illustrated in Figure 12.2a. The three items of Observation 12.5 correspond to there being many families of parallel non-negative lines that make first contact with the IPS at a single point. This is illustrated in Figure 12.2b, where we have shown two such families. (Figures 12.2a and 12.2b are slightly revised copies Figures 7.1b and 7.1c, respectively.)

Notice that parts c of the two observations hint at two other possibilities: a situation where M is one-one and a situation where M is many-many. Our next example takes care of one of these. We shall discuss the other shortly.

Example 12.6 The correspondence between the IPS of Figure 12.1ci and the RNS of Figure 12.1cii.

As we saw in our two preceding observations, a point in the RNS that is associated with a positive-measure piece of cake corresponds to a line segment on the outer boundary of the IPS, and a gap in the RNS corresponds to corner point on the outer boundary of the IPS. In Figure 12.1ci, the outer boundary of the IPS contains no line segments and no corner points and, hence, neither of the situations described in these observations occurs. This implies that the RNS is as indicated in Figure 12.1cii. No point is associated with a positive-measure piece of cake and there are no gaps (i.e., every interval of positive width corresponds to a positive-measure piece of cake). We shall refer to such an RNS (or a portion of the RNS, as will be the case in our next example), as being *spread out*.

This example leads us to the following observation.

Observation 12.7 The following three items correspond to each other:

a. a spread-out region of the RNS,
b. a (non-straight line) smooth curve on the outer boundary of the IPS, and
c. a situation where M is one-one.

To connect this observation with the notion of points of first contact of a family of parallel non-negative lines with the IPS, as we did for Observations 12.4 and 12.5, we simply note that for the IPS in Figure 12.1ci any family of parallel non-negative lines makes first contact with the IPS at exactly one point, and first contact with the IPS is made at a given point on the outer boundary of the IPS by exactly one family of parallel non-negative lines.

Example 12.8 The correspondence between the IPS of Figure 12.1di and the RNS of Figure 12.1dii.

This combines aspects of Examples 12.3 and 12.6. The outer boundary of the IPS in Figure 12.1di consists of a line segment and a curve that is not a line segment, and these two pieces meet at a corner point. Each point of the darkened line segment in the left half of the RNS in Figure 12.1dii corresponds to a measure-zero piece of cake, but this set of points together corresponds to a positive-measure piece of cake, with no measure-zero gaps. The isolated point in the right half of the RNS corresponds to a positive-measure piece of cake. The correspondence between the IPS and the RNS is as follows: the line segment on the left of the RNS corresponds to the first portion of the outer boundary of the IPS, beginning at the upper left of the picture and going until the corner point. The fact that this line segment of the RNS contains no individual points associated with a positive-measure piece of cake tells us that the corresponding curve on the outer boundary of the IPS contains no line segments. The fact that it has no gaps tells us that the corresponding curve on the outer boundary of the IPS contains no corner points. Moving to the right in the RNS, the gap between the line segment just discussed and the isolated point to the right corresponds to the corner point in the IPS. The isolated point to the right in the RNS corresponds to the final line segment on the outer boundary of the IPS. The gap to the right of this point corresponds to the fact that the outer boundary of the IPS meets the x axis in a non-vertical way. Note that there is no gap at the left of the RNS. This corresponds to the fact that the outer boundary of the IPS meets the y axis horizontally.

We close this section by noting what seems to be an omission in our discussion of the possibilities for the relation M.

- Observation 12.4, Example 12.2, and parts of Examples 12.3 and 12.8 illustrate a situation in which M is one-many.
- Observation 12.5 and parts of Examples 12.3 and 12.8 illustrate a situation in which M is many-one.
- Observation 12.7, Example 12.6, and part of Example 12.8 illustrate a situation in which M is one-one.

The obvious question is: can M be many-many? In other words, can there be a collection of points in the RNS, each of which corresponds, via M, to a collection of points on the outer boundary of the IPS? Or, equivalently, can there be a collection of convex combinations of measures, each of which is maximized by every member of some collection of p-classes of partitions? Intuitively, our examples and observations suggest that this is possible if and only if the situations described in Observations 12.4 and 12.5 can occur together. This requires, simultaneously, an individual point in the RNS that is associated with a positive-measure piece of cake, and a gap in the RNS. Or, equivalently, this requires, simultaneously, a line segment and a corner point on the outer boundary of the IPS. It is not hard to see that in the two-player context this is impossible. We will see that the situation is very different when there are more than two players.

The chores versions of the ideas presented in this section are similar. We shall examine these in the general n-player context in the next section.

12C. Relating the IPS and the RNS in the General *n*-Player Context

In this section, we draw on some of the perspective developed in the previous section to help us understand the relationship between the IPS and the RNS in the general n-player context. Theorem 12.12 gives a necessary and sufficient condition on the RNS for the existence of more than one p-class of partitions that is w-associated with the same point in S^+. Theorem 12.16 gives a necessary and sufficient condition on the RNS for the existence of a single p-class of partitions that is w-associated with more than one point in S^+. These theorems will reveal a kind of duality. Clearly, the notions being considered have a dual nature. As we shall see, the corresponding conditions on the RNS also have a kind of dual relationship. The condition in Theorem 12.12 says that part of the

RNS is concentrated together. The condition in Theorem 12.16 says that parts of the RNS are separated from each other.

Corollaries 12.13 and 12.17 use the RD function to connect these notions to the maximization of convex combinations of measures. Then, Theorems 12.14 and 12.18 use the geometric perspective developed in Chapter 7 (connecting the maximization of convex combinations of measures, with points of first contact of families of parallel hyperplanes with the IPS) to establish connections between the IPS and the RNS.

Observation 12.4 of the previous section involves a point in the RNS being associated with a positive-measure piece of cake. We generalize this notion as follows.

Definition 12.9

a. For distinct $i, j = 1, 2, \ldots, n$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$, the RNS is *$i,j$-concentrated with respect to ω* if and only if $\{a \in C : \frac{f_i(a)}{f_j(a)} = \frac{\omega_i}{\omega_j}$ and, for every $k = 1, 2, \ldots, n$ with $k \neq i$ and $k \neq j$, $\frac{f_i(a)}{f_k(a)} \geq \frac{\omega_i}{\omega_k}$ and $\frac{f_j(a)}{f_k(a)} \geq \frac{\omega_j}{\omega_k}\}$ has positive measure.

b. The RNS is *concentrated* if and only if, for some ij, and ω, the RNS is i,j-concentrated with respect to ω.

To illustrate, let us first see that for two players this notion corresponds with what we have already considered in the previous section. If there are only two players and the RNS is concentrated, then it must be 1,2-concentrated with respect to some $\omega = (\omega_1, \omega_2) \in S^+$. (Notice that Definition 12.9 is symmetric with respect to i and j. Thus, "1,2-concentrated with respect to ω" is the same as "2,1-concentrated with respect to ω.") This implies that $\{a \in C : \frac{f_1(a)}{f_2(a)} = \frac{\omega_1}{\omega_2}\}$ has positive measure. But, since $\omega_1 + \omega_2 = 1$ and $f_1(a) + f_2(a) = 1$ for every $a \in C$, it follows that $\{a \in C : \frac{f_1(a)}{f_2(a)} = \frac{\omega_1}{\omega_2}\} = \{a \in C : f_1(a) = \omega_1$ and $f_2(a) = \omega_2\} = \{a \in C : f(a) = (\omega_1, \omega_2)\}$. This implies that $(\omega_1, \omega_2) \in$ RNS and there is a positive-measure piece of cake that is associated with this point.

Next, we illustrate Definition 12.9 for three players, using Figure 12.3. Fix $\omega = (\omega_1, \omega_2, \omega_3) \in S^+$, $p = (p_1, p_2, p_3) \in S^+$, and consider the following three conditions:

a. $\frac{p_1}{p_2} = \frac{\omega_1}{\omega_2}$,
b. $\frac{p_1}{p_3} \geq \frac{\omega_1}{\omega_3}$, and
c. $\frac{p_2}{p_3} \geq \frac{\omega_2}{\omega_3}$.

These conditions hold if and only if p is on the solid line segment ℓ in the figure. This line segment is that portion of the line determined by ω and (0, 0, 1) that is between ω and the line segment connecting (1, 0, 0) and (0, 1, 0) (including ω

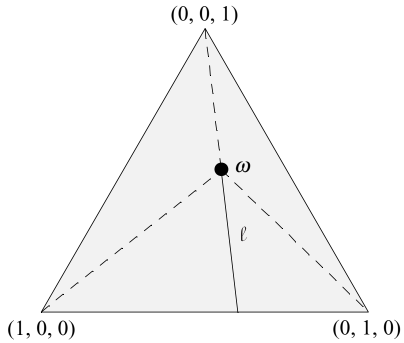


Figure 12.3

but not including the point on the line segment connecting (1, 0, 0) and (0, 1, 0)). The RNS is 1,2-concentrated with respect to ω if and only if there is a positive-measure $A \subseteq C$ such that, for all $a \in A$, the three preceding conditions hold with $f_1(a)$, $f_2(a)$, and $f_3(a)$ in place of p_1, p_2, and p_3, respectively. Thus, we see that the RNS is 1,2-concentrated with respect to ω if and only if there is a positive-measure piece of cake associated with ℓ.

If $m_1 = m_2 = m_3$, then the RNS consists of the single point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and, in this case, the RNS is certainly concentrated. In particular, it is 1,2-concentrated, 1,3-concentrated, and 2,3-concentrated, with respect to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This is the most extreme example of the RNS being concentrated.

We present two additional perspectives on the notion of concentrated.

Definition 12.10 Fix $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. For distinct $i, j = 1, 2, \ldots, n$, a point $p = (p_1, p_2, \ldots, p_n)$ is on the *i,j boundary associated with ω* if and only if

a. $\frac{p_i}{p_j} = \frac{\omega_i}{\omega_j}$ and
b. for any $k = 1, 2, \ldots, n$ with $k \neq i$ and $k \neq j$, $\frac{p_i}{p_k} \geq \frac{\omega_i}{\omega_k}$ and $\frac{p_j}{p_k} \geq \frac{\omega_j}{\omega_k}$.

In other words, the i,j boundary associated with ω is the boundary between the region of the simplex associated by ω with Player i and the region of the simplex associated by ω with Player j (as in the context of w-association, which we illustrated for three players in Figure 10.3). Condition a says that p is on the hyperplane determined by the $n - 1$ points consisting of the point ω together with the vertices of all players except for Player i and Player j. The first inequality of condition b says that p is either on the hyperplane determined by the $n - 1$ points consisting of the point ω together with the vertices of all players except for Player i and Player k or else is on Player i's side of this hyperplane. The second inequality says the same thing with j in place of i. We can also use Figure 12.3 to illustrate this idea. In the figure, the 1,2 boundary associated with ω is the solid line segment ℓ. The connection between this and the notion of i,j-concentrated is the following: for any $\omega \in S^+$ and distinct $i, j = 1, 2, \ldots, n$,

> the RNS is i,j-concentrated with respect to ω
> if and only if
> the set of bits of cake corresponding to points on the i,j boundary associated with ω has positive measure.

A point p in S can be on no boundary, one boundary, or more than one boundary associated with a given point ω. When we say that p is on the i,j boundary associated with ω, we do *not* assume that p is not also on some other boundary. We note that, for any $\omega \in S^+$, ω is on *every* i,j boundary associated with ω.

Definition 12.11 Suppose that $A \subseteq C$ has positive measure and fix distinct $i, j = 1, 2, \ldots, n$. We shall say that m_i and m_j are in *relative agreement on* A, or that Player i and Player j are in *relative agreement on* A, if, for every $B \subseteq A$, $\frac{m_i(B)}{m_i(A)} = \frac{m_j(B)}{m_j(A)}$.

In other words, if Player i and Player j are in relative agreement on A, then they may differ in their evaluations of the size of A, but they agree in their evaluations of how big a fraction any $B \subseteq A$ is of A. It is not hard to see that for distinct $i, j = 1, 2, \ldots, n$,

> Player i and Player j are in relative agreement on some positive-measure $A \subseteq C$
> if and only if
> for some $\omega \in S^+$ and some positive-measure $A \subseteq C$, all points of A correspond to points on the i,j boundary associated with ω
> if and only if
> for some $\omega \in S^+$, the RNS is i,j-concentrated with respect to ω.

A particular positive-measure $A \subseteq C$ satisfies the first of the two preceding statements if and only if it satisfies the second. We may think of such a set A as a name for any positive-measure subset of the set in part a of Definition 12.9. More generally,

> some two players are in relative agreement on some positive-measure set
> if and only if
> the RNS is concentrated.

Thus, we see that the notion of two players being in relative agreement on some positive-measure set and the notion of the RNS being concentrated are two ways of looking at the same phenomenon. In this chapter, where a central concern is the structure of the RNS, we shall focus on the notion of concentrated. In Chapter 14, we shall use the notion of relative agreement to study strong Pareto optimality.

Theorem 12.12
a. *There exists a point in S^+ with which more that one p-class of partitions is w-associated if and only if the RNS is concentrated.*
b. *More specifically: for any $\omega \in S^+$, more than one p-class of partitions is w-associated with ω if, and only if, for some i and j, the RNS is i, j-concentrated with respect to ω.*

Proof: Clearly, part b implies part a. We prove part b. Fix some $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$.

For the forward direction, suppose that more than one p-class of partitions is w-associated with ω. Then there exist partitions $P = \langle P_1, P_2, \ldots, P_n \rangle$ and $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ that are each w-associated with ω and are not p-equivalent. Since P and Q are not p-equivalent, we know that there is a set A of positive measure such that $A \subseteq P_i$ and $A \subseteq Q_j$ for some distinct i and j. Since P and Q are each w-associated with ω, it follows that $\frac{f_i(a)}{f_k(a)} \geq \frac{\omega_i}{\omega_k}$ for almost every $a \in P_i$ and every $k = 1, 2, \ldots, n$ with $k \neq i$, and that $\frac{f_j(a)}{f_k(a)} \geq \frac{\omega_j}{\omega_k}$ for almost every $a \in Q_j$ and every $k = 1, 2, \ldots, n$ with $k \neq j$. This implies that, for almost every $a \in A$, $\frac{f_i(a)}{f_j(a)} = \frac{\omega_i}{\omega_j}$. Since A has positive measure, $\{a \in C : \frac{f_i(a)}{f_j(a)} = \frac{\omega_i}{\omega_j}$ and, for every $k = 1, 2, \ldots, n$ with $k \neq i$ and $k \neq j$, $\frac{f_i(a)}{f_k(a)} \geq \frac{\omega_i}{\omega_k}$ and $\frac{f_j(a)}{f_k(a)} \geq \frac{\omega_j}{\omega_k}\}$ has positive measure. Hence, the RNS is i,j-concentrated with respect to ω.

For the reverse direction, suppose that the RNS is i,j-concentrated with respect to ω. Let $A = \{a \in C : \frac{f_i(a)}{f_j(a)} = \frac{\omega_i}{\omega_j}$ and, for every $k = 1, 2, \ldots, n$ with $k \neq i$ and $k \neq j$, $\frac{f_i(a)}{f_k(a)} \geq \frac{\omega_i}{\omega_k}$ and $\frac{f_j(a)}{f_k(a)} \geq \frac{\omega_j}{\omega_k}\}$. Then A has positive measure. Consider ω^*, the collection of all partitions that are w-associated with ω. In constructing this set, A can be divided arbitrarily between Player i and Player j. By giving all of A to Player i or all of A to Player j, we see that there exist at least two non-p-equivalent partitions in ω^* and, hence, there is more than one p-class of partitions that is w-associated with ω. □

We note that the "more than one" in parts a and b of the theorem can each be replaced by "infinitely many." For the forward direction of part b, this is obvious, since this change makes the premise stronger. For the reverse direction of part b, this follows because of the fact that the set A in the proof that is to be divided between Player i and Player j can be so divided in infinitely many different ways that result in non-p-equivalent partitions. The correctness of this change for part a follows from that for part b.

The RNS is concentrated if, and only if, for some $\omega \in S^+$, it is i,j-concentrated with respect to ω for at least one i,j pair. As we shall discuss shortly, the RNS being i,j-concentrated with respect to ω for more than one i,j pair corresponds to additional structure on the IPS.

By Theorem 10.6, we know that the function RD provides a one-to-one correspondence between points in S^+ that are to be used as in w-association and points in S^+ that provide coefficients for convex combinations of the measures. This correspondence yields the following corollary to Theorem 12.12.

Corollary 12.13

a. *There exists a convex combination of measures with coefficients from S^+ that is maximized by more than one p-class of partitions if and only if the RNS is concentrated.*
b. *More specifically: for any $\alpha \in S^+$, more than one p-class of partitions maximizes the convex combination of measures corresponding to α if and only if, for some i and j, the RNS is i,j-concentrated with respect to $RD(\alpha)$.*

Proof: Part a follows easily from part b. Part b follows from the theorem and part b of Corollary 10.7. □

We are now in a position to make our first connection in this section between the IPS and the RNS. We recall that the function m provides a one-to-one correspondence between Pareto maximal p-classes and points on the outer Pareto boundary of the IPS. We use this function, together with our geometric perspective on the maximization of convex combinations of measures (which we discussed in Chapter 7 and involves points of first contact of families of parallel hyperplanes with the IPS), and Corollary 12.13. Fix $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S^+$. Then,

> the family of parallel hyperplanes $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ makes first contact with the IPS at more than one point
>
> if and only if
>
> more than one p-class of partitions maximizes the convex combination of measures corresponding to α
>
> if and only if (Corollary 12.13)
>
> for some i and j, the RNS is i,j-concentrated with respect to RD(α).

These ideas lead us to the following result.

Theorem 12.14

a. *There is a line segment on the outer Pareto boundary of the IPS if and only if the RNS is concentrated.*

b. *More specifically: for any* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S^+$*, there is a line segment* ℓ *on the outer Pareto boundary of the IPS and every point of* ℓ *is a point of first contact with the IPS of the family of parallel hyperplanes* $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ *if and only if, for some* i *and* j*, the RNS is* i, j*-concentrated with respect to RD*(α).

Proof: We first prove part b. Fix $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S^+$. For the forward direction, assume that ℓ is a line segment on the outer Pareto boundary of the IPS and that every point of ℓ is a point of first contact with the IPS of the family of parallel hyperplanes $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$. This family obviously makes first contact with the IPS at more than one point, and it therefore follows from the preceding discussion that, for some i and j, the RNS is i,j-concentrated with respect to RD(α).

For the reverse direction of part b, we assume that for some i and j the RNS is i,j-concentrated with respect to RD(α). Our preceding discussion tells us that the family of parallel hyperplanes $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ makes first contact with the IPS at more than one point. We know that any such point of first contact is a Pareto maximal point and hence is on the outer Pareto boundary. It follows from the convexity of the IPS that this family of parallel hyperplanes makes first contact with the IPS at all points along some line segment ℓ that lies on the outer Pareto boundary of the IPS. (We have already illustrated this idea for two players in Figure 12.2a. When there are more than two players, the set of points of first contact may be more than a line. We shall deal with this issue shortly.)

Next, we consider part a. For the forward direction, we assume that there is a line segment ℓ on the outer Pareto boundary of the IPS. It is geometrically clear that, for some $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in S$, the family of parallel hyperplanes with coefficients given by β makes first contact with the IPS at every point of ℓ.

If $\beta \in S^+$, then the result follows from part b of the theorem. Assume then that $\beta \in S \backslash S^+$ and let $\delta_\beta = \{i \leq n : \beta_i > 0\}$. Then $\delta_\beta \neq \emptyset$ and $\{1, 2, \ldots, n\} \backslash \delta_\beta \neq \emptyset$. Let k be such that the member of the family of parallel hyperplanes with coefficients given by β that makes first contact with the IPS is the hyperplane $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = k$.

We claim that, for any point $(p_1, p_2, \ldots, p_n)$ on line segment ℓ, if $i \notin \delta_\beta$ then $p_i = 0$. Suppose, by way of contradiction, that this is not the case. Then there is a point $p = (p_1, p_2, \ldots, p_n)$ on ℓ such that, for some $i \notin \delta_\beta$, $p_i > 0$. Notice that since p is on the given hyperplane, $\beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_n p_n = k$ and, since p is in the IPS, $m(P) = p$ for some partition $P = \langle P_1, P_2, \ldots, P_n \rangle$.

Also, since $p_i > 0$, we know that $m_i(P_i) > 0$. Fix any $j \in \delta_\beta$ and let $Q = \langle Q_1, Q_2, \ldots, Q_n\rangle$ be the partition obtained by transferring all of P_i from Player i to Player j. Then

$$\begin{aligned}
&\beta_1 m_1(Q_1) + \beta_2 m_2(Q_2) + \cdots + \beta_n m_n(Q_n)\\
&\quad = \beta_1 m_1(P_1) + \beta_2 m_2(P_2) + \cdots + \beta_{i-1} m_{i-1}(P_{i-1}) + \beta_i m_i(\emptyset)\\
&\qquad + \beta_{i+1} m_{i+1}(P_{i+1}) + \cdots + \beta_{j-1} m_{j-1}(P_{j-1})\\
&\qquad + \beta_j m_j(P_j \cup P_i) + \beta_{j+1} m_{j+1}(P_{j+1}) + \cdots + \beta_n m_n(P_n)\\
&\quad = [\beta_1 m_1(P_1) + \beta_2 m_2(P_2) + \cdots + \beta_{i-1} m_{i-1}(P_{i-1}) + \beta_i m_i(\emptyset)\\
&\qquad + \beta_{i+1} m_{i+1}(P_{i+1}) + \cdots + \beta_{j-1} m_{j-1}(P_{j-1}) + \beta_j m_j(P_j)\\
&\qquad + \beta_{j+1} m_{j+1}(P_{j+1}) + \cdots + \beta_n m_n(P_n)] + \beta_j m_j(P_i)\\
&\quad = (\beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_{i-1} p_{i-1} + \beta_i(0) + \beta_{i+1} p_{i+1}\\
&\qquad + \cdots + \beta_n p_n) + \beta_j m_j(P_i).
\end{aligned}$$

Since $i \notin \delta_\beta$, and hence $\beta_i = 0$, we have

$$\beta_1 m_1(Q_1) + \beta_2 m_2(Q_2) + \cdots + \beta_n m_n(Q_n) = \left(\sum\nolimits_{i' \in \delta_\beta} \beta_{i'} p_{i'}\right) + \beta_j m_j(P_i).$$

Since $m(P) = p$ is on the line segment ℓ, and the family of parallel hyperplanes with coefficients given by β makes first contact with the IPS at every point of ℓ, it follows that P maximizes the convex combination of measures with coefficients given by β. Then, since $\beta_1 m_1(P_1) + \beta_2 m_2(P_2) + \cdots + \beta_n m_n(P_n) = \sum_{i' \in \delta_\beta} \beta_{i'} p_{i'}$ we know that $\sum_{i' \in \delta_\beta} \beta_{i'} p_{i'} \geq (\sum_{i' \in \delta_\beta} \beta_{i'} p_{i'}) + \beta_j m_j(P_i)$ and, hence, that $\beta_j m_j(P_i) = 0$. Since $j \in \delta_\beta$, we know that $\beta_j > 0$ and, therefore, $m_j(P_i) = 0$. But $m_i(P_i) > 0$ and, hence, this contradicts our assumption that the measures are absolutely continuous with respect to each other. This establishes that, for any point $(p_1, p_2, \ldots, p_n)$ on line segment ℓ, if $i \notin \delta_\beta$, then $p_i = 0$.

Let S_{δ_β} denote the face of the simplex corresponding to δ_β, and let $\text{IPS}_{\delta_\beta}$ be the restriction of the IPS associated with these players. Then $S_{\delta_\beta} = \{(x_1, x_2, \ldots, x_n) \in S : \text{for } i = 1, 2, \ldots, n, \text{ if } i \notin \delta_\beta, \text{ then } x_i = 0\}$ and $\text{IPS}_{\delta_\beta} = \{(x_1, x_2, \ldots, x_n) \in \text{IPS} : \text{for } i = 1, 2, \ldots, n, \text{ if } i \notin \delta_\beta, \text{ then } x_i = 0\}$. By our preceding work, the line segment ℓ lies in $\text{IPS}_{\delta_\beta}$.

We now change perspective slightly and view ℓ, S_{δ_β}, and $\text{IPS}_{\delta_\beta}$ as subsets of $\mathbf{R}^{|\delta_\beta|}$, making the order-preserving identification of coordinates, as usual. (This is legitimate, since every point in each of these sets has coordinate zero in position i for every $i \notin \delta_\beta$. This change in perspective involves simply ignoring all zeros in all such positions.) Then, we see that ℓ is on the outer Pareto boundary of $\text{IPS}_{\delta_\beta}$ and every point of ℓ is a point of first contact with $\text{IPS}_{\delta_\beta}$ of the family of parallel hyperplanes $\sum_{i \in \delta_\beta} \beta_i x_i = c$. Since $\beta_i > 0$ for every $i \in \delta_\beta$, it follows from part b of the theorem that $\text{IPS}_{\delta_\beta}$ is concentrated.

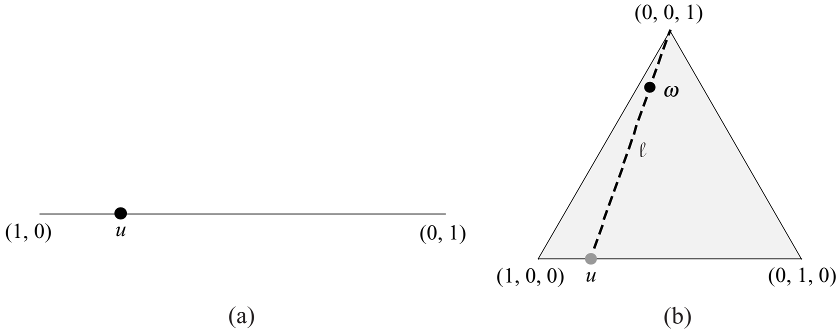


Figure 12.4

We claim that this implies that the full RNS is concentrated and, hence, establishes the forward direction of part a of the theorem. We shall not give a detailed proof of this fact, but shall instead present a simple illustration. Suppose there are three players and consider Figure 12.4. In Figure 12.4a, we have shown just the one-simplex where the restricted RNS corresponding to just Players 1 and 2 is located. We have not shown this entire RNS. We focus instead on the single point u. The existence of such a point in this restricted RNS tells us that there is a piece of cake of positive measure associated with u and hence that this restricted RNS is concentrated. In Figure 12.4b, we show the two-simplex where the full RNS is located. Again, we have not shown the entire RNS. What we have shown is the point u and a dashed line from Player 3's vertex to u. We have drawn the point u lighter in this figure since (by absolute continuity), we know that u is not in this RNS. In going from the two-simplex of Figure 12.4b to the one-simplex of Figure 12.4a, all points on the dashed line in Figure 12.4b are projected to the point u. Hence, the cake associated with this dashed line has positive measure. This implies that the RNS is 1,2-concentrated with respect to ω for any point ω that is sufficiently close to (0, 0, 1) so that the cake associated with the dashed line below this point has positive measure. We conclude that the full RNS is concentrated. It is not hard to see that this idea can be generalized to more than three players and can be made precise. This completes the proof of the forward direction of part a.

The reverse direction of part a follows immediately from the reverse direction of part b. □

Combining part b of the theorem with Theorem 10.6 (and again using the connection between points of first contact with the IPS of families of parallel hyperplanes, and maximization of convex combinations of measures), we see that, for any $\omega \in S^+$, the RNS is i, j-concentrated with respect to ω if and only

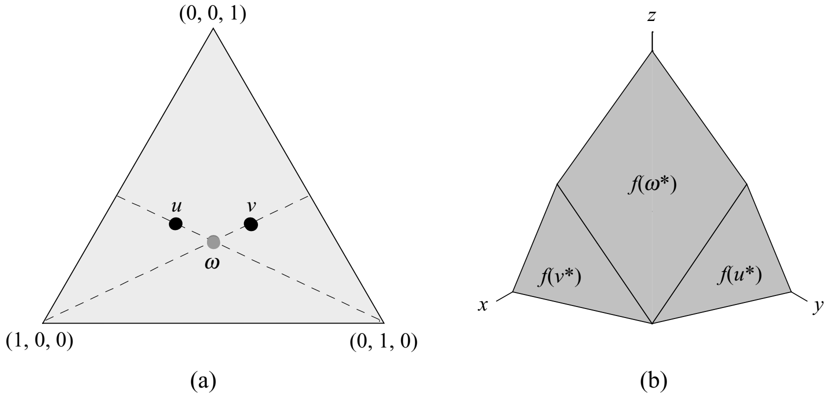


Figure 12.5

if there is a line segment on the outer Pareto boundary of the IPS consisting of points of the form $m(P)$ for partitions P that are w-associated with ω.

If the RNS is i,j-concentrated with respect to some ω then, by the theorem, there is a line segment on the outer Pareto boundary of the IPS that lies on a hyperplane of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = k$, where $\text{RD}(\omega) = \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. (Recall that $\text{RD}(\text{RD}(\omega)) = \omega$.) The proof of the reverse direction of part b of Theorem 12.12 implies that there is such a line segment with the property that any two points on the line segment have the same kth coordinate for each $k \neq i$, $k \neq j$. Suppose now that for some ω the RNS is i,j-concentrated with respect to ω for more than one i,j pair. Then there are at least two different line segments on the outer Pareto boundary of the IPS that lie on the same hyperplane. By the convexity of the IPS, this implies that there is a two-dimensional region on the outer Pareto boundary of the IPS. Although we shall not pursue this line of reasoning further, it is clear that if, for some ω, the RNS is i,j-concentrated with respect to ω for enough i,j pairs, then there will be a “flat” region on the outer Pareto boundary of the IPS. In other words, there will be a convex region on the outer Pareto boundary of maximal (i.e., $n-1$) dimension. In particular, if there is a point ω in the RNS that corresponds to a piece of cake of positive measure, then the RNS is i,j-concentrated with respect to ω for all pairs i,j, and in this case there is such a flat region on the outer Pareto boundary of the IPS. Conversely, if there is such a flat region on the outer Pareto boundary of the IPS then, for some ω, the RNS is i,j-concentrated with respect to ω for many i,j pairs. We illustrate this in Figure 12.5.

In the figures, we assume that there are three players. Figure 12.5a shows the RNS and Figure 12.5b shows the corresponding IPS. (It would be straightforward to define the associated cake and measures, but we shall not do so.)

The RNS consists of two points, u and v, and it follows that the RNS is 1,2-concentrated, 1,3-concentrated, and 2,3-concentrated with respect to u and with respect to v. By definition, the set u^* consists of all partitions that are w-associated with u. A partition is in this set if and only if it gives all cake associated with v to Player 2. (All cake associated with u can be partitioned arbitrarily among the three players.) For any such partition P, $f(P)$ is on the outer boundary of the IPS, and the set of all such $f(P)$ is the flat region on the outer boundary of the IPS that we have labeled "$f(u^*)$." Similarly, the point v in the RNS corresponds to the flat region on the outer boundary of the IPS that we have labeled "$f(v^*)$." In this situation, we see something quite different from what we saw in the two-player situations that we examined previously and illustrated in Figure 12.1. As we saw in Figures 12.1ai, 12.1aii, 12.1bi, and 12.1bii, if the RNS consists of points that are each associated with a piece of cake of positive measure, then the outer boundary of the corresponding IPS consists of line segments, one corresponding to each of these points, and nothing else. Thus, in Figures 12.1ai and 12.1aii, we see that an RNS consisting of a single point has a corresponding IPS with outer boundary consisting of a single line segment, and, in Figures 12.1bi and 12.1bii, we see that an RNS consisting of three points has a corresponding IPS with outer boundary consisting of three line segments. However, in our present example, we see that the RNS is 1,3-concentrated and 2,3-concentrated (but not 1,2-concentrated) with respect to the point ω in the figure (which is not a point in the RNS). The set of all $f(P)$ for partitions P that are w-associated with ω is the flat region on the outer boundary of the IPS that we have labeled "$f(\omega^*)$." Thus, in this case, the RNS consists of two points but the outer boundary of the IPS consists of three flat regions.

We now shift our focus to Observation 12.5. This observation involves the notion of a gap in the RNS. We generalize this notion in Definition 12.15. This definition, Theorem 12.16, Corollary 12.17, and Theorem 12.18, and will parallel our treatment of Definition 12.9, Theorem 12.12, Corollary 12.13, and Theorems 12.14.

Definition 12.15

a. Suppose that γ is a partition of $\{1, 2, \ldots, n\}$ into at least two pieces and $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of C. The RNS is *γ-separable with respect to P* if and only if for some $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$ and $\varepsilon > 0$
   i. P is w-associated with ω and
   ii. for any i and j that are in different pieces of partition γ, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j} + \varepsilon$ for almost every $a \in P_i$.

b. The RNS is *separable* if and only if, for some γ and P, the RNS is γ-separable with respect to P.

We first consider how the notion of separability generalizes the two-player notion of a gap in the RNS, as in Observation 12.5. Suppose that m_1 and m_2 are measures on C and the corresponding RNS has a gap. In particular, let us consider the RNS shown in Figure 12.1bii. We claim that this RNS is separable. More specifically, we claim that if $P = \langle P_1, P_2 \rangle$ is the partition that results from giving all of the cake associated with points $u = (\frac{4}{5}, \frac{1}{5})$ and $v = (\frac{4}{7}, \frac{3}{7})$ to Player 1, and all of the cake associated with point $w = (\frac{1}{4}, \frac{3}{4})$ to Player 2, then the RNS is γ-separable with respect to P, where $\gamma = \{\{1\}, \{2\}\}$. Let $\omega = (\omega_1, \omega_2)$ be any point strictly between v and w. Then, $\frac{1/4}{3/4} < \frac{\omega_1}{\omega_2} < \frac{4/7}{3/7}$ or, equivalently, $\frac{1}{3} < \frac{\omega_1}{\omega_2} < \frac{4}{3}$. This implies that $\frac{3}{4} < \frac{\omega_2}{\omega_1} < 3$. Let $\varepsilon = \min\{\frac{4}{3} - \frac{\omega_1}{\omega_2}, 3 - \frac{\omega_2}{\omega_1}\}$. Then $\varepsilon > 0$. We claim that this ω and ε show that the RNS is γ-separable with respect to P.

We must show that, for almost every $a \in P_1$, $\frac{f_1(a)}{f_2(a)} \geq \frac{\omega_1}{\omega_2} + \varepsilon$, and for almost every $a \in P_2$, $\frac{f_2(a)}{f_1(a)} \geq \frac{\omega_2}{\omega_1} + \varepsilon$. For almost every $a \in P_1$, either $f(a) = u = (\frac{4}{5}, \frac{1}{5})$ or $f(a) = v = (\frac{4}{7}, \frac{3}{7})$. It follows that, for any such a, either $\frac{f_1(a)}{f_2(a)} = \frac{4/5}{1/5} = 4$ or $\frac{f_1(a)}{f_2(a)} = \frac{4/7}{3/7} = \frac{4}{3}$ and, hence, $\frac{f_1(a)}{f_2(a)} \geq \frac{4}{3} = \frac{\omega_1}{\omega_2} + (\frac{4}{3} - \frac{\omega_1}{\omega_2}) \geq \frac{\omega_1}{\omega_2} + \varepsilon$. For almost every $a \in P_2$, $f(a) = w = (\frac{1}{4}, \frac{3}{4})$. Hence, for any such a, $\frac{f_2(a)}{f_1(a)} = \frac{3/4}{1/4} = 3$ and therefore $\frac{f_2(a)}{f_1(a)} = 3 = \frac{\omega_2}{\omega_1} + (3 - \frac{\omega_2}{\omega_1}) \geq \frac{\omega_2}{\omega_1} + \varepsilon$. Thus, the RNS of Figure 12.1bii is γ-separable with respect to P.

There is a simple intuition behind the notion of separability: the RNS is separable if and only if there is a point $\omega \in S^+$ such that we can move ω a little without affecting ω^*. In other words, the RNS is separable if and only if there is some "wiggle room" for ω. More specifically, if γ is a partition of $\{1, 2, \ldots, n\}$ into at least two pieces and $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of C, then the RNS is γ-separable with respect to P if and only if the following holds:

> For some $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$ and any chosen direction from ω in which the ratio between the ith and jth coordinates of ω does not change for any $i, j = 1, 2, \ldots, n$ that are in the same piece of the partition given by γ, it is possible to move a small distance from ω in this direction to a point $\omega' \in S^+$ so that $\omega^* = \omega'^*$.

This perspective on how to obtain ω' from ω will be central to the proof of Theorem 12.16.

Figure 12.6 presents three examples. In each figure, regions T_1, T_2, and T_3 of the simplex are the regions corresponding to some partition $P = \langle P_1, P_2, P_3 \rangle$.

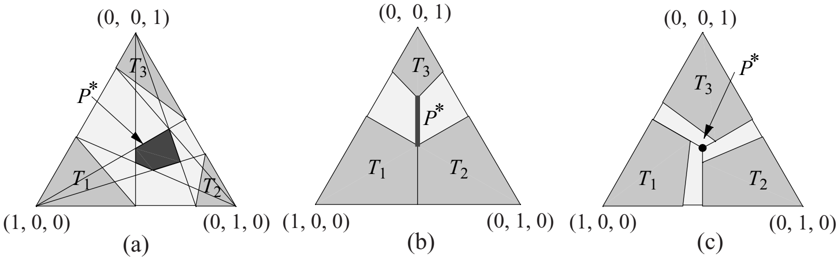


Figure 12.6

(In other words, $f(P_1) = T_1$, $f(P_2) = T_2$, and $f(P_3) = T_3$.) In addition, the set of all $\omega \in S^+$ with which P is w-associated, which we shall denote P^*, is shown in each figure. Notice that for any $\omega \in S^+$, $P \in \omega^*$ if and only if $\omega \in P^*$. We consider each of these figures:

- In Figure 12.6a, it is clear that, for any ω in the interior of P^*, ω has some wiggle room, and so this RNS is separable. Pick any such ω and any partition γ of $\{1, 2, 3\}$ into two or three (non-empty) pieces. Then, an appropriate ε can be found so that Definition 12.15 is satisfied.
- In Figure 12.6b, any ω in the interior of the line segment that is P^* has some wiggle room, and therefore this RNS is separable. Pick any such ω and set $\gamma = \{\{1, 2\}, \{3\}\}$. Then an appropriate ε can be found so that Definition 12.15 is satisfied. Notice that in this case such an ω cannot wiggle in every direction, but only up and down. This is because there is no gap between T_1 and T_2. This is permissible since 1 and 2 are in the same piece of the partition given by γ. If we set $\gamma = \{\{1\}, \{2, 3\}\}$ or $\gamma = \{\{2\}, \{1, 3\}\}$, then this RNS is not γ-separable with respect to P.
- In Figure 12.6c, P^* is a single point, and so this RNS is not separable. Notice that in this example there is a gap between T_1 and T_2, between T_1 and T_3, and between T_2 and T_3. Hence, this example illustrates that the existence of such gaps does not imply separability. Separability requires a kind of coherency between such gaps.

Theorem 12.16

a. There exists a p-class of partitions that is w-associated with more than one point in S^+ if and only if the RNS is separable.

b. More specifically: for any partition P, P is w-associated with more than one point in S^+ if and only if, for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is γ-separable with respect to P.

Proof: Clearly, part b implies part a. We prove part b. Fix some partition $P = \langle P_1, P_2, \ldots, P_n \rangle$.

For the forward direction, suppose that $\omega', \omega'' \in S^+, \omega' \neq \omega''$, and P is w-associated with ω' and with ω''. Let $\omega' = \langle \omega'_1, \omega'_2, \ldots, \omega'_n \rangle$, $\omega'' = \langle \omega''_1, \omega''_2, \ldots, \omega''_n \rangle$, and consider the following relation on the set $\{1, 2, \ldots, n\}$:

$$i \sim j \quad \text{if and only if} \quad \frac{\omega'_i}{\omega'_j} = \frac{\omega''_i}{\omega''_j}$$

This is an equivalence relation on $\{1, 2, \ldots, n\}$ and, since $\omega' \neq \omega''$, there are at least two equivalence classes. Let γ be the induced partition of $\{1, 2, \ldots, n\}$. We claim that the RNS is γ-separable with respect to P.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be any point strictly between ω' and ω''. (In other words, $\omega = \lambda\omega' + (1 - \lambda)\omega''$ for some λ with $0 < \lambda < 1$.) For any i and j that are in different pieces of partition γ, $\frac{\omega'_i}{\omega'_j} \neq \frac{\omega''_i}{\omega''_j}$ and $\frac{\omega_i}{\omega_j}$ is strictly between these two ratios. Set $\varepsilon = \min\{\min\{|\frac{\omega_i}{\omega_j} - \frac{\omega'_i}{\omega'_j}|, |\frac{\omega_i}{\omega_j} - \frac{\omega''_i}{\omega''_j}|, |\frac{\omega_j}{\omega_i} - \frac{\omega'_j}{\omega'_i}|, |\frac{\omega_j}{\omega_i} - \frac{\omega''_j}{\omega''_i}|\} : i$ and j are in different pieces of the partition $\gamma\}$. Note that each minimum is taken over a finite set of positive numbers; therefore, each minimum exists and $\varepsilon > 0$. We claim that the RNS is γ-separable with respect to P, with this ε used in Definition 12.15. It is easy to see that since P is w-associated with ω' and with ω'', and ω is strictly between ω' and ω'', then P is w-associated with ω. We must show that, for any i and j that are in different pieces of partition γ, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j} + \varepsilon$ for almost every $a \in P_i$.

Fix any i and j that are in different pieces of partition γ. Since P is w-associated with ω' and with ω'', we know that $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega'_i}{\omega'_j}$ and $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega''_i}{\omega''_j}$ for almost every $a \in P_i$. Since $\frac{\omega_i}{\omega_j}$ is strictly between $\frac{\omega'_i}{\omega'_j}$ and $\frac{\omega''_i}{\omega''_j}$, it follows that $\max\{\frac{\omega'_i}{\omega'_j}, \frac{\omega''_i}{\omega''_j}\} > \frac{\omega_i}{\omega_j}$, and so $\max\{\frac{\omega'_i}{\omega'_j}, \frac{\omega''_i}{\omega''_j}\} - \frac{\omega_i}{\omega_j} = |\max\{\frac{\omega'_i}{\omega'_j}, \frac{\omega''_i}{\omega''_j}\} - \frac{\omega_i}{\omega_j}| \geq \varepsilon$. This implies that, for almost every $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \max\{\frac{\omega'_i}{\omega'_j}, \frac{\omega''_i}{\omega''_j}\} = \frac{\omega_i}{\omega_j} + (\max\{\frac{\omega'_i}{\omega'_j}, \frac{\omega''_i}{\omega''_j}\} - \frac{\omega_i}{\omega_j}) \geq \frac{\omega_i}{\omega_j} + \varepsilon$. Hence, the RNS is γ-separable with respect to P.

For the reverse direction of part b, assume that we are given a partition γ of $\{1, 2, \ldots, n\}$ such that the RNS is γ-separable with respect to P. Let $\langle \delta_a, \delta_b \rangle$ be a partition of $\{1, 2, \ldots, n\}$ into two non-empty pieces such that any piece of γ is either completely contained in δ_a or else is completely contained in δ_b. Let ω and ε be as in part a of Definition 12.15. Then P is w-associated with ω. We must find some $\omega' \neq \omega$ such that P is w-associated with ω'. We shall obtain such an ω' from ω by doing the following to ω:

i. not changing the ratios between pairs of coordinates corresponding to elements of δ_a,

ii. not changing the ratios between pairs of coordinates corresponding to elements of δ_b, and
iii. changing the ratios between coordinates corresponding to elements of δ_a and coordinates corresponding to elements of δ_b.

Notice that, since $\omega \in S^+$ and δ_a is a proper subset of $\{1, 2, \ldots, n\}$, it follows that $\sum_{i \in \delta_a} \omega_i < 1$. Hence, $1 < \frac{1}{\sum_{i \in \delta_a} \omega_i}$. For each real number r with $1 < r < \frac{1}{\sum_{i \in \delta_a} \omega_i}$, let $\delta(r) = \frac{1-(r \sum_{i \in \delta_a} \omega_i)}{\sum_{i \in \delta_b} \omega_i}$. For any such r, $\sum_{i \in \delta_a} \omega_i < r \sum_{i \in \delta_a} \omega_i < 1$ and, hence, $0 < (1 - r \sum_{i \in \delta_a} \omega_i) < (1 - \sum_{i \in \delta_a} \omega_i) = \sum_{i \in \delta_b} \omega_i$. This implies that $0 < \delta(r) < 1$ and, hence, $\delta(r) < r$. Also,

$$\begin{aligned}\lim_{r \to 1} \delta(r) &= \lim_{r \to 1} \left(\frac{1 - \left(r \sum_{i \in \delta_a} \omega_i\right)}{\sum_{i \in \delta_b} \omega_i} \right) \\ &= \frac{1 - \sum_{i \in \delta_a} \omega_i}{\sum_{i \in \delta_b} \omega_i} = \frac{\sum_{i \in \delta_b} \omega_i}{\sum_{i \in \delta_b} \omega_i} = 1.\end{aligned}$$

For each $i = 1, 2, \ldots, n$, and each real number r with $1 < r < \frac{1}{\sum_{i \in \delta_a} \omega_i}$, set

$$\omega_i'(r) = \begin{cases} r\omega_i & \text{if } i \in \delta_a \\ \delta(r)\omega_i & \text{if } i \in \delta_b \end{cases}.$$

Then, for each such i and r, $\omega_i'(r) > 0$. Also,

$$\begin{aligned}\sum_{i=1}^{n} \omega_i'(r) &= r \sum_{i \in \delta_a} \omega_i + \delta(r) \sum_{i \in \delta_b} \omega_i \\ &= r \sum_{i \in \delta_a} \omega_i + \left(\frac{1 - \left(r \sum_{i \in \delta_a} \omega_i\right)}{\sum_{i \in \delta_b} \omega_i} \right) \sum_{i \in \delta_b} \omega_i \\ &= r \sum_{i \in \delta_a} \omega_i + 1 - r \sum_{i \in \delta_a} \omega_i = 1.\end{aligned}$$

Thus, for each such r, if we define $\omega'(r) = (\omega_1'(r), \omega_2'(r), \ldots, \omega_n'(r))$ then $\omega'(r) \in S^+$. Note that we have obtained $\omega'(r)$ from ω in accordance with conditions i, ii, and iii.

For any $i_a \in \delta_a$, $i_b \in \delta_b$, and r with $1 < r < \frac{1}{\sum_{i \in \delta_a} \omega_i}$, we have $\frac{\omega_{i_a}'(r)}{\omega_{i_b}'(r)} = \frac{r\omega_{i_a}}{\delta(r)\omega_{i_b}} > \frac{\omega_{i_a}}{\omega_{i_b}}$ and, hence, $\omega'(r) \neq \omega$. Also, since $\mathrm{Lim}_{r \to 1} \delta(r) = 1$, $\frac{\omega_{i_a}'(r)}{\omega_{i_b}'(r)}$ can be made as close to $\frac{\omega_{i_a}}{\omega_{i_b}}$ as desired by choosing r sufficiently close to one. Since δ_a and δ_b are each finite, we can fix r so that, for every $i_a \in \delta_a$ and $i_b \in \delta_b$, $\frac{\omega_{i_a}'(r)}{\omega_{i_b}'(r)} - \frac{\omega_{i_a}}{\omega_{i_b}} \leq \varepsilon$ and set $\omega' = (\omega_1', \omega_2', \ldots, \omega_n') = \omega'(r)$. Then $\omega' \neq \omega$. We claim that P is w-associated with ω'.

Fix distinct $i, j = 1, 2, \ldots, n$. We must show that $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega'_i}{\omega'_j}$ for almost every $a \in P_i$. We consider four cases:

Case 1: $i, j \in \delta_a$. Since P is w-associated with ω, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every $a \in P_i$. But $i, j \in \delta_a$ implies that $\frac{\omega'_i}{\omega'_j} = \frac{\omega_i{}'(r)}{\omega_j{}'(r)} = \frac{r\omega_i}{r\omega_j} = \frac{\omega_i}{\omega_j}$. Hence, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega'_i}{\omega'_j}$ for almost every $a \in P_i$.

Case 2: $i, j \in \delta_b$. This is as in Case 1.

Case 3: $i \in \delta_a$ and $j \in \delta_b$. By assumption, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j} + \varepsilon$ for almost every $a \in P_i$. Our construction of ω' tells us that $\frac{\omega_i}{\omega_j} + \varepsilon \geq \frac{\omega'_i}{\omega'_j}$. Hence, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega'_i}{\omega'_j}$ for almost every $a \in P_i$.

Case 4: $i \in \delta_b$ and $j \in \delta_a$. Since P is w-associated with ω, we know that $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every $a \in P_i$. By our construction of ω', we have $\frac{\omega'_j}{\omega'_i} > \frac{\omega_j}{\omega_i}$ and, hence, $\frac{\omega_i}{\omega_j} > \frac{\omega'_i}{\omega'_j}$. It follows that $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega'_i}{\omega'_j}$ for almost every $a \in P_i$.

This completes the proof of the theorem. □

As was the case for Theorem 12.12, the "more than one" in parts a and b of the theorem can each be replaced by "infinitely many." For the forward direction of part b, this is obvious, since this change makes the premise stronger. For the reverse direction of part b, this follows because of the fact that there are infinitely many choices of r in the proof, and each choice results in a different ω' with which P is w-associated. Or, we simply note that if P is w-associated with distinct ω and ω', and ω'' is on the line segment between ω and ω' (i.e., $\omega'' = \lambda\omega + (1 - \lambda)\omega'$ for some $0 < \lambda < 1$), then P is w-associated with ω'' and there are infinitely many such ω''. As was the case for Theorem 12.12, the correctness of this change for part a follows from that for part b.

Theorem 12.16 put no restriction on the partition γ (other than that, by Definition 12.15, γ must consist of at least two pieces). As we shall see shortly, $\gamma = \{\{1\}, \{2\}, \ldots, \{n\}\}$ is an important special case.

As we did following the proof of Theorem 12.12, we obtain a corollary to Theorem 12.16 by using Corollary 10.7.

Corollary 12.17

a. *There exists a p-class of partitions that maximizes more than one convex combination of measures corresponding to points in* S^+ *if and only if the RNS is separable.*

b. *More specifically: for any partition* P, P *maximizes more than one convex combination of measures corresponding to points in* S^+ *if and only if, for some partition* γ *of* $\{1, 2, \ldots, n\}$, *the RNS is* γ*-separable with respect to* P.

Proof: Part a follows easily from part b. We prove part b.

Fix a partition P. By part a of Corollary 10.7, P maximizes more than one convex combination of measures corresponding to points of S^+ if and only if P is w-associated with more that one point in S^+ and, by part b of Theorem 12.16, this occurs if and only if, for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is γ-separable with respect to P. □

We continue to discuss the RNS property of being separable in a way that parallels our earlier discussion of the RNS being concentrated. As before, we use the one-to-one correspondence given by the function m between Pareto maximal p-classes and points on the outer Pareto boundary of the IPS, together with our geometric perspective on the maximization of convex combinations of measures (involving points of first contact with the IPS of families of parallel hyperplanes). These ideas, together with Corollary 12.17, yield our next connection between the IPS and the RNS.

Fix a partition P. Then $m(P)$ is a point in the IPS and

> more than one family of parallel hyperplanes with coefficients from S^+ makes first contact with the IPS at $m(P)$
>
> if and only if
>
> P maximizes more than one convex combination of measures corresponding to elements of S^+
>
> if and only if (Corollary 12.17)
>
> for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is γ-separable with respect to P.

Our next task is somewhat harder. Previously, we were led to consider the existence of a family of parallel hyperplanes that makes first contact with the IPS at more than one point. It was easy to see that this occurs if and only if there is a line segment on the outer Pareto boundary of the IPS, and this led us to Theorem 12.14. Now, we wish to consider the existence of more than one family of parallel hyperplanes that makes first contact with the IPS at the same point. What does this tell us about the shape of the IPS?

Choose distinct $\alpha, \beta \in S^+$ and suppose that the families of parallel hyperplanes with coefficients given by α and by β each make first contact with the IPS at some point p (and, perhaps, at other points too). Then it is geometrically clear that the family of parallel hyperplanes corresponding to any point on the line segment between α and β makes first contact with the IPS at p. We shall say that p is an *edge point* of the IPS if this occurs. In other words, p is an edge point of the IPS if and only if there is some line segment ℓ in S^+ such that for each $\alpha \in \ell$ the family of parallel hyperplanes corresponding to α makes first contact with the IPS at p. In the language of calculus, p is an edge point if and only if, at the point p, the outer Pareto boundary of the IPS has an undefined directional derivative in at least one direction.

The intuitive idea here is that p is an edge point of the IPS if and only if there is a family of parallel hyperplanes that makes first contact with the IPS at p and is such that the direction of this family can be changed some small amount so that p is still a point of first contact with the IPS of the family of parallel hyperplanes that results from this change. We have already illustrated this idea for two players in Figure 12.2b. (The notion of "edge point" generalizes the two-player notion of "corner point.") The following theorem is immediate from our preceding work.

Theorem 12.18

a. *There is a edge point on the outer Pareto boundary of the IPS if and only if the RNS is separable.*
b. *More specifically: for any partition P, $m(P)$is an edge point on the outer Pareto boundary of the IPS if and only if, for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is γ-separable with respect to P.*

Suppose that P is a partition of C and, for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is γ-separable with respect to P. Fix i and j that are in different pieces of partition γ. Then there are distinct $\omega, \omega' \in S^+$, such that P is w-associated with ω and with ω', and the ratios between the ith and jth coordinates of ω and of ω' are different. (This follows from the proof of the reverse direction of Theorem 12.16, if we choose δ_a and δ_b so that $i \in \delta_a$ and $j \in \delta_b$.) Then $m(P)$ is an edge point on the outer Pareto boundary of the IPS and is a point of first contact with the IPS of the family of parallel hyperplanes with coefficients given by RD(ω) and of the family of parallel hyperplanes with coefficients given by RD(ω'), and these families have different ratios between their ith and jth coefficients. Suppose now that $m(P)$ is also a point of first contact with the IPS of other families of parallel hyperplanes, and these families disagree with each other, and with RD(ω) and RD(ω'), not only in the ratios between their ith and jth coefficients but also in the ratios between their ith and kth coefficients, and their jth and kth coefficients, for many k. This corresponds to the partition γ of $\{1, 2, \ldots, n\}$ making more distinctions, i.e., having more members of $\{1, 2, \ldots, n\}$ that are in different pieces of γ. If we insist that γ make all possible such distinctions, in other words, that $\gamma = \{\{1\}, \{2\}, \ldots, \{n\}\}$, then, for *any* distinct $i, j = 1, 2, \ldots, n$, there are families of parallel hyperplanes that have different ratios between their ith and jth coefficients and make first contact with the IPS at $m(P)$. We call such a point $m(P)$ a *jagged point*. Thus, $m(P)$ is

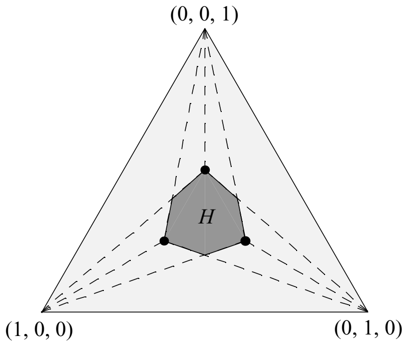


Figure 12.7

a jagged point on the outer Pareto boundary of the IPS if and only if the RNS is γ-separable with respect to P, where $\gamma = \{\{1\}, \{2\}, \ldots, \{n\}\}$.

As in our discussion of edge points, we can describe jagged points using the language of calculus. The point p is a jagged point on the outer Pareto boundary of the IPS if and only if, at the point p, the outer Pareto boundary of the IPS has an undefined directional derivative in *every* direction. Also in analogy with our discussion of edge points, we can describe a jagged point informally as follows: p is a jagged point of the IPS if and only if there is a family of parallel hyperplanes that makes first contact with the IPS at p and is such that the direction of this family can be changed some small amount *in any direction* so that p is still a point of first contact with the IPS of the family of parallel hyperplanes that results from this change.

Figure 12.7 gives a perspective, in terms of the RNS, of what a jagged point of the IPS looks like. In the figure, we assume that there are three players and that the RNS consists of three points. We have drawn dashed line segments connecting each vertex of the simplex with each of the three points of the RNS. Consider the darkened region H, where we consider H to be open (i.e., H does not include its boundary). Then H does not contain any points of the RNS and no dashed line segment intersects H. It is not hard to see that any point chosen from H has exactly one partition that is w-associated with it and any two points chosen from H have the same partition w-associated with them. (In particular, the partition associated with any point in H is the partition that gives to each player the cake associated with the point of the RNS that is closest to that player's vertex.) The fact that a point ω can move around H and still correspond to the same partition, together with the correspondence between such points ω and families of parallel hyperplanes, connects this perspective with that discussed in the previous paragraph. Thus, we see that if P is the unique partition that is w-associated with every $\omega \in H$, then $m(P)$ is a jagged point on the outer Pareto boundary of the IPS.

We now return to a theme we introduced in Section 12A and examined for two players in Section 12B. This involves the relation M. In Section 12B, we discussed, for the two-player context, when M is one-one, when it is many-one, and when it is one-many. We also saw that it is impossible, in that context, for M to be many-many. We now consider these possibilities in the general n-player context. We recall that, as explained on Section 12A, the first coordinate of M is a point in S^+, to be thought of as in w-association or to be thought of as in the maximization of convex combinations of measures, and the second coordinate of M is a p-class of partitions or is a point on the outer Pareto boundary of the IPS. Previous results in this section allow us to characterize when M is one-many and when it is many-one. The following is an immediate corollary to Theorems 12.12 and 12.16.

Corollary 12.19

a. *For any $\omega \in S^+$, there is more than one point p on the outer Pareto boundary of the IPS such that $M(\omega, p)$ holds if and only if, for some i and j, the RNS is i, j-concentrated with respect to ω.*
b. *For any point p on the outer Pareto boundary of the IPS, there is more than one point $\omega \in S^+$ such that $M(\omega, p)$ holds if and only if, for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is γ-separable with respect to the partition P of C, wherem$(P) = p$.*

Next we consider, somewhat informally, the cases not covered by Corollary 12.19, i.e., when M is one-one and when M is many-many. Concerning M being one-one, it will be convenient for us to think of the first coordinate of M as being a point in S^+, as in the maximization of convex combinations of measures. We wish to know under what circumstances there is an $\alpha \in S^+$ and a point p on the outer Pareto boundary of the IPS, such that $M(\alpha, p)$ holds, but for no $\beta \in S^+$ with $\beta \neq \alpha$ does $M(\beta, p)$ hold, and for no q on the outer Pareto boundary of the IPS with $q \neq p$ does $M(\alpha, q)$ hold. Or, equivalently, we wish to know under what circumstances there is a point p on the outer Pareto boundary of the IPS and a family F of parallel hyperplanes such that

a. F makes first contact with the IPS at p,
b. F makes first contact with the IPS at no point other than p, and
c. no other family of parallel hyperplanes makes first contact with the IPS at p.

Assume that condition a holds. As discussed earlier in this section, we know that if p is not on a line segment on the outer Pareto boundary of the IPS, then condition b holds, and if p is not an edge point, then condition c holds. (The converse of the second statement is true. The converse of the first statement may or may not be true. If condition b holds, then p is not an interior point of a line segment on the outer Pareto boundary of the IPS. However, it is possible that p could be an endpoint of such a line segment.) Thus, if p is a point on the outer Pareto boundary of the IPS at which the IPS is smooth in all directions (i.e., p is not an edge point) and flat in no direction (i.e., p is not on a line segment on the outer Pareto boundary), and if $\alpha \in S^+$ yields the coefficients of some family of parallel hyperplanes that makes first contact with the IPS at p, then $M(\alpha, p)$ holds and M is a one-one relation in this case. In other words, given these assumptions, for no $\beta \in S^+$ with $\beta \neq \alpha$ does $M(\beta, p)$ hold and for no q on the outer Pareto boundary of the IPS with $q \neq p$ does $M(\alpha, q)$ hold. This is clearly consistent with (the natural generalization to the n-player context of) parts b and c of Observation 12.7. However, the correspondence with part a of Obervation 12.7 does not generalize to the context of more than two players, and we can see this by considering Figure 12.6c again. In the figure, suppose that P is w-associated with ω. Then $M(\omega, [P]_p)$ holds. It is clear from the figure that, for any $\omega' \in S^+$ and any partition Q, $M(\omega', [P]_p)$ holds if and only if $\omega' = \omega$, and $M(\omega, [Q]_p)$ holds if and only if P and Q are p-equivalent. Hence, M is one-one in this situation. However, the RNS is not "spread out" in the sense described in Section 12B for two players. In Chapter 14, we shall construct an example in which M is one-one on the entire cake. (See the discussion following the proof of Theorem 14.14.)

It remains for us to consider when the relation M is many-many. By Corollary 12.19, this requires that, for some $\omega \in S^+$ and some partition P that is w-associated with ω,

- the RNS is i, j-concentrated with respect to ω for some i and j and
- the RNS is γ-separable with respect to P for some partition γ of $\{1, 2, \ldots, n\}$.

It follows from our discussion earlier in this section that this occurs if and only if the family of parallel hyperplanes with coefficients given by $\text{RD}(\omega)$ makes first contact with the IPS at $m(P)$ and

- $m(P)$ is on a line segment on the outer Pareto boundary of the IPS and
- $m(P)$ is an edge point of the IPS.

We illustrate this situation for three players in Figure 12.8.

First consider the RNS shown in Figure 12.8a. This RNS consists of two points, p and q, both of which are on the line $x = y$. This implies that Player 1 and Player 2 agree on almost all of the cake (i.e., for almost every $a \in C$, $f_1(a) = f_2(a)$). Although we shall not do so, it is quite easy to explicitly define a cake C and associated measures so that the corresponding RNS is as in

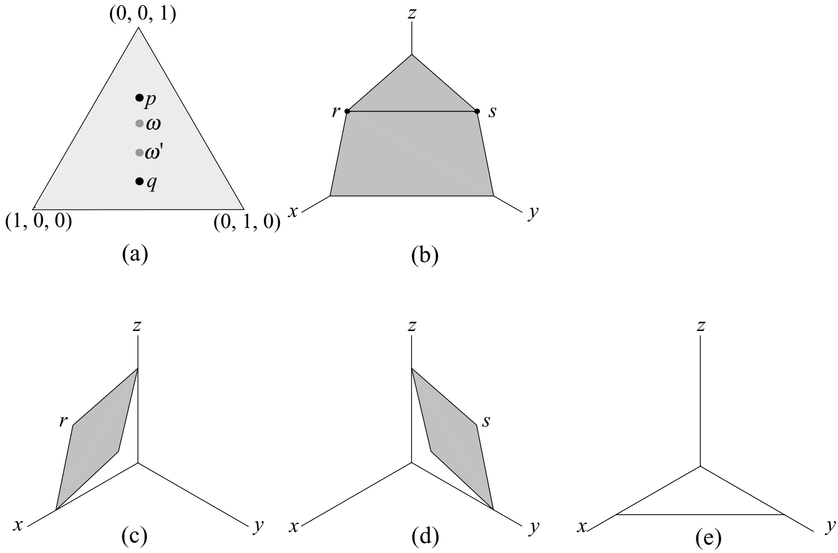


Figure 12.8

the figure. Fix any two points ω and ω' that are on the open line segment between p and q, as in the figure. Any partition associated with either of these two points gives all of the cake associated with point p to Player 3 and divides the cake associated with point q between Player 1 and Player 2. Since the cake associated with point q has positive measure, it follows that the RNS is 1,2-concentrated with respect to ω and also with respect to ω'. Also, a partition is w-associated with ω if and only if it is w-associated with ω'. Let P be any partition that is w-associated with these two points and let $\gamma = \{\{1, 2\}, \{3\}\}$. Then the RNS is γ-separable with respect to P. This is the situation discussed in the preceding paragraph, in which the concentration and the separability of the RNS coincide and, thus, the relation M is many-many. We can be more specific here. Let P be the partition that is w-associated with these points and gives all of the cake associated with q to Player 1, and let Q be the partition that is w-associated with these points and gives all of the cake associated with q to Player 2. Then, P and Q are each w-associated with both ω and ω' and are not p-equivalent. Then, $M(\omega, [P]_p)$, $M(\omega, [Q]_p)$, $M(\omega', [P]_p)$, and $M(\omega', [Q]_p)$ all hold, illustrating that M is many-many in this situation. These relations are clearly true with first coordinate any point on the line segment between p and q, and with second coordinate the p-class of any partition that gives the cake associated with point p to Player 3 and divides the cake associated with point q between Player 1 and Player 2. Hence, M is "infinitely many" to "infinitely many."

We claim that the corresponding IPS is as pictured in Figure 12.8b. To see this, we have shown the intersection of the IPS with each of the three coordinate planes in Figures 12.8c, 12.8d, and 12.8e. Concerning Figure 12.8c, we note that the RNS corresponding to the measures m_1 and m_3 consists of two points in the interior of the one-simplex. Then, arguing as we did earlier in this section when discussing Figures 12.1bi and 12.1bii, it is easy to see that the corresponding IPS, which is the same as the intersection of the origin three-dimensional IPS with the xz plane, is as pictured in Figure 12.8c. The argument for Figure 12.8d is the same, with the roles of Player 1 and Player 2 (and therefore the coordinates of x and y) reversed. For Figure 12.8e, we simply note that, since m_1 and m_2 are equal on almost all of C, the corresponding IPS in the xy plane is the line segment between (1, 0, 0) and (0, 1, 0). Putting these three figures together yields the full IPS of Figure 12.8b.

In Figure 12.8b, consider the open line segment between points r and s. Every point on this line segment is an edge point. This simultaneous "line segment and edge point" situation is needed for M to be many-many but, as we saw in Section 12B, is impossible if there are only two players. (Recall that what we now call an "edge point" in our general n-player context, we called a "corner point" in the two-player context of Section 12B.)

We close this section by considering chores versions of the ideas and results of this section. The following is the natural adjustment of the definition of concentrated to the chores setting.

Definition 12.20

a. For distinct $i, j = 1, 2, \ldots, n$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$, the RNS is *chores i, j-concentrated with respect to ω* if and only if $\{a \in C : \frac{f_i(a)}{f_j(a)} = \frac{\omega_i}{\omega_j}$ and, for every $k = 1, 2, \ldots, n$ with $k \neq i$ and $k \neq j$, $\frac{f_i(a)}{f_k(a)} \leq \frac{\omega_i}{\omega_k}$ and $\frac{f_j(a)}{f_k(a)} \leq \frac{\omega_j}{\omega_k}\}$ has positive measure.
b. The RNS is *chores concentrated* if and only if, for some i, j, and ω, the RNS is chores i, j-concentrated with respect to ω.

We have already discussed the fact that some two players are in relative agreement on some positive-measure set if and only if the RNS is concentrated. It is also true that some two players are in relative agreement on some positive-measure set if and only if the RNS is chores concentrated. Thus, the RNS is concentrated if and only if it is chores concentrated. This is illustrated in Figure 12.9. We assume that there are three players, that Player 1 and Player 2 are in relative agreement on some set A of positive measure, and that A corresponds to points in the RNS on the solid line segment between ω and ω' in Figures 12.9a and 12.9b. (We just show this portion of the RNS. The RNS may

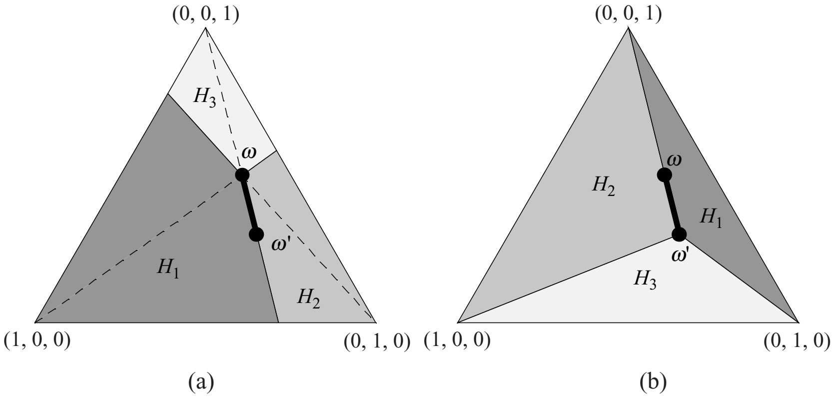


Figure 12.9

consist of many more points. Note that the relative agreement of Player 1 and Player 2 on A implies that the line segment corresponding to A is collinear with Player 3's vertex.) In Figure 12.9a we have shown the regions of the RNS determined by ω using the notion of w-associated, and in Figure 12.9b we have shown the regions of the RNS determined by ω' using the notion of chores w-associated. It is clear from these figures that the given RNS is $1,2$-concentrated with respect to ω and is chores $1,2$-concentrated with respect to ω'.

The natural adjustments of Theorem 12.12 and Corollary 12.13 to the chores setting are the following. The proofs are analogous and we omit them.

Theorem 12.21

a. *There exists a point in S^+ with which more than one p-class of partitions is chores w-associated if and only if the RNS is chores concentrated.*

b. *More specifically: for any $\omega \in S^+$, more than one p-class of partitions is chores w-associated with ω if and only if, for some i and j, the RNS is chores i, j-concentrated with respect to ω.*

Corollary 12.22

a. *There exists a convex combination of measures with coefficients from S^+ that is minimized by more than one p-class of partitions if and only if the RNS is chores concentrated.*

b. *More specifically: for any $\alpha \in S^+$, more than one p-class of partitions minimizes the convex combination of measures corresponding to α if and only if, for some i and j, the RNS is chores i, j-concentrated with respect to $RD(\alpha)$.*

We continue to parallel our previous discussion by next using these results to connect the IPS and the RNS. Fix $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S^+$. Then, the family of parallel hyperplanes $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ makes last contact with the IPS at more than one point

if and only if

more than one p-class of partitions minimizes the convex combination of measures corresponding to α

if and only if (Corollary 12.22)

for some i and j, the RNS is chores i, j-concentrated with respect to RD(α).

The following is the natural adjustment of Theorem 12.14 to the chores setting. It follows easily from the preceding ideas.

Theorem 12.23

a. *There is a line segment on the inner Pareto boundary of the IPS if and only if the RNS is chores concentrated.*

b. *More specifically: for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S^+$, there is a line segment ℓ on the inner Pareto boundary of the IPS and every point of ℓ is a point of last contact with the IPS of the family of parallel hyperplanes $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ if and only if, for some i and j, the RNS is chores i, j-concentrated with respect to RD(α).*

We have previously observed that the RNS is concentrated if and only if it is chores concentrated. This, together with Theorems 12.14 and 12.23, yields the following result.

Corollary 12.24 (to Theorems 12.14 and 12.23) *There is a line segment on the outer Pareto boundary of the IPS if and only if there is a line segment on the inner Pareto boundary of the IPS.*

Next, we present the natural adjustment of the notion of separable to the chores setting.

Definition 12.25

a. Suppose that γ is a partition of $\{1, 2, \ldots, n\}$ into at least two pieces and $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of C. The RNS is *chores γ-separable with respect to P* if and only if for some $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$ and $\varepsilon > 0$
   i. P is chores w-associated with ω and
   ii. for any i and j that are in different pieces of partition γ, $\frac{f_i(a)}{f_j(a)} \leq \frac{\omega_i}{\omega_j} - \varepsilon$ for almost every $a \in P_i$.

b. The RNS is *chores separable* if and only if, for some γ and P, the RNS is chores γ-separable with respect to P.

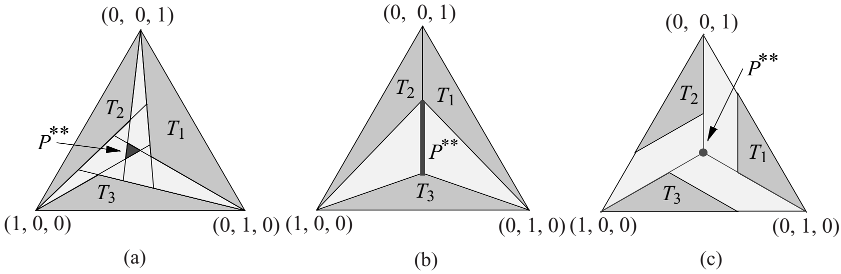


Figure 12.10

We used Figure 12.6 to illustrate the notion of separable for three players. We recall that each of these three figures shows an RNS consisting of three regions, T_1, T_2, and T_3, that $P = \langle P_1, P_2, P_3 \rangle$ is a partition, and that P_1, P_2, and P_3 correspond to T_1, T_2, and T_3, respectively. In each region, we let P^* denote the set of points with which P is w-associated. Then, it is clear from the figures that the RNSs in Figures 12.6a and 12.6b are separable, while the RNS in Figure 12.6c is not. In a similar manner, we can illustrate the notion of chores separable, using Figure 12.10. As before, each RNS consists of three regions, T_1, T_2, and T_3, $P = \langle P_1, P_2, P_3 \rangle$ is a partition, and P_1, P_2, and P_3 correspond to T_1, T_2, and T_3, respectively. Letting P^{**} denote the set of points with which P is chores w-associated in each figure, we see that the RNSs in Figures 12.10a and 12.10b are chores separable, whereas the RNS in Figure 12.10c is not. The explanations are analogous to those used for Figure 12.6, and we omit them, except to note that the RNS in Figure 12.10a is chores γ-separable with respect to P for any partition γ of $\{1, 2, 3\}$ into at least two pieces, whereas the RNS in Figure 12.10b is chores γ-separable with respect to P only for the partition $\gamma = \{\{1, 2\}, \{3\}\}$ of $\{1, 2, 3\}$.

There is a correspondence between the notions of separable and chores separable that holds in the two-player context. First note that in that context the γ in the definition of separable and of chores separable must be the partition $\{\{1\}, \{2\}\}$ of $\{1, 2\}$. Suppose that $\langle P_1, P_2 \rangle$ is a partition. Then the RNS is $\{\{1\}, \{2\}\}$-separable with respect to $\langle P_1, P_2 \rangle$ if and only if the RNS is chores $\{\{1\}, \{2\}\}$-separable with respect to $\langle P_2, P_1 \rangle$. Thus, the RNS is separable if and only if it is chores separable, and these notions are equivalent to the existence of a gap in the RNS, as discussed in Section 12B. No such correspondence holds, in general, if there are more than two players.

The natural adjustments of Theorem 12.16 and Corollary 12.17 to the chores setting are the following. The proofs are analogous and we omit them.

Theorem 12.26

a. *There exists a p-class of partitions that is chores w-associated with more than one point in S^+ if and only if the RNS is chores separable.*

b. *More specifically: for any partition P, P is chores w-associated with more than one point in S^+ if and only if, for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is chores γ-separable with respect to P.*

Corollary 12.27

a. *There exists a p-class of partitions that minimizes more than one convex combination of measures corresponding to points in S^+ if and only if the RNS is chores separable.*

b. *More specifically: for any partition P, P minimizes more than one convex combination of measures corresponding to points in S^+ if and only if, for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is chores γ-separable with respect to P.*

Continuing to parallel our previous discussion, we next combine these ideas with the notion of points of last contact with the IPS of families of parallel hyperplanes. Fix a partition P. Then, $m(P)$ is a point in the IPS and

more than one family of parallel hyperplanes with coefficients from S^+ makes last contact with the IPS at $m(P)$

if and only if

P minimizes more than one convex combination of measures corresponding to elements of S^+

if and only if (Corollary 12.27)

for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is chores γ-separable with respect to P.

Next, we present the natural adjustment of Theorem 12.18 to the chores setting. The proof follows from the preceding ideas. The notion of "edge point on the inner Pareto boundary of the IPS" is defined in a way analogous to how we defined "edge point on the outer Pareto boundary of the IPS."

Theorem 12.28

a. *There is an edge point on the inner Pareto boundary of the IPS if and only if the RNS is chores separable.*

b. *More specifically: for any partition P, $m(P)$ is an edge point on the inner Pareto boundary of the IPS if and only if, for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is chores γ-separable with respect to P.*

We adjust the relation M to the chores setting as follows.

Definition 12.29 We define the relation M_C between S^+ and the set of p-classes of Pareto minimal partitions as follows: for $\omega \in S^+$ and P a Pareto minimal partition, $M_C(\omega, [P]_p)$ holds if and only if P is chores w-associated with ω.

In analogy with our rules for the relation M, we will allow the first coordinate of M_C to be a point in S^+ either to be thought of as in chores w-association or to be thought of as in the minimization of convex combinations of measures, and the second coordinate of M_C to be a p-class of Pareto minimal partitions or a point on the inner Pareto boundary of the IPS. The adjustment of Corollary 12.19 to the chores setting is the following. The proof is immediate from Theorems 12.21 and 12.26.

Corollary 12.30

a. *For any $\omega \in S^+$, there is more than one point p on the inner Pareto boundary of the IPS such that $M_C(\omega, p)$ holds if and only if, for some i and j, the RNS is chores i, j-concentrated with respect to ω.*
b. *For any point p on the inner Pareto boundary of the IPS, there is more than one point $\omega \in S^+$ such that $M_C(\omega, p)$ holds if and only if, for some partition γ of $\{1, 2, \ldots, n\}$, the RNS is chores γ-separable with respect to the partition P of C, where $m(P) = p$.*

We briefly and informally consider the cases not covered by Corollary 12.30, i.e., when M_C is one-one and when it is many-many:

- If p is a point on the inner Pareto boundary of the IPS at which the IPS is smooth in all directions (i.e., p is not an edge point) and flat in no direction (i.e., p is not on a line segment on the inner Pareto boundary), and if $\alpha \in S^+$ yields the coefficients of some family of parallel hyperplanes that makes last contact with the IPS at p, then $M_C(\alpha, p)$ holds and M_C is a one-one relation in this case. In other words, given these assumptions, for no $\beta \in S^+$ with $\beta \neq \alpha$ does $M_C(\beta, p)$ hold, and for no q on the outer Pareto boundary of the IPS with $q \neq p$ does $M_C(\alpha, q)$ hold.
- If p is an interior point of a line segment ℓ on the inner Pareto boundary of the IPS and is an edge point on the inner Pareto boundary of the IPS, and if $\alpha \in S^+$ gives the coefficients of some family of parallel hyperplanes that makes last contact with the IPS at p, then M_C is a many-many relation in this case. In particular, if $T = \{\beta \in S^+ : \beta$ yields the coefficients of some family of parallel hyperplanes that makes last contact with the IPS at $p\}$, then $M_C(\beta, q)$ holds for every $\beta \in T$ and $q \in \ell$. (Note that ℓ obviously contains many points and T contains many points since p is an edge point on the inner Pareto boundary of the IPS.)

12D. The Situation Without Absolute Continuity

In this section we make no general assumptions about absolute continuity.

In Section 12B, we revisited the IPSs in Figures 2.1a, 2.1b, 2.1c, and 2.1d. We repeated these in Figures 12.1ai, 12.1bi, 12.1ci, and 12.1di and displayed the corresponding RNSs in Figures 12.1aii, 12.1bii, 12.1cii, and 12.1dii, respectively. We begin this section by revisiting the IPSs from Figure 2.1 that we did not include in Section 12B because they involve the failure of absolute continuity.

Consider Figure 12.11. Figures 12.11ai and 12.11bi are copies of Figures 2.1e and 2.1f, respectively. As in Figure 12.1, we have darkened the outer boundary

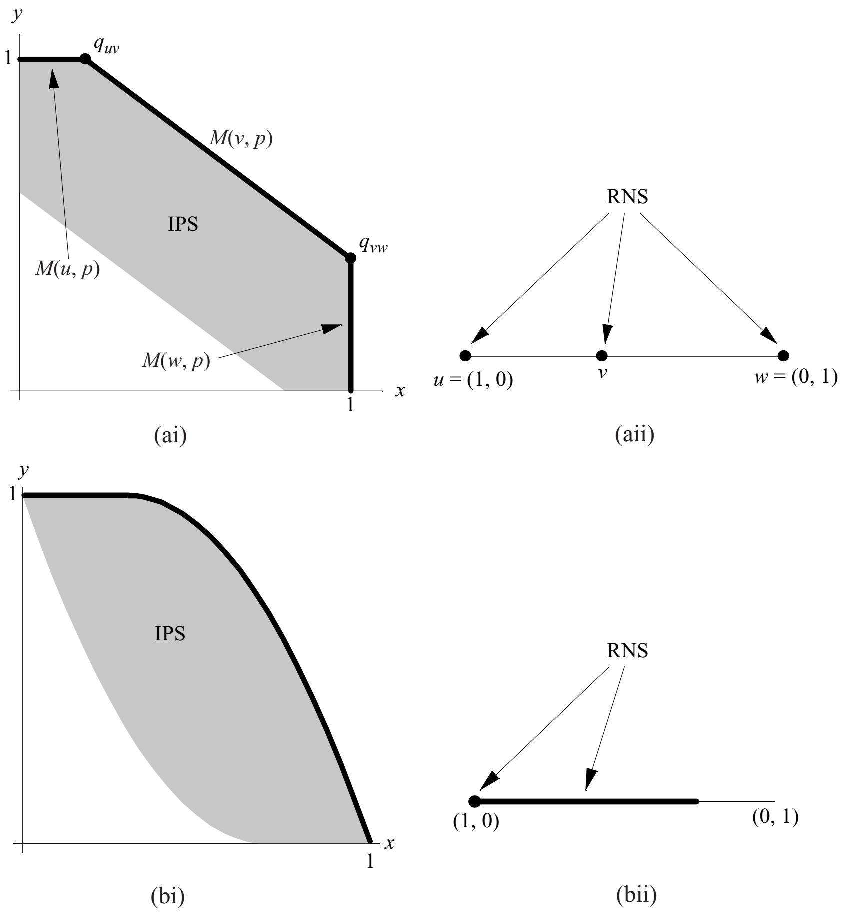


Figure 12.11

(which, since the measures are not to be absolutely continuous with respect to each other, is not the same as the outer Pareto boundary). Recall that, by Theorem 11.1, for each of the two regions pictured, we know that there is a cake C and measures m_1 and m_2 on C so that the given figure is the corresponding IPS. We claim that Figures 12.11aii and 12.11bii are the RNSs corresponding to the IPSs in Figures 12.11ai and 12.11bi, respectively. The ideas are similar to those used in our discussion of Figure 12.1 and, hence, we shall just sketch them here.

First consider Figures 12.11ai and 12.11aii. We continue to use the relation M, as in previous sections of this chapter.

- The point u in the RNS corresponds to a piece of cake that has positive value to Player 1 and no value to Player 2. This corresponds, via M, to the horizontal line segment on the outer boundary of the IPS, which we have labeled "$M(u, p)$."
- The gap in the RNS between u and v corresponds, via M, to the point q_{uv} on the outer boundary of the IPS.
- The point v in the RNS corresponds to the line segment on the outer boundary of the IPS that we have labeled "$M(v, p)$."
- The gap in the RNS between v and w corresponds to the point q_{vw} on the outer boundary of the IPS.
- The point w in the RNS corresponds to a piece of cake that has positive value to Player 2 and no value to Player 1. This corresponds to the vertical line segment on the outer boundary of the IPS, which we have labeled "$M(w, p)$."

Next, we consider Figures 12.11bi and 12.11bii. The RNS of Figure 12.11bii consists of a point and a line segment. (The point corresponds to a piece of cake of positive measure, as does any non-empty interval of the line segment.)

- Precisely as before, the point u in the RNS corresponds to the horizontal line segment on the outer boundary of the IPS.
- The line segment in the RNS (which we have previously called a "spread-out" region of the RNS) corresponds to the (non-straight line) curve on the outer boundary of the IPS.
- The lack of a gap between the point and the line segment in the RNS corresponds to the fact that the line segment and the curve in the IPS meet "smoothly." In other words, the IPS does not have a corner point at the meeting of this line and curve. (Or, in the language of calculus, the relevant one-sided derivatives of this line and curve are equal at their meeting point.)
- The lack of any points in the RNS to the right of the line segment corresponds to the fact that the curve in the IPS does not meet the x axis in a vertical manner.

It is not hard to see that the chores version of these ideas (which relate to the inner boundary) are similar, and we omit them.

Next, we wish to consider the issues of concentration and separability. We begin by noting that neither the definitions of concentration and separability, nor the associated theorems and corollaries, relied on absolute continuity. Hence, we may take Definitions 12.9 and 12.15 to be our definitions of concentration and separability, respectively, regardless of whether the measures are absolutely continuous with respect to each other and, in this general context, Theorems 12.12, 12.14, 12.16, and 12.18, and Corollaries 12.13 and 12.17, are still valid. Similarly, the corresponding chores definitions (Definitions 12.20 and 12.25) and results (Theorems 12.21, 12.23, 12.26, and 12.28, and Corollaries 12.22 and 12.27) are still valid in our present context.

We do not need to alter our definition of relative agreement (Definition 12.11), but we do need a slight adjustment in our use of this notion. In the last section, we discussed the relationship between this notion and the notion of concentrated. We found that, for distinct $i, j = 1, 2, \dots, n$,

> Player i and Player j are in relative agreement on some positive-measure $A \subseteq C$ if and only if, for some $\omega \in S^+$, the RNS is i,j-concentrated with respect to ω.

In adjusting this statement to our present context, we find that our usual practice of interpreting "positive measure" to mean "positive measure with respect to $\mu = m_1 + m_2 + \cdots + m_n$" does not suffice. We need to insist that the set under consideration has positive measure to each of the two involved players. In other words, for distinct $i, j = 1, 2, \dots, n$,

> Player i and Player j are in relative agreement on some $A \subseteq C$ that has positive measure to each of these two players if and only if, for some $\omega \in S^+$, the RNS is i,j-concentrated with respect to ω.

Or, more generally,

> some two players are in relative agreement on some set that has positive measure to each of these two players if and only if the RNS is concentrated.

The proof of these statements is straightforward and we omit it.

Notice that all of the results that explicitly concern the shape of the IPS (i.e., Theorems 12.14, 12.18, 12.23, and 12.28) only comment on the Pareto boundary, rather than the full boundary. Except in the special case of two players when the two measures are absolutely continuous with respect to each other, we know (see Theorems 3.9, 3.22, and 5.35) that the Pareto boundary is a proper subset of the boundary. Hence, it would be desirable to know when there are line segments or edge points on the boundary, not just the Pareto boundary of the IPS. We first examine and give a complete solution to this problem for two players, and then discuss why a solution is problematic when there are more than two players.

Assume that there are two players, Player 1 and Player 2. As discussed in the previous paragraph, if the measures are absolutely continuous with respect to each other, then the Pareto boundary is the same as the boundary, and our previous results suffice. If the measures are not absolutely continuous with respect to each other, then (as illustrated in Figure 12.11) there is a line segment of the outer boundary of the IPS that is not on the outer Pareto boundary of the IPS. For the case of two players, the following result can be viewed as a generalization of Theorems 12.14 and 12.23.

Theorem 12.31 *Assume that there are two players. There is a line segment on the outer boundary and on the inner boundary of the IPS if and only if either the RNS is concentrated or at least one of the measures is not absolutely continuous with respect to the other.*

Before beginning the proof, we make the following three observations:

- By the symmetry of the IPS when there are two players (see Theorem 2.4), there is a line segment on the outer boundary of the IPS if and only if there is a line segment on the inner boundary of the IPS.
- Measure m_1 fails to be absolutely continuous with respect to measure m_2 if and only if there is a horizontal line segment on the outer boundary of the IPS. (The left endpoint of this line segment is the point $(0, 1)$).
- Measure m_2 fails to be absolutely continuous with respect to measure m_1 if and only if there is a vertical line segment on the outer boundary of the IPS. (The bottom endpoint of this line segment is the point $(1, 0)$.)

Proof of Theorem 12.31: By the first of our preceding observations, we may ignore the "inner boundary" condition in the statement of the theorem.

For the forward direction, we assume that there is a line segment on the outer boundary of the IPS. If this line segment is either vertical or horizontal, then our second and third observations tell us that at least one of the measures is not absolutely continuous with respect to the other. If the line segment is neither vertical nor horizontal, then it must be on the outer Pareto boundary of the IPS. Then part a of Theorem 12.14 implies that the RNS is concentrated.

For the reverse direction, we first assume that the RNS is concentrated. Then, by part a of Theorem 12.14, there is a line segment on the outer Pareto boundary of the IPS, and so there is certainly a line segment of the outer boundary of the IPS. Next, suppose that the measures are not each absolutely continuous with respect to the other. Then our second and third observations imply that there is a line segment on the outer boundary of the IPS. □

We note that the two conditions given by the theorem ("the RNS is concentrated" and "at least one of the measures is not absolutely continuous with respect to the other") may appear to be quite different sorts of conditions, but are actually very closely related. "The RNS is concentrated" says that there is a point in S^+, the open interval between the points $(1, 0)$ and $(0, 1)$, that is associated with a piece of cake of positive measure. "At least one of the measures is not absolutely continuous with respect to the other" says that the point $(1, 0)$ or the point $(0, 1)$ is associated with a piece of cake of positive measure. It follows that the disjunction of these two conditions is equivalent to the statement that some point of S is associated with a piece of cake of positive measure.

Thus we see that when there are two players it is possible to extend our analysis of line segments on the Pareto boundary of the IPS to line segments on the boundary of the IPS. The analogous issue for corner points is trivial, since it is geometrically clear that a corner point can only occur at a point on the Pareto boundary. Thus, for the two-player context, we can extend part a of Theorem 12.18 and part a of Theorem 12.28 in the obvious way: there is a corner point on the outer boundary and on the inner boundary of the IPS if and only if the RNS is separable. (This combining of the results for the outer and inner boundaries uses the symmetry of the IPS given by Theorem 2.4. Or, taking a different perspective, we recall that in the two-player context the RNS is separable if and only if it is chores separable. We also recall that an "edge point" in our general n-player context is a "corner point" in the two-player context.)

Suppose now that there are more than two players. Although we shall not study this issue in detail, we give an example to show that extending our previous results from the Pareto boundary to the full boundary is problematic. Suppose that there are three players, Player 1, Player 2, and Player 3, with measures m_1, m_2, and m_3, respectively, and assume that $m_1 \neq m_2$. Then the IPS corresponding to just Players 1 and 2 is more than the one-simplex and, hence, the intersection of the full IPS with the xy plane is more than just the line segment between $(1, 0, 0)$ and $(0, 1, 0)$. We illustrate this situation in Figure 12.12. In Figure 12.12a, we have shown the IPS, where we have darkened the outer boundary of the restricted IPS corresponding to Players 1 and 2. In Figure 12.12b, we have shown just the region of the xy plane that is enclosed by the outer boundary of this restricted IPS and the relevant one-simplex (i.e., the line segment between $(1, 0, 0)$ and $(0, 1, 0)$). Notice that this region is part of the outer boundary of the full IPS (since the sum of the coordinates of any point

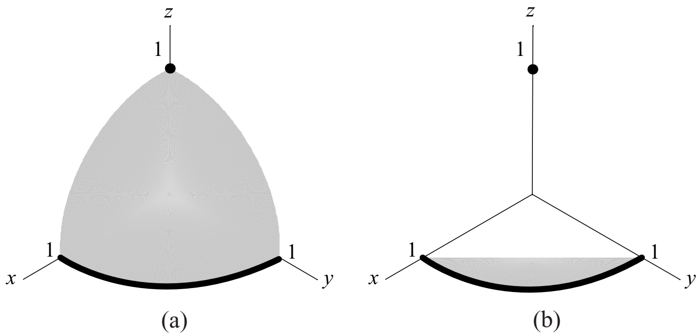


Figure 12.12

in this region is at least one), but not the outer Pareto boundary. Obviously, this region contains many line segments. Thus, we see that, as long as the measures are not all equal, there will always be line segments on the outer boundary of the IPS, regardless of whether the RNS is concentrated or whether the measures are absolutely continuous with respect to each other. It is therefore not clear how, for example, Theorem 12.14 can be generalized to apply to the full outer boundary.

12E. Fairness and Efficiency Together: Part 2

We make no general assumptions about absolute continuity in this section. In Section 5C (where we assumed that absolute continuity held) and in Section 5D (where we assumed that absolute continuity failed), we investigated the existence of partitions that are proportional and Pareto maximal, partitions that are strongly proportional and Pareto maximal, and partitions that satisfy the analogous chores properties. In this section, we shall establish the following result.

Theorem 12.32 *There exists a partition that is envy-free and Pareto maximal.*

The proof involves the relationship between the IPS and the RNS. This result is attributable to D. Weller [43]. Although our terminology and notation is somewhat different from that used by Weller, the proof is essentially his. Our presentation differs in certain ways due to our ability to take advantage of material we have previously developed. A closely related result was proved by M. Berliant, W. Thomson, and K. Dunz [13].

The proof uses the following result, attributable to S. Kakutani [28].

Theorem 12.33 (Kakutani's Fixed Point Theorem) *Let S be a closed, bounded, and convex subset of $\mathbf{R}^n$ and assume that the function $J : S \to \mathbf{P}(S)$ satisfies the following two properties:*

a. For every $\omega \in S$, $J(\omega)$ is closed and convex.
b. If $\langle \omega^t \rangle$ is a sequence of points in S that converges to ω, $\langle p^t \rangle$ is a sequence of points in S that converges to p, and $p^t \in J(\omega^t)$ for every $t = 1, 2, \ldots$, then $p \in J(\omega)$.

Then, for some $\omega \in S$, $\omega \in J(\omega)$.

A function mapping S to its power set is called a *point-to-set mapping* of S, and if such a function satisfies property b of the theorem, it is said to be *upper semicontinuous*. A point ω satisfying the conclusion of the theorem is said to be a *fixed point* of J. Then, the theorem says that any upper semicontinuous point-to-set mapping on a closed, bounded, and convex subset of $\mathbf{R}^n$, whose range consists of sets that are closed and convex, has a fixed point.

We have previously used "S" to denote the simplex. Theorem 12.33 applies to any closed, bounded, and convex set. We have used the letter "S" in the theorem since we shall be applying this result only to the simplex.

Next, we define a function that will be central to the proof of Theorem 12.32. Let *OPB* denote the outer Pareto boundary of the IPS. There is a natural mapping from OPB to S. This mapping is illustrated, for the case of two players, in Figure 12.13. We simply draw the one-simplex S and the outer boundary of the IPS using the same coordinate system. (In the figure, we have darkened S and the OPB and have drawn the part of the outer boundary that is not part of the OPB as a dashed line. Recall that the existence of this dashed horizontal line tells us that measure m_1 is not absolutely continuous with respect to m_2.) Any line connecting a point in OPB to the origin will contain exactly one point of S. This provides the desired mapping. More precisely, and in the general n-player context, we describe this mapping by defining $g : \text{OPB} \to S$ as follows:

If $p = (p_1, p_2, \ldots, p_n) \in \text{OPB}$, then for each $i = 1, 2, \ldots, n$, set $g_i(p) = \frac{p_i}{p_1 + p_2 + \cdots + p_n}$ and set $g(p) = (g_1(p), g_2(p), \ldots, g_n(p))$.

It is easy to see that, for every $p \in \text{OPB}$, $g(p) \in S$, and that when there are two players, g yields the mapping illustrated in Figure 12.13. (Notice that g is clearly a one-to-one function and is continuous. It is onto if and only if the measures are absolutely continuous with respect to each other.)

We use g to define a function $H : S^+ \to \mathbf{P}(S)$ as follows:

$$\text{for any } \omega \in S^+, H(\omega) = \{g(m(P)) : P \in \omega^*\}$$

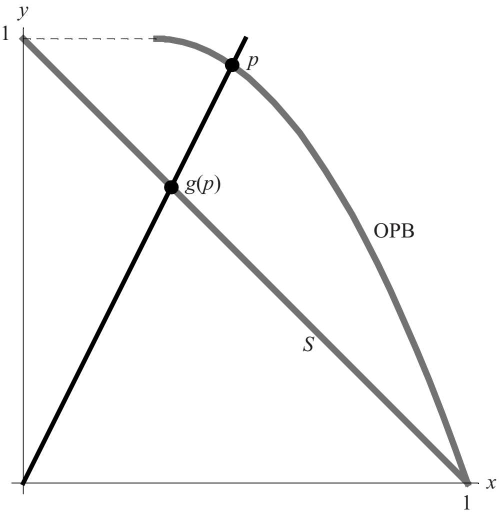


Figure 12.13

Notice that, for any $\omega \in S^+$ and $P \in \omega^*$, P is Pareto maximal and, hence, $m(P) \in$ OPB. Thus, for any such ω, $m(P)$ is in the domain of g.

In general, H sends a point near a boundary of S to a set of points far away from that boundary. For example, consider the case of three players. If ω is a point near $(1, 0, 0)$ then, in general, a partition w-associated with ω gives little cake to Player 1. Hence, for $P \in \omega^*$, $m(P)$ will have a small first coordinate, as will $g(m(P))$, and so $g(m(P))$ will be near the line segment between $(0, 1, 0)$ and $(0, 0, 1)$. We shall be interested in a fixed point of H and this perspective suggests that such a fixed point will not be near a boundary of S.

We shall obtain our desired fixed point by using Kakutani's fixed point theorem. However, the function H does not satisfy the premises of this result. We shall need to extend H to a function from S to $\mathbf{P}(S)$. We do so by first defining a function I from $S \backslash S^+$, the boundary of S, to $\mathbf{P}(S)$. We then combine this I with H to define a function J from S to $\mathbf{P}(S)$. The proof of Theorem 12.32 will center on finding a fixed point of J.

Informally stated, the idea is to define $I(\omega)$, for any $\omega \in S \backslash S^+$, to be most of S. We leave out a small region of S that includes the face containing ω. This guarantees that ω will not be a fixed point of I, and thus not a fixed point of J.

For any $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S \backslash S^+$, let $\delta_\omega = \{i \leq n : \omega_i > 0\}$ and let S_{δ_ω} denote the face of S corresponding to δ_ω. Then $S_{\delta_\omega} = \{(p_1, p_2, \ldots, p_n) \in S : p_i = 0 \text{ if } i \notin \delta_\omega\}$ and ω is an interior point of S_{δ_ω}. Let $T_{\delta_\omega} =$

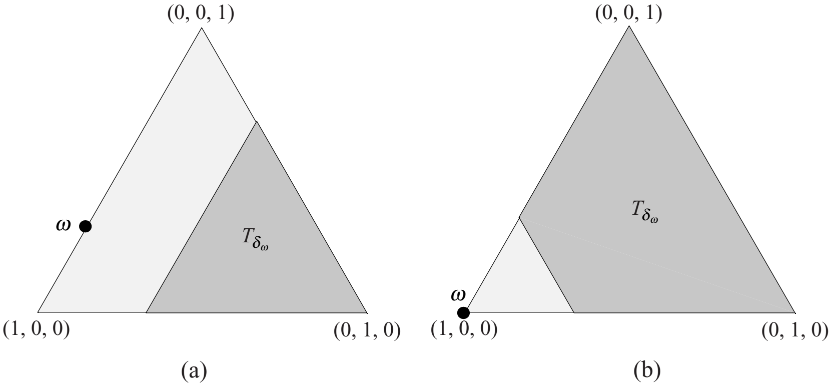


Figure 12.14

$\{(p_1, p_2, \ldots, p_n) \in S : \sum_{i \in \delta_\omega} p_i \leq \frac{n-1}{n}\}$ and define $I : S \backslash S^+ \to \mathbf{P}(S)$ as follows:

$$\text{for any } \omega \in S \backslash S^+, I(\omega) = T_{\delta_\omega}$$

Notice that for any $\omega \in S \backslash S^+$, $\{(p_1, p_2, \ldots, p_n) \in \mathbf{R}^n : \sum_{i \in \delta_\omega} p_i = \frac{n-1}{n}\}$ is a hyperplane in $\mathbf{R}^n$. Hence, for any such ω, $I(\omega) = T_{\delta_\omega}$ is the intersection of one of the closed half-spaces determined by this hyperplane, and S. Two examples of T_{δ_ω} for three players are illustrated in Figure 12.14.

We now define $J : S \to \mathbf{P}(S)$ as follows: for any $\omega \in S$,

$$J(\omega) = \begin{cases} H(\omega) & \text{if } \omega \in S^+ \\ I(\omega) & \text{if } \omega \in S \backslash S^+ \end{cases}$$

Observe that the functions H and I fit together in a nice way. Suppose that ω and ω' are points in S that are close to each other and that $\omega \in S \backslash S^+$ and $\omega' \in S^+$. Then ω' is close to S_{δ_ω}. As discussed previously, $H(\omega')$ will, in general, be far from this face. Clearly, $I(\omega) = T_{\delta_\omega}$ is disjoint from, but close to, this face (since $\sum_{i \in \delta_\omega} p_i = 1$ for any $(p_1, p_2, \ldots, p_n) \in S_{\delta_\omega}$, $\sum_{i \in \delta_\omega} p_i \leq \frac{n-1}{n}$ for any $(p_1, p_2, \ldots, p_n) \in T_{\delta_\omega}$, and $\sum_{i \in \delta_\omega} p_i = \frac{n-1}{n}$ for some $(p_1, p_2, \ldots, p_n) \in T_{\delta_\omega}$) and contains all points of S that are far from this face. Hence, $H(\omega')$ will be contained in $I(\omega)$. This fitting together of H and I will be made more precise when we discuss the upper semicontinuity of J.

It may seem peculiar that we have defined $I(\omega)$ to be so large, compared to our definition of $H(\omega)$. The idea here is simply that what we want is a fixed point of H, which means we want a fixed point of J that is in S^+. We have extended H to J in order to satisfy the premises of Kakutani's fixed point theorem. We have done so in a rather crude way, but this is justified since it is clear from the definition of I that we have not introduced any new fixed points.

Lemma 12.34 *J satisfies the premises of Kakutani's fixed point theorem.*

Proof: We must show that

a. for every $\omega \in S$, $J(\omega)$ is closed and convex and
b. J is upper semicontinuous.

Fix $\omega \in S$. We must show that $J(\omega)$ is closed and convex. Suppose first that $\omega \in S^+$. Then $J(\omega) = H(\omega) = \{g(m(P)) : P \in \omega^*\}$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = \text{RD}(\omega)$. (For the definition of the function RD, see Definition 10.5.) For any partition P, Theorem 10.6 implies that $P \in \omega^*$ if and only if P maximizes the convex combination of measures corresponding to α. By our work in Chapter 7, we know that this occurs if and only if $m(P)$ is a point of first contact of the family of parallel hyperplanes $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ with the IPS.

Define $K : S^+ \to \mathbf{P}(\text{OPB})$ as follows: for $\omega \in S^+, K(\omega) = \{m(P) : P \in \omega^*\}$. Then, for ω as in the previous paragraph, $K(\omega) \subseteq \text{OPB}$ is the set of points of first contact of the given family of parallel hyperplanes with the IPS. Since $J(\omega) = H(\omega) = \{g(m(P)) : P \in \omega^*\} = \{g(p) : p \in K(\omega)\}$, it suffices to show that $K(\omega)$ is closed and convex, since it is clear that g takes closed sets to closed sets and convex sets to convex sets. This is trivial, since $K(\omega)$ is the intersection of a hyperplane and the IPS, each of these sets is closed and convex, and the intersection of closed and convex sets is closed and convex.

Next, we suppose that $\omega \in S \backslash S^+$. Then $J(\omega) = I(\omega) = T_{\delta_\omega}$. As discussed earlier, T_{δ_ω} is the intersection of one of the closed half-spaces determined by a hyperplane, and S. Both of these sets are closed and convex and, hence, their intersection, $I(\omega)$, is closed and convex.

We must show that J is upper semicontinuous. Let $K : S^+ \to \mathbf{P}(\text{OPB})$ be as before. Then, as noted, for any $\omega \in S^+$, $H(\omega) = \{g(m(P)) : P \in \omega^*\} = \{g(p) : p \in K(\omega)\}$, and so we can view $H(\omega)$ as being obtained by applying the function g to each element of $K(\omega)$. We shall use this fact shortly.

Claim Suppose that $\langle \omega^t \rangle$ is a sequence of points in S^+ that converges to $\omega \in S^+$, that $\langle q^t \rangle$ is a sequence of points in OPB that converges to q, and that $q^t \in K(\omega^t)$ for every $t = 1, 2, \ldots$. Then, $q \in K(\omega)$.

Proof of Claim: Let $\langle \omega^t \rangle, \langle q^t \rangle, \omega,$ and q be as in the claim. Set $\text{RD}(\omega) = (\alpha_1, \alpha_2, \ldots, \alpha_n), q = (q_1, q_2, \ldots, q_n)$, and, for each $t = 1, 2, \ldots,$ set $\text{RD}(\omega^t) = (\alpha_1^t, \alpha_2^t, \ldots, \alpha_n^t), q^t = (q_1^t, q_2^t, \ldots, q_n^t)$, and let Q^t be any partition satisfying that $m(Q^t) = q^t$. Since for any $t, q^t \in K(\omega^t)$, we know that Q^t is w-associated with ω^t. (This uses the fact that w-association respects p-equivalence. See the paragraph following the proof of Theorem 10.6.) By Theorem 10.6, this implies that Q^t maximizes the convex combination of measures corresponding to $\text{RD}(\omega^t) = (\alpha_1^t, \alpha_2^t, \ldots, \alpha_n^t)$, and it follows that the family of parallel hyperplanes given by $\alpha_1^t x_1 + \alpha_2^t x_2 + \cdots + \alpha_n^t x_n = c$ makes first contact with the IPS at q^t (and possibly at other points too). Similarly, we see that it suffices for us to show that the family of parallel hyperplanes $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ makes first contact with the IPS at q.

Notice that, since the function RD is certainly continuous and the sequence $\langle \omega^t \rangle$ converges to ω, we know that the sequence $\langle \text{RD}(\omega^t) \rangle = \langle (\alpha_1^t, \alpha_2^t, \ldots, \alpha_n^t) \rangle$ converges to $\text{RD}(\omega) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then, since $\langle q^t \rangle$ converges to q, it follows that the sequence $\langle \alpha_1^t \text{q}_1^t + \alpha_2^t \text{q}_2^t + \cdots + \alpha_n^t \text{q}_n^t \rangle$ converges to $\alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n$. Set $k = \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n$. Then $q = (q_1, q_2, \ldots, q_n)$ is on the hyperplane $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = k$. We claim that this is the hyperplane of first contact of the family of parallel hyperplanes $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ with the IPS.

Suppose, by way of contradiction, that this family of parallel hyperplanes makes contact with the point $q' = (q_1', q_2', \ldots, q_n')$ of the IPS before q. Set $k' = \alpha_1 q_1' + \alpha_2 q_2' + \cdots + \alpha_n q_n'$. Then, $k' > k$. Let $\varepsilon = \frac{k'-k}{2}$. Then $\varepsilon > 0$.

Since $\alpha_1 q_1' + \alpha_2 q_2' + \cdots + \alpha_n q_n' = k'$ and the sequence $\langle (\alpha_1^t, \alpha_2^t, \ldots, \alpha_n^t) \rangle$ converges to $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, it follows that, for some sufficiently large r, $t \geq r$ implies that $\alpha_1^t q_1{}' + \alpha_2^t q_2{}' + \cdots + \alpha_n^t q_n{}' > k' - \varepsilon$. Also, since the sequence $\langle \alpha_1^t \text{q}_1^t + \alpha_2^t q_2^t + \cdots + \alpha_n^t \text{q}_n^t \rangle$ converges to $\alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n = k$, it follows that, for some sufficiently large s, $t \geq s$ implies that $\alpha_1^t \text{q}_1^t + \alpha_2^t \text{q}_2^t + \cdots + \alpha_n^t \text{q}_n^t < k + \varepsilon$. Hence, for any $t \geq \max\{r, s\}$,

$$\begin{aligned} \alpha_1^t q_1' + \alpha_2^t q_2' + \cdots + \alpha_n^t q_n' &> k' - \varepsilon \\ &= k + \varepsilon > \alpha_1^t q_1^t + \alpha_2^t q_2^t + \cdots + \alpha_n^t q_n^t. \end{aligned}$$

This contradicts the fact that the family of parallel hyperplanes given by $\alpha_1^t x_1 + \alpha_2^t x_2 + \cdots + \alpha_n^t x_n = c$ makes first contact with the IPS at q^t. This contradiction completes the proof of the claim.

We return to the proof that $J : S \to \mathbf{P}(S)$ is upper semicontinuous. Suppose that $\langle \omega^t \rangle$ is a sequence of points in S that converges to ω, that $\langle p^t \rangle$ is a sequence of points in S that converges to p, and that $p^t \in J(\omega^t)$ for every t. We must show that $p \in J(\omega)$. We consider three cases:

Case 1: $\omega \in S^+$. Then some tail of the sequence $\langle \omega^t \rangle$ is in S^+ and we need only consider this tail. For notational convenience, we shall simply assume that the entire sequence $\langle \omega^t \rangle$ is in S^+. Since, for every $t =$

$1, 2, \ldots, p^t \in J(\omega^t) = H(\omega^t) = \{g(m(P)) : P \in (\omega^t)^*\}$, we know that each such p^t is in the range of the function g. Hence, $g^{-1}(p^t)$ exists and, since $K(\omega^t) = \{m(P) : P \in (\omega^t)^*\}$, we see that $g^{-1}(p^t) \in K(\omega^t)$. Consider the sequence $\langle g^{-1}(p^t)\rangle$, which is a sequence of points in OPB. Since the sequence $\langle p^t \rangle$ converges to p, and g^{-1} is certainly continuous, $\langle g^{-1}(p^t)\rangle$ converges to $g^{-1}(p)$. By the claim, $g^{-1}(p) \in K(\omega)$. This implies that, for some $P \in \omega^*$, $g^{-1}(p) = m(P)$ and, hence, $p = g(m(P))$. It follows that $p \in H(\omega) = J(\omega)$.

Case 2: $\omega \in S\backslash S^+$ and some tail of the sequence $\langle \omega^t \rangle$ is in S^+. As in Case 1, we shall simply assume that the entire sequence is in S^+. As we did previously, let $\delta_\omega = \{i \leq n : \omega_i > 0\}$ and let S_{δ_ω} be the face of S corresponding to δ_ω. Then $\omega \in S_{\delta_\omega}$ and $J(\omega) = I(\omega) = T_{\delta_\omega}$. In addition, let $\delta^c_\omega = \{i \leq n : \omega_i = 0\}$. For each $t = 1, 2, \ldots, p^t \in J(\omega^t) = H(\omega^t)$ and, hence, we can pick a partition $P^t = \langle P^t_1, P^t_2, \ldots, P^t_n\rangle$ of C such that $P^t \in (\omega^t)^*$ and $g(m(P^t)) = p^t$.

We claim that $\text{Lim}_{t\to\infty}(\sum_{i\in\delta^c_\omega} m_i(P^t_i)) \geq 1$. To see this, first note that, for each $i \in \delta^c_\omega$ and $j \in \delta_\omega$, the sequence of ith components of $\langle \omega^t \rangle$ converges to zero and the sequence of jth components of $\langle \omega^t \rangle$ converges to some positive number. Hence, if a is any bit of cake that corresponds to a point in $S\backslash S_{\delta_\omega}$, then for every sufficiently large t, every partition in $(\omega^t)^*$ gives a to some player named by δ^c_ω. Since all players named by δ^c_ω believe that almost every $a \in C$ corresponds to a point in $S\backslash S_{\delta_\omega}$, this implies that each such player believes that for almost every $a \in C$, a is given to some player named by δ^c_ω in every partition in every $(\omega^t)^*$, for sufficiently large t. In other words, as t goes to infinity, each player named by δ^c_ω believes that the measure of $\bigcup_{i\in\delta^c_\omega} P^t_i$ approaches one. Then, for any $\varepsilon > 0$, it is possible to choose s sufficiently large so that $t \geq s$ implies that, for every $j \in \delta^c_\omega$, $m_j(\bigcup_{i\in\delta^c_\omega} P^t_i) > 1 - \varepsilon$. Since, for each t, P^t is a Pareto maximal partition of C, it follows from Theorem 6.2 that $\langle P^t_i : i \in \delta^c_\omega\rangle$ is a Pareto maximal partition of $\bigcup_{i\in\delta^c_\omega} P^t_i$ among the players named by δ^c_ω. This implies that, for every $t \geq s$, $\sum_{i\in\delta^c_\omega} m_i(P^t_i) > 1 - \varepsilon$. (One way to see this is to imagine, for any $t \geq s$, the IPS associated with the cake $\bigcup_{i\in\delta^c_\omega} P^t_i$ and the players named by δ^c_ω. This is a slightly different IPS from what we have usually considered, since players may think that the cake, $\bigcup_{i\in\delta^c_\omega} P^t_i$, has measure less than one. However, each player does think that the cake has measure greater than $1 - \varepsilon$. By convexity, it is easy to see that any point on the outer Pareto boundary of this IPS has coordinate sum greater than $1 - \varepsilon$. The point $(m_i(P^t_i) : i \in \delta^c_\omega)$ is on this outer Pareto boundary.) Since $\varepsilon > 0$ was arbitrary, it follows that $\text{Lim}_{t\to\infty}(\sum_{i\in\delta^c_\omega} m_i(P^t_i)) \geq 1$.

Set $p = (p_1, p_2, \ldots, p_n)$ and, for each t, set $p^t = (p_1^t, p_2^t, \ldots, p_n^t)$. Continuing with Case 2, we see that

$$\begin{aligned}
\sum_{i\in\delta_\omega} p_i &= \mathrm{Lim}_{t\to\infty}\left(\sum_{i\in\delta_\omega} p_i^t\right) = \mathrm{Lim}_{t\to\infty}\left(\sum_{i\in\delta_\omega} g_i(m(P^t))\right)\\
&= \mathrm{Lim}_{t\to\infty}\left(\sum_{i\in\delta_\omega}\left(\frac{m_i\left(P_i^t\right)}{m_1\left(P_1^t\right)+m_2\left(P_2^t\right)+\cdots+m_n\left(P_n^t\right)}\right)\right)\\
&= \mathrm{Lim}_{t\to\infty}\left(\frac{\sum\limits_{i\in\delta_\omega} m_i\left(P_i^t\right)}{\left(\sum\limits_{i\in\delta_\omega} m_i\left(P_i^t\right)\right)+\left(\sum\limits_{i\in\delta_\omega^c} m_i\left(P_i^t\right)\right)}\right)\\
&\le \mathrm{Lim}_{t\to\infty}\left(\frac{\sum\limits_{i\in\delta_\omega} m_i\left(P_i^t\right)}{\left(\sum\limits_{i\in\delta_\omega} m_i\left(P_i^t\right)\right)+1}\right)\\
&= \mathrm{Lim}_{t\to\infty}\left(\frac{1}{1+\left(\frac{1}{\sum_{i\in\delta_\omega} m_i(P_i^t)}\right)}\right) \le \frac{1}{1+\left(\frac{1}{n-1}\right)} = \frac{n-1}{n}.
\end{aligned}$$

The first equality holds since $\mathrm{Lim}_{t\to\infty}(p^t) = p$, the second equality follows by our choice of the P^t, the third equality follows from the definition of m and of the g_i, and the fourth equality is obvious. The first inequality follows from our work in the preceding paragraph. The fifth equality is obvious. For the second inequality, we note that, since $\omega \in S\backslash S^+$, δ_ω^c is non-empty and hence δ_ω has at most $n-1$ elements. Since for each $i \in \delta_\omega$ and $t = 1, 2, \ldots, m_i(P_i^t) \le 1$, it follows that, for each such t, $\sum_{i\in\delta_\omega}(m_i(P_i^t)) \le n-1$ and, hence, $\mathrm{Lim}_{t\to\infty}(\sum_{i\in\delta_\omega}(m_i(P_i^t))) \le n-1$. This yields the second inequality. The last inequality is obvious. This computation tells us that $p = (p_1, p_2, \ldots, p_n) \in T_{\delta_\omega} = I(\omega) = J(\omega)$.

Case 3: $\omega \in S\backslash S^+$ and no tail of the sequence $\langle\omega^t\rangle$ is in S^+. Then an infinite subsequence of $\langle\omega^t\rangle$ is in $S\backslash S^+$, the boundary of S. Since there are only a finite number of subsets of $\{1, 2, \ldots, n\}$, there is a $\delta \subseteq \{1, 2, \ldots, n\}$ such that an infinite subsequence of $\langle\omega^t\rangle$ lies in the interior of the face of S corresponding to δ. Let this infinite subsequence be $\langle\omega^{k_t}\rangle$, let S_δ denote this face, let $\delta^c = \{1, 2, \ldots, n\}\backslash\delta$, let S_{δ^c} be the face of S corresponding to δ^c, and let $T_\delta = \{(p_1, p_2, \ldots, p_n) \in S : \sum_{i\in\delta} p_i \leq \frac{n-1}{n}\}$. Then, $\langle \omega^{k_t} \rangle$ converges to ω, $\langle p^{k_t} \rangle$ converges to p, and, for each k_t, $p^{k_t} \in J(\omega^{k_t}) = I(\omega^{k_t}) = T_\delta$.

As we have done previously, let $\delta_\omega = \{i \leq n : \omega_i > 0\}$ and let S_{δ_ω} denote the face of S corresponding to δ_ω. Since $\langle \omega^{k_t} \rangle$ converges to ω and each ω^{k_t} is an interior point of S_δ, it follows that ω is on, though not necessarily an interior point of, S_δ. This implies that $\delta_\omega \subseteq \delta$ and, hence, that $T_\delta \subseteq T_{\delta_\omega}$. Since $p^{k_t} \in T_\delta$ for every k_t, and T_δ is closed, it follows that $p \in T_\delta$. Then, since $T_\delta \subseteq T_{\delta_\omega}$, we know that $p \in T_{\delta_\omega} = I(\omega) = J(\omega)$.

We have shown that J is upper semicontinuous. This completes the proof of Lemma 12.34. □

Proof of Theorem 12.32: Lemma 12.34 allows us to apply Kakutani's fixed point theorem (Theorem 12.33) to obtain a fixed point of J. As we have previously discussed, for any $\omega \in S \backslash S^+$, $J(\omega) = I(\omega) = T_{\delta_\omega}$ is disjoint from S_{δ_ω}. Since $\omega \in S_{\delta_\omega}$, it follows that $\omega \notin J(\omega)$. Hence, no element of $S \backslash S^+$ is a fixed point of J and, thus, there exists an $\omega \in S^+$ such that $\omega \in J(\omega)$. Since $J(\omega) = H(\omega)$, it follows that $\omega \in H(\omega)$. By the definition of H, we know that for some partition $P = \langle P_1, P_2, \ldots, P_n \rangle \in \omega^*$, $g(m(P)) = \omega$. We claim that P is envy-free and Pareto maximal.

Since $P \in \omega^*$, it follows immediately from previous work (i.e., from part b of Theorem 10.23) that P is Pareto maximal. We must show that P is envy-free. Choose distinct $i, j = 1, 2, \ldots, n$. We must show that $m_i(P_i) \geq m_i(P_j)$.

Set $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. Then $g(m(P)) = g((m_1(P_1), m_2(P_2), \ldots, m_n(P_n)) = (\omega_1, \omega_2, \ldots, \omega_n)$ and, since g certainly preserves ratios of pairs of terms, it follows that, for all i and j, $\frac{m_j(P_j)}{m_i(P_i)} = \frac{\omega_j}{\omega_i}$.

Since P is w-associated with ω, it follows that $\frac{m_j(P_j)}{m_i(P_j)} \geq \frac{\omega_j}{\omega_i}$. Putting this together with the relationship given in the previous paragraph, we have $\frac{m_j(P_j)}{m_i(P_j)} \geq \frac{\omega_j}{\omega_i} = \frac{m_j(P_j)}{m_i(P_i)}$ and, hence, $m_i(P_i) \geq m_i(P_j)$. This establishes that P is envy-free and completes the proof the theorem. □

There are natural questions that we can ask regarding strengthenings of Theorem 12.32. We can ask for a partition that is Pareto maximal and strongly envy-free, or we can ask for a partition that is Pareto maximal and super envy-free. We first establish a result about the partition of Theorem 12.32 and the shape of the outer boundary of the IPS.

Lemma 12.35 *Let P be as in the proof of Theorem 12.32. If $m(P)$ is not on a line segment on the outer Pareto boundary of the IPS, then P is strongly envy-free.*

Proof: Let P and ω be as in the proof of Theorem 12.32. Then P is w-associated with ω.

Assume that P is not strongly envy-free. We know that P is envy-free and, hence, for some distinct i and j, $m_i(P_i) = m_i(P_j)$. Envy-freeness implies that $m_i(P_i) > 0$ and thus $m_i(P_j) > 0$. Recall that as part of our proof of Theorem 12.32 we established that $\frac{\omega_j}{\omega_i} = \frac{m_j(P_j)}{m_i(P_i)}$. Hence, since $m_i(P_i) = m_i(P_j)$, it follows that $\frac{m_j(P_j)}{m_i(P_j)} = \frac{\omega_j}{\omega_i}$.

Claim For every $A \subseteq P_j$ with $m_i(A) > 0$, $\frac{m_j(A)}{m_i(A)} = \frac{\omega_j}{\omega_i}$.

Proof of Claim: Since P is w-associated with ω, we know that for every $A \subseteq P_j$ with $m_i(A) > 0$, $\frac{m_j(A)}{m_i(A)} \geq \frac{\omega_j}{\omega_i}$. Suppose, by way of contradiction, that for some $A \subseteq P_j$ with $m_i(A) > 0$, $\frac{m_j(A)}{m_i(A)} > \frac{\omega_j}{\omega_i}$. Then $m_j(A) > (\frac{\omega_j}{\omega_i})m_i(A)$. Since $\frac{m_j(P_j)}{m_i(P_j)} = \frac{\omega_j}{\omega_i}$, we know that $m_j(P_j) = (\frac{\omega_j}{\omega_i})m_i(P_j)$. Hence, $m_j(P_j \backslash A) = m_j(P_j) - m_j(A) < (\frac{\omega_j}{\omega_i})m_i(P_j) - (\frac{\omega_j}{\omega_i})m_i(A) = (\frac{\omega_j}{\omega_i})m_i(P_j \backslash A)$. This implies that $\frac{m_j(P_j \backslash A)}{m_i(P_j \backslash A)} < \frac{\omega_j}{\omega_i}$. This contradicts the fact that $P_j \backslash A \subseteq P_j$ and P is w-associated with ω and, thus, establishes the claim.

The claim implies that, for almost every $a \in P_j$, $\frac{f_j(a)}{f_i(a)} = \frac{\omega_j}{\omega_i}$. This, and the fact that P is w-associated with ω, implies that almost every $a \in C$ that is given to Player j corresponds (via f) to a point along the i, j boundary associated with ω (see Definition 12.10). Then, since the envy-freeness of partition P implies that P_j has positive measure, it follows that the RNS is i, j-concentrated with respect to ω. Theorem 12.12 implies that there is a partition Q that is not p-equivalent to P and is w-associated with ω. Then $m(P)$ and $m(Q)$ are distinct points on the outer Pareto boundary of the IPS and, by Theorem 10.6 and our geometric perspective of the maximization of convex combinations of measures, it follows that these points are each points of first contact with the IPS of the family of parallel hyperplanes that has coefficients given by RD(ω). This implies that all points on the line segment connecting these two points are on the outer Pareto boundary of the IPS. Therefore, $m(P)$ is on a line segment on the outer Pareto boundary of the IPS. □

Although we do not have a natural characterization for the existence of a partition that is both strongly envy-free and Pareto maximal, Lemma 12.35 leads us to an existence result.

Theorem 12.36 *If no two players are in relative agreement on any set that has positive measure to each of these two players (or, equivalently, if the RNS is not concentrated), then there exists a partition that is strongly envy-free and Pareto maximal.*

Proof: Assume that no two players are in relative agreement on any set that has positive measure to each of these two players and, hence, that the RNS is not concentrated. Let P be as in the proof of Theorem 12.32. Then P is envy-free and Pareto maximal. We claim that P is strongly envy-free.

Since the RNS is not concentrated, Theorem 12.14 tells us that there are no line segments on the outer Pareto boundary of the IPS. Then certainly $m(P)$ is not on a line segment on the outer Pareto boundary of the IPS. Therefore, by Lemma 12.35, P is strongly envy-free. □

In Chapter 14 (as part of the proof of situation c of Theorem 14.14), we shall see an example where the premises of the theorem are satisfied and, hence, where there exists a partition that is strongly envy-free and Pareto maximal.

It is not hard to see that Theorem 12.36 is far from a characterization. The assumption that no two players are in relative agreement on any set that has positive measure to each of these two players is considerably stronger than necessary. However, we do not have a simple and natural weakening of this assumption that allows us to establish the existence of a partition that is strongly envy-free and Pareto maximal. We also do not have any existence result for super envy-freeness and Pareto maximality together.

In Chapter 14, we shall study a strengthening of Pareto maximality called strong Pareto maximality and shall consider a strengthening of Theorem 12.32 using this notion.

We conclude by stating chores versions of the two main results of this section. The proofs are completely analogous and we omit them.

Theorem 12.37 *There is a partition that is c-envy-free and Pareto minimal.*

Theorem 12.38 *If no two players are in relative agreement on any set that has positive measure to each of these two players (or, equivalently, if the RNS is not concentrated), then there exists a partition that is strongly c-envy-free and Pareto minimal.*

13

Other Issues Involving Weller's Construction, Partition Ratios, and Pareto Optimality

In this chapter, we explore an assortment of issues that did not fit naturally into previous chapters. In Sections 13A, 13B, 13C, and 13D, we assume that the measures are absolutely continuous with respect to each other. In Section 13E, we reconsider the results of these sections without this assumption.

13A. The Relationship between Partition Ratios and *w*-Association

Suppose that a partition P of the cake C is Pareto maximal. For simplicity, we also assume that $P \in \text{Part}^+$. (We recall that Part^+ denotes the set of all partitions of C that give each player a piece of cake of positive measure.) For distinct $i, j = 1, 2, \ldots, n$, let pr_{ij} be the ij partition ratio. (These are given by Definition 8.6. By Theorem 8.9, for any associated cyclic sequence φ, $\text{CP}(\varphi)$, the cyclic product of φ, is at most one.) By Theorem 10.9, P is w-associated with some $\omega \in S^+$. In this section, we investigate the relationship between ω and the pr_{ij}.

Let us first consider the two-player context with Player 1 and Player 2 and associated measures m_1 and m_2, respectively. The relevant RNS is the one-simplex, which is the line segment from Player 1's vertex, $(1, 0)$, to Player 2's vertex, $(0, 1)$. Re-examining Examples 10.2 and 10.3 will provide us with useful perspective. We illustrated these two examples in Figures 10.1. and 10.2. For convenience, we repeat these here as Figures 13.1a and 13.1b.

In Example 10.2, with the RNS as in Figure 13.1a, we fixed a number κ with $0 < \kappa < 1$ and we considered the partition $P = \langle P_1, P_2 \rangle$ of the cake in which Player 1 receives all bits of cake that are associated with points of the RNS between $(1, 0)$ and $(\kappa, 1 - \kappa)$ and Player 2 receives all bits of cake that are associated with points of the RNS between $(\kappa, 1 - \kappa)$ and $(0, 1)$. We assumed,

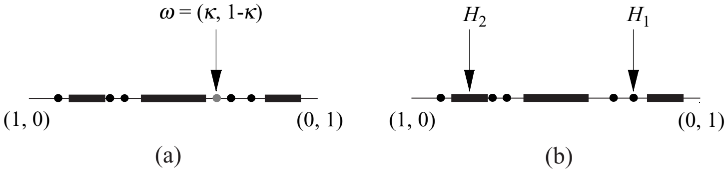


Figure 13.1

for simplicity, that the RNS did not contain the point $(\kappa, 1-\kappa)$. We showed that P is Pareto maximal. After the relevant work later in Chapter 10, we can see that P is w-associated with the point $(\kappa, 1-\kappa)$.

In Example 10.3, with the RNS as in Figure 13.1b, we let A_1 be the collection of all bits of cake associated with the point H_1, we let A_2 be the collection of all bits of cake associated with points in H_2, and we let $P = \langle P_1, P_2 \rangle$ be any partition of C such that $A_1 \subseteq P_1$ and $A_2 \subseteq P_2$. We showed that P is not Pareto maximal.

We went on to discuss the contrast between these two examples. We saw that in Example 10.2 all points of the RNS associated with P_1 lie to the left of all points of the RNS associated with P_2. And, there is a gap between the rightmost extent of the cake associated with P_1 and the leftmost extent of the cake associated with P_2. If ω is any point in this gap, then P is w-associated with ω. This is in contrast with Example 10.3, in which there is a subset of the RNS associated with a positive-measure subset of P_1 that lies completely to the right of some subset of the RNS associated with a positive-measure subset of P_2. Hence, there is no place to put a point ω with which P is w-associated, and so P is not Pareto maximal.

These examples illustrate the fact that a partition $P = \langle P_1, P_2 \rangle$ is Pareto maximal if and only if, in the RNS, the rightmost extent of the cake associated with P_1 is at or to the left of the leftmost extent of the cake associated with P_2. We next explore the connection between these points (of leftmost extent and rightmost extent) and partition ratios.

We recall that $\mathrm{pr}_{12} = \sup\{\frac{m_2(A)}{m_1(A)} : A \subseteq P_1 \text{ and } A \text{ has positive measure}\}$. Fix any $(x, y) \in S^+$ (i.e., (x, y) is strictly between $(1, 0)$ and $(0, 1)$) and consider the following two cases:

Case 1: $\frac{y}{x} < \mathrm{pr}_{12}$. Then, for some $A \subseteq P_1$ of positive measure, $\frac{m_2(A)}{m_1(A)} > \frac{y}{x}$. Let $B = \{a \in A : \frac{f_2(a)}{f_1(a)} > \frac{y}{x}\}$. Then $B \subseteq P_1$, B has positive measure and, for every $a \in B$, $f(a)$ is to the right of (x, y). Hence, in this case, there is a subset of P_1 of positive measure that is associated with points to the right of (x, y).

Case 2: $\frac{y}{x} \geq \text{pr}_{12}$. Then there is no $A \subseteq P_1$ of positive measure such that $\frac{m_2(A)}{m_1(A)} > \frac{y}{x}$. It follows that there is no $B \subseteq P_1$ of positive measure such that, for every $a \in B$, $f(a)$ is to the right of (x, y). Hence, in this case, there is no subset of P_1 of positive measure that is associated with points to the right of (x, y).

These two cases imply the following:

$\text{pr}_{12} = \sup\{\frac{y}{x}$: the subset of P_1 associated with points of the RNS to the right of (x, y) has positive measure$\}$

Similarly,

$\text{pr}_{21} = \sup\{\frac{x}{y}$: the subset of P_2 associated with points of the RNS to the left of (x, y) has positive measure$\}$.

Less formally (and somewhat imprecisely), the idea is this. Consider the set of points of the RNS associated with P_1. Then pr_{12} is the ratio of the second coordinate to the first coordinate of the right limit of this set (possibly excluding a set of points associated with a piece of cake of measure zero). Similarly, if we consider the points of the RNS associated with P_2, pr_{21} is the ratio of the first coordinate to the second coordinate of the left limit of this set (again, possibly excluding a set of points associated with a piece of cake of measure zero). This is illustrated in Figure 13.2, using the same RNS as in Figure 13.1 and the same partition $P = \langle P_1, P_2 \rangle$ discussed previously using Figure 13.1a. (P_1 is the set of all bits of cake associated with points of the RNS to the left of $(\kappa, 1-\kappa)$ and P_2 is the set of all bits of cake associated with points of the RNS to the right of $(\kappa, 1-\kappa)$.) In the figure, $(\omega_{R1}, \omega_{R2})$ is the right limit of the set of points of the RNS associated with P_1 and, hence, $\text{pr}_{12} = \frac{\omega_{R2}}{\omega_{R1}}$; $(\omega_{L1}, \omega_{L2})$ is the left limit of the set of points of the RNS associated with P_2 and, hence, $\text{pr}_{21} = \frac{\omega_{L1}}{\omega_{L2}}$.

We wish to connect this perspective on partition ratios with Theorem 8.9, which tells us that P is Pareto maximal if and only if $\text{pr}_{12}\,\text{pr}_{21} \leq 1$. Suppose that $\omega_R = (\omega_{R1}, \omega_{R2}) \in S^+$, $\omega_L = (\omega_{L1}, \omega_{L2}) \in S^+$, $\text{pr}_{12} = \frac{\omega_{R2}}{\omega_{R1}}$, and $\text{pr}_{21} = \frac{\omega_{L1}}{\omega_{L2}}$. (So ω_R is the right limit of the points in the RNS associated with P_1,

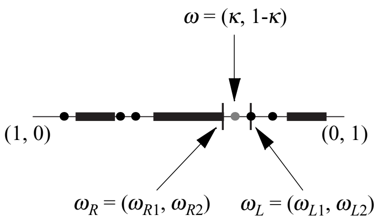


Figure 13.2

and ω_L is the left limit of the points in the RNS associated with P_2, as described earlier.) Then

P is Pareto maximal
if and only if
$\text{pr}_{12}\text{pr}_{21} \leq 1$
if and only if
$\left(\frac{\omega_{R2}}{\omega_{R1}}\right)\left(\frac{\omega_{L1}}{\omega_{L2}}\right) \leq 1$
if and only if
$\frac{\omega_{R2}}{\omega_{R1}} \leq \frac{\omega_{L2}}{\omega_{L1}}$
if and only if (since $\omega_{R1} + \omega_{R2} = 1$ and $\omega_{L1} + \omega_{L2} = 1$)
$\omega_{R2} \leq \omega_{L2}$ and $\omega_{R1} \geq \omega_{L1}$
if and only if
ω_R is to the left of or is equal to ω_L.

Let us refer to ω_R and ω_L as "the points of S^+ that correspond to pr_{12} and pr_{21}," respectively. Then, what we have discovered is that P is Pareto maximal if and only if the point corresponding to pr_{12} is to the left of or is equal to the point corresponding to pr_{21}.

As discussed before, there will be an $\omega \in S^+$ with which P is w-associated if and only if there is a gap (into which ω can be placed) that is to the right of or equal to the right limit of the points in the RNS associated with P_1 and is to the left of or equal to the left limit of the points in the RNS associated with P_2. (This "gap" can be a single point. In other words, if this right limit point and left limit point are equal, then we may let ω be this point.) Thus, P is Pareto maximal if and only if $(\omega_{R1}, \omega_{R2})$ is to the left of or is equal to $(\omega_{L1}, \omega_{L2})$, where, as before, we let $(\omega_{R1}, \omega_{R2})$ and $(\omega_{L1}, \omega_{L2})$ be the points of S^+ that correspond to pr_{12} and pr_{21}, respectively. In this case, P is w-associated with ω if and only if ω is between (though not necessarily strictly between) $(\omega_{R1}, \omega_{R2})$ and $(\omega_{L1}, \omega_{L2})$. This tells us that, for any $\omega = (\omega_1, \omega_2)$,

P is w-associated with ω
if and only if
$\frac{\omega_{R2}}{\omega_{R1}} \leq \frac{\omega_2}{\omega_1}$ and $\frac{\omega_{L1}}{\omega_{L2}} \leq \frac{\omega_1}{\omega_2}$
if and only if
$\text{pr}_{12} \leq \frac{\omega_2}{\omega_1}$ and $\text{pr}_{21} \leq \frac{\omega_1}{\omega_2}$.

This is the desired relationship between ω and the pr_{ij}.

Next, we consider the n-player context. We find that the preceding result generalizes in the obvious way, with a small adjustment arising from the fact that we no longer insist on having $P \in \text{Part}^+$. We recall that, for any partition

$P = \langle P_1, P_2, \ldots, P_n \rangle$ and distinct $i, j = 1, 2, \ldots, n$, the corresponding partition ratio p_{ij} is given by $\text{pr}_{ij} = \sup\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i \text{ and } A \text{ has positive measure}\}$ and that pr_{ij} is undefined if and only if P_i has measure zero.

Theorem 13.1 *Suppose that $P \in \text{Part}$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. Then P is w-associated with ω if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $\text{pr}_{ij} \leq \frac{\omega_j}{\omega_i}$ or pr_{ij} is undefined.*

Proof: Fix $P = \langle P_1, P_2, \ldots, P_n \rangle \in \text{Part}$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. Then

P is w-associated with ω
if and only if
for all distinct $i, j = 1, 2, \ldots, n$, and almost every $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$
if and only if
for all distinct $i, j = 1, 2, \ldots, n$, and almost every $a \in P_i$, $\frac{f_j(a)}{f_i(a)} \leq \frac{\omega_j}{\omega_i}$
if and only if
for all distinct $i, j = 1, 2, \ldots, n$, and any $A \subseteq P_i$ of positive measure, $\frac{m_j(A)}{m_i(A)} \leq \frac{\omega_j}{\omega_i}$
if and only if
for all distinct $i, j = 1, 2, \ldots, n$, $\sup\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i \text{ and } A \text{ has positive measure}\} \leq \frac{\omega_j}{\omega_i}$, or P_i has measure zero
if and only if
for all distinct $i, j = 1, 2, \ldots, n$, $\text{pr}_{ij} \leq \frac{\omega_j}{\omega_i}$ or pr_{ij} is undefined. □

By Theorem 10.9, the conditions of the theorem hold if and only if P is Pareto maximal.

Suppose that $P \in \text{Part}^+$. Theorem 13.1 provides an easy proof of the forward direction of Theorem 8.9 ("If partition P is Pareto maximal, then for any $\varphi \in \text{CS}$, $\text{CP}(\varphi) \leq 1$.") from the forward direction of Theorem 10.9 ("If P is Pareto maximal then P is w-associated with ω for some $\omega \in S^+$."). To see this, assume the truth of Theorem 10.9, suppose that P is Pareto maximal, and pick any cyclic sequence φ. We must show that $\text{CP}(\varphi) \leq 1$. Suppose $\varphi = \langle \text{pr}_{i_t i_1}, \text{pr}_{i_1 i_2}, \ldots, \text{pr}_{i_{t-2} i_{t-1}}, \text{pr}_{i_{t-1} i_t} \rangle$.

Since P is Pareto maximal, Theorem 10.9 implies that P is w-associated with some $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. By Theorem 13.1, for each $j = 1, 2, \ldots, t$, $\text{pr}_{i_j i_{j+1}} \leq \frac{\omega_{i_{j+1}}}{\omega_{i_j}}$, where we set $i_{t+1} = i_1$. Then we have

$$\begin{aligned}\text{CP}(\varphi) &= \text{pr}_{i_t i_1} \text{pr}_{i_1 i_2} \ldots \text{pr}_{i_{t-2} i_{t-1}} \text{pr}_{i_{t-1} i_t} \\ &\leq \left(\frac{\omega_{i_1}}{\omega_{i_t}}\right)\left(\frac{\omega_{i_2}}{\omega_{i_1}}\right) \ldots \left(\frac{\omega_{i_{t-1}}}{\omega_{i_{t-2}}}\right)\left(\frac{\omega_{i_t}}{\omega_{i_{t-1}}}\right) = 1\end{aligned}$$

as desired.

These ideas can also be used to establish the following result:

> For any partition P, $m(P)$ is a jagged point on the outer Pareto boundary of the IPS if and only if $\mathrm{CP}(\varphi) < 1$ for every $\varphi \in \mathrm{CS}$.

(For the definition of jagged point, see the discussion following the statement of Theorem 12.18. For the definition of CP and CS, see Definition 8.7.) We omit the proof. This answers a natural question that arises from Theorem 8.9. This result told us that a partition is Pareto maximal if and only if all associated cyclic products are less than or equal to one. We now see that if we insist instead that all cyclic products be less than one, then this characterizes partitions P such that $m(P)$ is a jagged point on the outer boundary of the IPS. As we saw in Chapter 12, this is also equivalent to the RNS being γ-separable with respect to P, where $\gamma = \{\{1\}, \{2\}, \ldots, \{n\}\}$.

Next, we consider an illustration of Theorem 13.1 in the three-player context.

Example 13.2 Suppose that $P = \langle P_1, P_2, P_3 \rangle$ is a partition that is w-associated with $\omega = (\omega_1, \omega_2, \omega_3)$ and consider Figure 13.3. In the figure, we have shown $\omega = (\omega_1, \omega_2, \omega_3)$ and the set T_1, which is the set of all points in the RNS that are associated with P_1. Then pr_{12}, which is the supremum of the ratio of Player 2's evaluation to Player 1's evaluation of positive-measure subsets of P_1, is associated with the left dashed line segment that contains Player 3's vertex. In particular, the value of pr_{12} is given by the ratio of the second to the first coordinate of points along this line segment. (Since this line segment goes through Player 3's vertex, this ratio is the same for every point on the

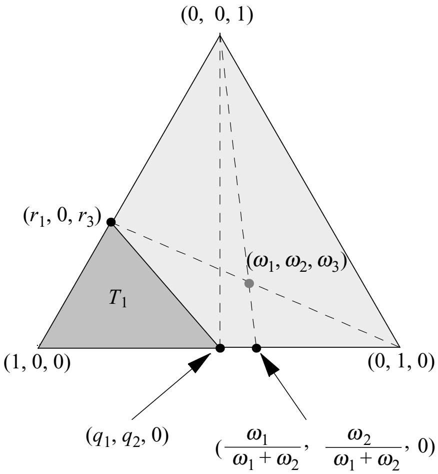


Figure 13.3

line segment.) For simplicity, we focus on the ratio of the second to the first coordinate of the point of intersection of this line segment with the line segment containing the vertices of Player 1 and Player 2. Let this point be $(q_1, q_2, 0)$, as indicated in the figure. Then $\text{pr}_{12} = \frac{q_2}{q_1}$.

Next, consider the point of intersection of the right dashed line segment that contains Player 3's vertex. This line segment contains the point $\omega = (\omega_1, \omega_2, \omega_3)$, and it is not hard to see that $(\frac{\omega_1}{\omega_1+\omega_2}, \frac{\omega_2}{\omega_1+\omega_2}, 0)$ is the point of intersection of this line segment with the line segment containing the vertices of Player 1 and Player 2. It is clear that $\text{pr}_{12} = \frac{q_2}{q_1} < \frac{\omega_2/(\omega_1+\omega_2)}{\omega_1/(\omega_1+\omega_2)} = \frac{\omega_2}{\omega_1}$, which is consistent with the theorem.

If we apply the same sort of analysis to pr_{13}, we find that the situation is slightly different. The partition ratio pr_{13}, which is the supremum of the ratio of Player 3's evaluation to Player 1's evaluation of positive-measure subsets of P_1, is associated with the dashed line segment that contains Player 2's vertex. Let $(r_1, 0, r_3)$ be the intersection of this line segment with the line segment containing the vertices of Player 1 and Player 3. In contrast with the situation considered in the previous paragraph, this dashed line segment is the same as the line segment determined by ω and Player 2's vertex. Hence, $\frac{r_3}{r_1} = \frac{\omega_3}{\omega_1}$, and so $\text{pr}_{13} = \frac{r_3}{r_1} = \frac{\omega_3}{\omega_1}$, which is consistent with the theorem.

The difference between the situations considered in the previous two paragraphs can be described as follows. The point ω determines three regions of the RNS, and this tells us how to partition the cake, using the notion of w-association. In the present example, Player 1's piece of cake includes pieces of cake of positive measure that are associated with points in the RNS arbitrarily close to Player 3's region of the RNS. In contrast, points in the RNS that are associated with Player 1's piece of cake are bounded away from Player 2's region of the RNS.

The chores version of Theorem 13.1 is the following. The proof is analogous and we omit it.

Theorem 13.3 *Suppose that* $P \in \text{Part}$ *and* $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. *Then* P *is chores* w*-associated with* ω *if and only if, for all distinct* $i, j = 1, 2, \ldots, n$, $\text{qr}_{ij} \geq \frac{\omega_j}{\omega_i}$ *or* qr_{ij} *is undefined.*

13B. Trades and Efficiency

In this section, we no longer focus on Pareto optimality. We consider the following problem. Let $P = \langle P_1, P_2, \ldots, P_n \rangle$ be some partition of C and fix distinct $i, j = 1, 2, \ldots, n$. Suppose that we are required to transfer some cake from

Player i to Player j. How efficient a transfer is possible and how can this be accomplished? (Throughout this section, we will be using the term "efficiency" in a different sense than in outer sections of this book, where "efficiency" refers to Pareto optimality.)

We need to elaborate on a few aspects of this question. First, what is a "transfer?" In other places, we have used "transfer" to mean what, in this section, we will call a *direct transfer*, i.e., a single shifting of some cake from one player to another. On the other hand, by a *generalized transfer* of cake from some Player i to some Player j, we mean a process in which Player i gives up some cake, Player j gets some additional cake, and the other players are indifferent. Of course, the obvious way to accomplish such a generalized transfer of cake is by means of a direct transfer. However, this is not the only possibility. For example, it might be better (in a sense to be made precise) to take a piece of cake from Player i, give it to some third player, Player k, and then take a piece of cake from Player k (a piece which Player k had at the start, i.e., one that is disjoint from the piece that Player k received from Player i) and give it to Player j. To satisfy our notion of generalized transfer, the piece of cake given to Player k and the piece of cake taken from Player k must be the same size, according to Player k. In this case, we shall refer to Player k as an *intermediate player*. It is easy to see that a generalized transfer can involve more than one intermediate player. It will be convenient to introduce notation for such transfers.

Definition 13.4 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$, fix some $t = 0, 1, \ldots,$ fix distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$, and, for notational convenience, set $k_0 = i$. For each $s = 0, 1, \ldots, t$, suppose $Q_{k_s} \subseteq P_{k_s}$ is such that $m_{k_s}(Q_{k_{s-1}}) = m_{k_s}(Q_{k_s})$ for every $s = 1, 2, \ldots, t$. Then, $Tr(\langle i, k_1, k_2, \ldots, k_t, j \rangle | \langle Q_{k_0}, Q_{k_1}, \ldots, Q_{k_t} \rangle)$ denotes the *generalized transfer* in which Player i gives Q_{k_0} to Player k_1, Player k_1 gives Q_{k_1} to Player $k_2, \ldots,$ and Player k_t gives Q_{k_t} to Player j. A *positive generalized transfer* is one in which all involved transfers are of positive measure.

A direct transfer is a special case of a generalized transfer and is given by an expression of this form with $t = 0$. So, for example, $\mathrm{Tr}(\langle i, j \rangle, \langle Q \rangle)$ is the direct transfer in which Q is transferred from Player i to Player j.

Notice that, by absolute continuity, we could have defined a positive generalized transfer to be one in which *at least* one involved transfer is of positive measure.

Next we consider the efficiency of a generalized transfer. Intuitively, a generalized transfer of cake from Player i to Player j has high efficiency if Player j places high value on the piece received compared to the value Player i places on the piece given up. Conversely, a generalized transfer has low efficiency if

Player j places low value on the piece received compared to the value Player i places on the piece given up. Definition 13.5 makes this notion precise.

Definition 13.5 The *efficiency* of a positive generalized transfer from Player i to Player j is given by $\frac{\Delta_j}{\Delta_i}$, where Δ_i is the (positive) loss to Player i (according to m_i) and Δ_j is the gain to Player j (according to m_j).

Then the efficiency of the positive generalized transfer $\text{Tr}(\langle i, k_1, k_2, \ldots, k_t, j\rangle | \langle Q_{k_0}, Q_{k_1}, \ldots, Q_{k_t}\rangle)$ is given by $\frac{\Delta_j}{\Delta_i} = \frac{m_j(Q_{k_t})}{m_i(Q_{k_0})}$. We set no requirements on the size of Δ_i, other than that it be positive. We shall consider such requirements later in this section. Absolute continuity guarantees that the efficiency of any positive generalized transfer is a positive number.

For distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$, we shall be interested in the supremum of the efficiencies, $\frac{\Delta_j}{\Delta_i}$, taken over all possible positive generalized transfers from Player i to Player j, using intermediate Players $k_1, k_2, \ldots, k_t$, in that order.

Definition 13.6 Fix distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$ and suppose that $P = \langle P_1, P_2, \ldots, P_n\rangle$ is a partition. We set

$\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \sup\{\frac{\Delta_j}{\Delta_i} : \frac{\Delta_j}{\Delta_i}$ is the efficiency of a positive generalized transfer from Player i to Player j, using intermediate players $k_1, k_2, \ldots, k_t$, in that order$\}$.

Or, equivalently,

$\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \sup\{\frac{m_j(Q_{k_t})}{m_i(Q_{k_0})} : \text{Tr}(\langle i, k_1, k_2, \ldots, k_t, j\rangle | \langle Q_{k_0}, Q_{k_1}, \ldots, Q_{k_t}\rangle)$ is a positive generalized transfer from Player i to Player $j\}$ (where we continue to set $k_0 = i$).

Note that $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) > 0$ and $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$ can be infinite, in which case we write "$\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \infty$." We also observe that $\text{Ef}(\langle i, j\rangle)$ is the supremum of the efficiencies of all direct transfers from Player i to Player j.

We wish to relate the efficiencies of positive generalized transfers to partition ratios. First, we observe that for direct transfers the relationship is almost obvious.

Lemma 13.7 *For distinct* $i, j = 1, 2, \ldots, n$, $\text{Ef}(\langle i, j\rangle) = \text{pr}_{ij}$.

The proof of the lemma follows trivially from the definitions. Note that "$\infty = \infty$" is one of the possibilities for the lemma.

Next we consider generalized transfers that may involve intermediate players.

Theorem 13.8 *For distinct* $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$, $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj}$.

We recall that each partition ratio is greater than zero and can be infinite, and that the product of a positive number and infinity is infinity.

Proof of Theorem 13.8: For notational convenience throughout the proof, we continue to set $k_0 = i$.

Fix distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$ and suppose first, by way of contradiction, that $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) < \text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj}$. By the definition of the partition ratios, it follows that, for each $s = 0, 1, \ldots, t$, there exists $Q_{k_s} \subseteq P_{k_s}$ such that

$$\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) < \frac{m_{k_1}(Q_{k_0})}{m_i(Q_{k_0})}\frac{m_{k_2}(Q_{k_1})}{m_{k_1}(Q_{k_1})}\cdots\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_j(Q_{k_t})}{m_{k_t}(Q_{k_t})}.$$

We work with the right-hand side. By rearranging terms, we obtain

$$\frac{1}{m_i(Q_{k_0})}\frac{m_{k_1}(Q_{k_0})}{m_{k_1}(Q_{k_1})}\frac{m_{k_2}(Q_{k_1})}{m_{k_2}(Q_{k_2})}\cdots\frac{m_{k_{t-1}}(Q_{k_{t-2}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_t}(Q_{k_t})}m_j(Q_{k_t}).$$

Claim For each $s = 0, 1, \ldots, t$, there exists $R_{k_s} \subseteq Q_{k_s}$ such that

$$\begin{aligned}&\frac{1}{m_i(Q_{k_0})}\frac{m_{k_1}(Q_{k_0})}{m_{k_1}(Q_{k_1})}\frac{m_{k_2}(Q_{k_1})}{m_{k_2}(Q_{k_2})}\cdots\frac{m_{k_{t-1}}(Q_{k_{t-2}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_t}(Q_{k_t})}m_j(Q_{k_t})\\ &= \frac{1}{m_i(R_{k_0})}\frac{m_{k_1}(R_{k_0})}{m_{k_1}(R_{k_1})}\frac{m_{k_2}(R_{k_1})}{m_{k_2}(R_{k_2})}\cdots\frac{m_{k_{t-1}}(R_{k_{t-2}})}{m_{k_{t-1}}(R_{k_{t-1}})}\frac{m_{k_t}(R_{k_{t-1}})}{m_{k_t}(R_{k_t})}m_j(R_{k_t})\end{aligned}$$

and each term in the expression on the right-hand side of the equality is equal to one, with the possible exception of the first and last terms.

The proof of the claim is a trivial variation of the proof of Lemma 8.3, and we omit it.

Let $R_{k_0}, R_{k_1}, \ldots, R_{k_t}$ be as in the claim. Then

$$\begin{aligned}\frac{m_j(R_{k_t})}{m_i(R_{k_0})} &= \frac{1}{m_i(R_{k_0})}\frac{m_{k_1}(R_{k_0})}{m_{k_1}(R_{k_1})}\frac{m_{k_2}(R_{k_1})}{m_{k_2}(R_{k_2})}\cdots\frac{m_{k_{t-1}}(R_{k_{t-2}})}{m_{k_{t-1}}(R_{k_{t-1}})}\frac{m_{k_t}(R_{k_{t-1}})}{m_{k_t}(R_{k_t})}m_j(R_{k_t})\\ &= \frac{1}{m_i(Q_{k_0})}\frac{m_{k_1}(Q_{k_0})}{m_{k_1}(Q_{k_1})}\frac{m_{k_2}(Q_{k_1})}{m_{k_2}(Q_{k_2})}\cdots\frac{m_{k_{t-1}}(Q_{k_{t-2}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_t}(Q_{k_t})}m_j(Q_{k_t})\\ &= \frac{m_{k_1}(Q_{k_0})}{m_i(Q_{k_0})}\frac{m_{k_2}(Q_{k_1})}{m_{k_1}(Q_{k_1})}\cdots\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_j(Q_{k_t})}{m_{k_t}(Q_{k_t})}.\end{aligned}$$

Since

$$\frac{m_{k_1}(Q_{k_0})}{m_i(Q_{k_0})}\frac{m_{k_2}(Q_{k_1})}{m_{k_1}(Q_{k_1})}\cdots\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_j(Q_{k_t})}{m_{k_t}(Q_{k_t})} > \text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$$

it follows that

$$\frac{m_j(R_{k_t})}{m_i(R_{k_0})} > \text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle).$$

But $\text{Tr}(\langle i, k_1, k_2, \ldots, k_t, j\rangle | \langle R_{k_0}, R_{k_1}, \ldots, R_{k_t}\rangle)$ is a positive generalized transfer from Player i to Player j having efficiency $\frac{m_j(R_{k_t})}{m_i(R_{k_0})}$. This is a contradiction, since $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$ is the supremum of the efficiencies of such positive generalized transfers and, therefore, must be greater than or equal to each.

Suppose next, by way of contradiction, that $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) > \text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj}$. It follows from the definition of $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$ that there is a positive generalized transfer $\text{Tr}(\langle i, k_1, k_2, \ldots, k_t, j\rangle | \langle Q_{k_0}, Q_{k_1}, \ldots, Q_{k_t}\rangle)$ from Player i to Player j having efficiency $\frac{m_j(Q_{k_t})}{m_i(Q_{k_0})} > \text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj}$. The definition of generalized transfer implies that, for every $s = 1, 2, \ldots, t$, $m_{k_s}(Q_{k_{s-1}}) = m_{k_s}(Q_{k_s})$. It follows that

$$\begin{aligned}
\text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj} &< \frac{m_j(Q_{k_t})}{m_i(Q_{k_0})} \\
&= \frac{m_j(Q_{k_t})}{m_i(Q_{k_0})}\frac{m_{k_1}(Q_{k_0})}{m_{k_1}(Q_{k_1})}\frac{m_{k_2}(Q_{k_1})}{m_{k_2}(Q_{k_2})} \\
&\quad \cdots\frac{m_{k_{t-1}}(Q_{k_{t-2}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_t}(Q_{k_t})} \\
&= \frac{m_{k_1}(Q_{k_0})}{m_i(Q_{k_0})}\frac{m_{k_2}(Q_{k_1})}{m_{k_1}(Q_{k_1})}\cdots\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_j(Q_{k_t})}{m_{k_t}(Q_{k_t})} \\
&\leq \text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj}
\end{aligned}$$

where the last inequality follows from the definition of the partition ratios. This is a contradiction. Hence, we have shown that $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj}$. □

The theorem tells us that for any two players i and j, and any choice of distinct intermediate players $k_1, k_2, \ldots, k_t$ (in that order),

a. if $\text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj} < \infty$ then, for any $\varepsilon > 0$, there is a positive generalized transfer from Player i to Player j using the given intermediate players that has efficiency greater than $\text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj} - \varepsilon$ and

b. if $\text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj} = \infty$ then, for any number λ, there is a positive generalized transfer from Player i to Player j using the given intermediate players that has efficiency greater than λ.

Concerning statement a, it can be shown (using ideas similar to those used in the proof of the theorem) that there exists a generalized transfer with efficiency equal to $\text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj}$ if and only if all of the suprema of the relevant partition ratios are achieved.

Theorem 13.8 provides us with a tool for determining the most efficient way to transfer cake from Player i to Player j. We consider the list of all expressions of the form $\text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj}$. This is a finite list since, by assumption, repeat intermediate players are not allowed. Each such expression is equal to a number. We simply pick the largest of these numbers. The corresponding expression tells us the precise path via intermediate players that will provide the most efficient generalized transfer. The efficiency of this generalized transfer can be made arbitrarily close to the given number, and can be made equal to the given number if and only if all suprema of the involved partition ratios are achieved.

Theorem 13.8 tells us that $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$ is like a cyclic product, except that it is equal to a non-cyclic product of partition ratios rather than a cyclic product of partition ratios.

What would it mean to assert that for some partition P all positive generalized transfers have efficiency less than or equal to one? This is equivalent to the assertion that every product of partition ratios of the type given in Theorem 13.8 is less than or equal to one. A single partition ratio is such a product. (In particular, it will appear in the equality of this theorem if the generalized transfer is a direct transfer. This is as in Lemma 13.7.) Thus, this assertion is equivalent to the assertion that each partition ratio is less than or equal to one. It turns out that this is equivalent to the notion of maximization of total utility, which we discussed in Chapter 7.

Definition 13.9 Let $P = \langle P_1, P_2, \ldots, P_n\rangle$ be a partition.

a. The *total utility* of P is given by $m_1(P_1) + m_2(P_2) + \cdots + m_n(P_n)$.
b. *P maximizes total utility* if the total utility of P is at least as large as the total utility of any other partition.

It is easy to see that a generalized transfer increases total utility if its efficiency is greater that one, leaves total utility unchanged if its efficiency is equal to one, and decreases total utility if its efficiency is less than one. Theorem 13.11 will establish the connection between maximization of total utility and partition ratios. First, we establish some preliminary facts.

Clearly, a partition maximizes total utility if and only if it maximizes the convex combination of measures corresponding to $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. We recall from our work in Chapter 7 that if P is a partition and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in S$, then P maximizes the convex combination of measures corresponding to α if and only if the family of parallel hyperplanes given by $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = c$ makes first contact with the IPS at $m(P)$ (and possibly at other points too). Hence, P maximizes total utility if and only if the family of parallel hyperplanes given by $(\frac{1}{n})x_1 + (\frac{1}{n})x_2 + \cdots + (\frac{1}{n})x_n = c$ makes first contact with the IPS at $m(P)$.

Theorem 13.10

a. There exists a partition that maximizes total utility.

b. If a partition maximizes total utility, then it is Pareto maximal.

c. If the measures are not all equal, then not every Pareto maximal partition maximizes total utility.

Proof: Part a follows immediately from the preceding discussion together with Theorem 7.5.

The proof of part b is trivial.

For part c, we simply note that if the measures are not all equal then it follows easily from our work in Chapter 12 that the outer Pareto boundary of the IPS does not consist of a single flat region. This implies that the family of parallel hyperplanes $(\frac{1}{n})x_1 + (\frac{1}{n})x_2 + \cdots + (\frac{1}{n})x_n = c$ does not make first contact with the IPS at all Pareto maximal points. It follows that there are Pareto maximal partitions that do not maximize total utility. □

If the number of points of first contact of a family of parallel hyperplanes with the IPS is more than one, then clearly it is infinite. Hence, since a partition P maximizes total utility if and only if the family of parallel hyperplanes given by $(\frac{1}{n})x_1 + (\frac{1}{n})x_2 + \cdots + (\frac{1}{n})x_n = c$ makes first contact with the IPS at $m(P)$, it follows that there is either one p-class of partitions that maximizes total utility or else there are infinitely many. Each of these two situations is possible when there are two players. To see that this is so, note that by Theorem 11.1 there is an IPS for which a single point is the only point of first contact of the family of parallel planes $(\frac{1}{2})x + (\frac{1}{2})y = c$ with this IPS, and an IPS for which there are infinitely many points that are each points of first contact of this family of parallel planes with this IPS. It is not hard to construct examples to see that each of these situations is possible when there are more than two players.

With regard to part c of Theorem 13.10, the geometric perspective used in the proof makes it clear that if the measures are not all equal then there are infinitely many points on the outer Pareto boundary of the IPS that are not points of first contact with the IPS of the family of parallel hyperplanes $(\frac{1}{n})x_1 + (\frac{1}{n})x_2 + \cdots + (\frac{1}{n})x_n = c$. This implies that there are infinitely many non-p-equivalent partitions that are Pareto maximal but do not maximize total utility.

The connection between maximization of total utility and partition ratios is the following.

Theorem 13.11 *A partition $P = \langle P_1, P_2, \ldots, P_n \rangle \in$ Part maximizes total utility if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $\text{pr}_{ij} \leq 1$ or pr_{ij} is undefined.*

Proof: Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$. Then

P maximizes total utility
if and only if
P maximizes the convex combination of measures corresponding to $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$
if and only if (Theorem 10.6)
P is w-associated with $\text{RD}(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$
if and only if (Theorem 13.1)
for all distinct $i, j = 1, 2, \ldots, n$, $\text{pr}_{ij} \leq \frac{(1/n)}{(1/n)} = 1$ or pr_{ij} is undefined. □

As we see in the preceding proof, a partition $P \in$ Part maximizes total utility if and only if, it is w-associated with the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, the centroid of the simplex. This suggests another perspective on the maximization of total utility. A partition $P = \langle P_1, P_2, \ldots, P_n \rangle \in$ Part maximizes total utility if and only if, for each $i = 1, 2, \ldots, n$, almost every point in P_i is associated with a point in the RNS that is at least as close to Player i's vertex as it is to any other player's vertex. In other words, all partitions in Part that maximize total utility are obtained by giving (the cake associated with) points in the RNS to the player whose vertex is closest to that point, with ties broken arbitrarily.

Next, we comment on why we insist that $k_1, k_2, \ldots, k_t$ be distinct, i.e., why we do not allow repeat intermediate players. Using repeat intermediate players would mean that our generalized transfer would involve at least one cyclic trade. If the partition is Pareto maximal, then such a cyclic trade cannot help the efficiency of the generalized transfer (since, if it did, then performing just this cyclic trade would lead to a Pareto bigger partition). On the other hand, if the partition is not Pareto maximal, then a cyclic trade inside a generalized transfer can help efficiency. The problem is that it can help too much. Suppose that the partition is not Pareto maximal. Then, by Theorem 8.2, there is a cyclic trade that creates a Pareto bigger partition. It is not hard to see that by incorporating repeated cyclic trades (involving the same players each time) it is possible to produce generalized transfers having arbitrarily large efficiencies. We could have either taken the view that Theorem 13.8 should only be applied to partitions that are Pareto maximal (so that repeat intermediate players do not help) or else we could insist, as a reasonable and natural assumption, that intermediate players be used at most once. We chose to take the latter perspective.

The generalized transfers we have discussed so far might involve very small quantities of cake. We next consider the efficiency of generalized transfers in which Player i is required to give up a specified quantity of cake. We need to generalize the notion of partition ratio and of the function Ef.

Definition 13.12 Fix distinct $i, j = 1, 2, \ldots, n$.

a. The *ij partition ratio function* Pr_{ij} is defined as follows: for any κ with $0 < \kappa \leq m_i(P_i)$, $\mathrm{Pr}_{ij}(\kappa) = \sup\{m_j(A) : A \subseteq P_i \text{ and } m_i(A) = \kappa\}$.
b. For distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$, and any κ with $0 < \kappa \leq m_i(P_i)$, let $Ef(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa) = \sup\{\frac{\Delta_j}{\Delta_i} : \Delta_i = \kappa$ and $\frac{\Delta_j}{\Delta_i}$ is the efficiency of a positive generalized transfer from Player i to Player j using intermediate players $k_1, k_2, \ldots, k_t$, in that order$\}$. (As in Definition 13.5, Δ_i denotes loss to Player i and Δ_j denotes the gain to Player j.)

$\mathrm{Pr}_{ij}(\kappa)$ corresponds to pr_{ij}, except that instead of considering all positive-measure subsets of P_i we now only consider sets of size κ (according to Player i). For ease of notation in what follows, we have chosen (in contrast with what we did in the definition of the pr_{ij}) to not divide $m_j(A)$ by $m_i(A) = \kappa$ in the definition of the Pr_{ij}.

$\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$ is the same as $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$, except that we now insist that $\Delta_i = \kappa$. It is not hard to see that, for fixed $i, k_1, k_2, \ldots, k_t$, and j, $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$ is a decreasing (though not necessarily strictly decreasing) function of κ and that

$$\operatorname*{Lim}_{\kappa \to 0} [\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)] = \mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle).$$

In certain cases $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$ may be undefined. This is certainly true if $m_i(P_i) < \kappa$. However, even if $m_i(P_i) \geq \kappa$, $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$ may be undefined. For example, consider the pieces of cake that Player k_1 could receive from Player i as part of a generalized transfer corresponding to $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$. Any such piece has size κ according to Player i. It may be that all such pieces are, according to Player k_1, bigger than all of P_{k_1}. Then Player k_1 would have no piece of cake to give to Player k_2, and so there would be no generalized transfers of the type desired.

The following result generalizes Theorem 13.8.

Theorem 13.13 *For distinct* $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$*, and any* κ *with* $0 < \kappa \leq m_i(P_i)$, $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa) = (\frac{1}{\kappa})((\mathrm{Pr}_{k_t j} \circ \mathrm{Pr}_{k_{t-1}k_t} \circ \cdots \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa))$.

Proof: The proof is similar to the proof of Theorem 13.8. We sketch the proof and leave the details to the reader. Fix distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$ and fix κ with $0 < \kappa \leq m_i(P_i)$.

Suppose first, by way of contradiction, that

$$\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa) < \left(\frac{1}{\kappa}\right)(\mathrm{Pr}_{k_t j} \circ \mathrm{Pr}_{k_{t-1}k_t} \circ \cdots \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa)$$

and hence that

$$\kappa \mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa) < (\mathrm{Pr}_{k_t j} \circ \mathrm{Pr}_{k_{t-1}k_t} \circ \cdots \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa).$$

By successively using the definition of each partition ratio function, it can be shown that there exists a generalized transfer $\mathrm{Tr}(\langle i, k_1, k_2, \ldots, k_t, j\rangle | \langle Q_{k_0}, Q_{k_1}, \ldots, Q_{k_t}\rangle)$ such that $m_i(Q_{k_0}) = \kappa$ and $m_j(Q_{k_t}) > \kappa \mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$. The efficiency of this generalized transfer is $\frac{m_j(Q_{k_t})}{\kappa}$. But this contradicts the definition of $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$, since $\frac{m_j(Q_{k_t})}{\kappa} > \mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$.

Next suppose, by way of contradiction, that $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa) > \frac{1}{\kappa}((\mathrm{Pr}_{k_t j} \circ \mathrm{Pr}_{k_{t-1}k_t} \circ \cdots \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa))$. Then there is a generalized transfer $\mathrm{Tr}(\langle i, k_1, k_2, \ldots, k_t, j\rangle | \langle Q_{k_0}, Q_{k_1}, \ldots, Q_{k_t}\rangle)$ with $m_i(Q_{k_0}) = \kappa$ that has efficiency greater than $\frac{1}{\kappa}((\mathrm{Pr}_{k_t j} \circ \mathrm{Pr}_{k_{t-1}k_t} \circ \cdots \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa))$. The efficiency of this generalized transfer is $\frac{m_j(Q_{k_t})}{\kappa}$ and, hence, $m_j(Q_{k_t}) > (\mathrm{Pr}_{k_t j} \circ \mathrm{Pr}_{k_{t-1}k_t} \circ \cdots \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa)$. But the definition of the partition ratio functions, and the fact that for each $s = 1, 2, \ldots, t$, $m_{k_s}(Q_{k_{s-1}}) = m_{k_s}(Q_{k_s})$ (where, as usual, we set $k_0 = i$), tells us that

$m_{k_1}(Q_{k_0}) \leq \mathrm{Pr}_{ik_1}(\kappa)$,and hence,
$m_{k_1}(Q_{k_1}) \leq \mathrm{Pr}_{ik_1}(\kappa)$, and hence,
$m_{k_2}(Q_{k_1}) \leq (\mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa)$, and hence,
$m_{k_2}(Q_{k_2}) \leq (\mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa)$, and hence,
$m_{k_3}(Q_{k_2}) \leq (\mathrm{Pr}_{k_2k_3} \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa)$, and hence,
$\ldots,$
$m_{k_t}(Q_{k_{t-1}}) \leq (\mathrm{Pr}_{k_{t-1}k_t} \circ \cdots \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa)$, and hence,
$m_{k_t}(Q_{k_t}) \leq (\mathrm{Pr}_{k_{t-1}k_t} \circ \cdots \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa)$, and hence,
$m_j(Q_{k_t}) \leq (\mathrm{Pr}_{k_t j} \circ \mathrm{Pr}_{k_{t-1}k_t} \circ \cdots \circ \mathrm{Pr}_{k_1k_2} \circ \mathrm{Pr}_{ik_1})(\kappa)$.

This is a contradiction; hence, we have established the theorem. □

There was no particular reason why we chose to fix the amount of cake that Player i was required to give up, rather than the amount that Player j was required to receive. If we had chosen to fix the amount of cake that Player j was required to receive, the presentation would have been similar.

We close this section by considering the chores versions of these ideas. We do not need to revise Definition 13.5, the definition of efficiency. However, in the chores context, small numbers for efficiency are good. Thus, we need to revise Definition 13.6, the definition of the Ef function, to what we shall call the ChEf function. We do so by taking an infimum instead of a supremum.

Definition 13.14 Fix distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$ and suppose that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition. We set

> $\mathrm{ChEf}(\langle i, k_1, k_2, \ldots, k_t, j \rangle) = \inf\{\frac{\Delta_j}{\Delta_i} : \frac{\Delta_j}{\Delta_i}$ is the efficiency of a positive generalized transfer from Player i to Player j, using intermediate players $k_1, k_2, \ldots, k_t$, in that order$\}$.

The chores versions of Lemma 13.7 and Theorem 13.8 are Lemma 13.15 and Theorem 13.16, respectively. The proofs are analogous and we omit them.

Lemma 13.15 *For distinct* $i, j = 1, 2, \ldots, n$, $\mathrm{ChEf}(\langle i, j \rangle) = \mathrm{qr}_{ij}$.

Theorem 13.16 *For distinct* $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$, $\mathrm{ChEf}(\langle i, k_1, k_2, \ldots, k_t, j \rangle) = \mathrm{qr}_{ik_1} \mathrm{qr}_{k_1 k_2} \ldots \mathrm{qr}_{k_{t-1} k_t} \mathrm{qr}_{k_t j}$.

There is no need to revise the definition of total utility. We are now interested in minimizing, rather than maximizing, total utility.

Definition 13.17 A partition P *minimizes total utility* if the total utility of P is at most as large as the total utility of any other partition.

The natural adjustments of Theorems 13.10 and 13.11 are Theorems 13.18 and 13.19, respectively. The proofs are analogous and we omit them.

Theorem 13.18

a. There exists a partition that minimizes total utility.
b. If a partition minimizes total utility, then it is Pareto minimal.
c. If the measures are not all equal, then not every Pareto minimal partition minimizes total utility.

Theorem 13.19 *A partition* $P = \langle P_1, P_2, \ldots, P_n \rangle \in \mathrm{Part}$ *minimizes total utility if and only if, for all distinct* $i, j = 1, 2, \ldots, n$, $\mathrm{qr}_{ij} \geq 1$ *or* qr_{ij} *is undefined.*

Next, we adjust the definitions of the partition ratio function Pr_{ij} and the function $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$ (see Definition 13.12) as follows.

Definition 13.20 Fix distinct $i, j = 1, 2, \ldots, n$.

a. The ij *chores partition ratio function* Qr_{ij} is defined as follows: for any κ with $0 < \kappa \leq m_i(P_i)$, $\mathrm{Qr}_{ij}(\kappa) = \inf\{m_j(A) : A \subseteq P_i \text{ and } m_i(A) = \kappa\}$.
b. For distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$ and any κ with $0 < \kappa \leq m_i(P_i)$, let $\mathrm{ChEf}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa) = \inf\{\frac{\Delta_j}{\Delta_i} : \Delta_i = \kappa$ and $\frac{\Delta_j}{\Delta_i}$ is the efficiency of a positive generalized transfer from Player i to Player j using intermediate players $k_1, k_2, \ldots, k_t$, in that order$\}$. (Recall that Δ_i denotes loss to Player i and Δ_j denotes the gain to Player j.)

Finally, the appropriate adjustment of Theorem 13.13 to the chores setting is as follows. The proof is analogous and we omit it.

Theorem 13.21 *For distinct* $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$ *and any* κ *with* $0 < \kappa \leq m_i(P_i)$, $\mathrm{ChEf}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa) = (\frac{1}{\kappa})((\mathrm{Qr}_{k_t j} \circ \mathrm{Qr}_{k_{t-1} k_t} \circ \cdots \circ \mathrm{Qr}_{k_1 k_2} \circ \mathrm{Qr}_{i k_1})(\kappa))$.

13C. Classifying the Failure of Pareto Optimality

In this section, we consider a method of classifying the failure of Pareto maximality and Pareto minimality.

By Theorem 8.2, if a partition is not Pareto maximal, then there is a positive cyclic trade that produces a Pareto bigger partition. The length of such a cyclic trade is at least two and at most n. Hence, we can classify the failure of Pareto maximality by how long a cyclic trade is needed to produce a Pareto bigger partition. (Of course, this is only meaningful if $n > 2$.) It is tempting to conjecture that this produces a hierarchy of non-Pareto maximal partitions, with one end of the hierarchy consisting of partitions that are "almost Pareto maximal" and the other end consisting of partitions that are as far as possible from being Pareto maximal. However, it is not clear whether to conjecture that "there exists a cyclic trade of length two that produces a partition Pareto bigger than P" or "there exists a cyclic trade of length n that produces a partition Pareto bigger than P" is closer to Pareto maximal. Of course, this is all informal, since it is not clear what "closer to Pareto maximal" means. We provide no answer or even a more precise question here, but we do present two examples that suggest there is no such nice hierarchy.

In a hierarchy, there should be some sort of implication that holds between levels. However, the examples in this section will show that, in general, there is no implication between the following two statements about a partition P, for distinct $k_1, k_2 = 2, 3, \ldots, n$.

- There exists a cyclic trade of length k_1 that produces a partition Pareto bigger that P.
- There exists a cyclic trade of length k_2 that produces a partition Pareto bigger that P.

Each of our two examples involves three players. Let C be the interval $[0, 3)$ on the real number line and let m_L be Lebesgue measure on this set. We will use m_L to define measures $m_1, m_2,$ and m_3 that will be absolutely continuous with respect to each other and with respect to m_L. First, we present a general method for defining $m_1, m_2,$ and m_3, and then we use this method in Examples 13.22 and 13.23. These examples are similar to examples used in previous chapters, for other purposes.

For each $i = 1, 2, 3$, let α_{i1}, α_{i2}, and α_{i3} be positive real numbers such that $\alpha_{i1} + \alpha_{i2} + \alpha_{i3} = 1$, and define m_i as follows: for any $A \subseteq C$,

$$m_i(A) = \alpha_{i1} m_L(A \cap [0, 1)) + \alpha_{i2} m_L(A \cap [1, 2)) + \alpha_{i3} m_L(A \cap [2, 3))$$

It is straightforward to check that each m_i is a (countably additive, non-atomic, probability) measure and is absolutely continuous with respect to the other two m_i and with respect to m_L. The idea here is simply to weight the measures differently (using the α_{ij}) on the three different parts of C. (We have used the interval $[0, 3)$ and Lebesgue measure for convenience. We could have started with any measure μ on any cake C, partitioned C into three pieces of equal size according to μ, and then defined m_1, m_2, and m_3 as in the preceding paragraph.)

We wish to consider the corresponding RNS. Suppose that m_1, m_2, and m_3, are as previously defined. Then the three players are in relative agreement (see Definition 12.11) on each of the sets $[0, 1)$, $[1, 2)$, and $[2, 3)$. This implies that (possibly excluding a set of measure zero) the density functions of the m_i with respect to $\mu = m_1 + m_2 + m_3$ are constant on each of these sets. In particular, if $m_1, m_2,$ and m_3 are defined as before, and f_1, f_2, and f_3, respectively, are the corresponding density functions with respect to μ, then for $i, j = 1, 2, 3$, $f_i(a) = \frac{\alpha_{ij}}{\alpha_{1j}+\alpha_{2j}+\alpha_{3j}}$ for all but possibly a measure-zero collection of $a \in [j-1, j)$. In our examples that follow, we shall always assume that the f_i have been redefined on this measure-zero set, if necessary, so that the given equality holds for all $a \in [j-1, j)$. It follows that, for each $j = 1, 2, 3$, all of the points in $[j-1, j)$ correspond to a single point in the simplex.

Examples 13.22 and 13.23 each involve three players. Together, they show that for a partition P there need not be an implication in either direction between the statement "There exists a cyclic trade of length two that produces a partition Pareto bigger than P" and the statement "There exists a cyclic trade of length three that produces a partition Pareto bigger than P."

Example 13.22 A partition P for which there is a cyclic trade of length two that produces a partition Pareto bigger than P, but no cyclic trade of length three that produces a partition Pareto bigger that P.

Let $C = [0, 3)$ and define α_{ij} for $i, j = 1, 2, 3$ as follows:

$$\begin{array}{lll} \alpha_{11} = .4 & \alpha_{12} = .5 & \alpha_{13} = .1 \\ \alpha_{21} = .5 & \alpha_{22} = .4 & \alpha_{23} = .1 \\ \alpha_{31} = .1 & \alpha_{32} = .1 & \alpha_{33} = .8 \end{array}$$

Let $m_1, m_2,$ and m_3 be defined from these α_{ij} as described previously and let $P = \langle P_1, P_2, P_3\rangle = \langle [0, 1), [1, 2), [2, 3)\rangle$. Then, for each $i, j = 1, 2, 3$ and any $A \subseteq P_i$ of positive measure, $\frac{m_j(A)}{m_i(A)} = \frac{\alpha_{ji} m_L(A)}{\alpha_{ii} m_L(A)} = \frac{\alpha_{ji}}{\alpha_{ii}}$, and this implies that, for distinct $i, j = 1, 2, 3$, $\text{pr}_{ij} = \sup\{\frac{m_j(A)}{m_i(A)} : A \subseteq [i-1, i)$ and A has positive measure$\} = \frac{\alpha_{ji}}{\alpha_{ii}}$. Hence, we have:

$$\begin{array}{lll} & \text{pr}_{12} = \frac{5}{4} & \text{pr}_{13} = \frac{1}{4} \\ \text{pr}_{21} = \frac{5}{4} & & \text{pr}_{23} = \frac{1}{4} \\ \text{pr}_{31} = \frac{1}{8} & \text{pr}_{32} = \frac{1}{8} & \end{array}$$

Since $\text{pr}_{12}\text{pr}_{21} = (\frac{5}{4})(\frac{5}{4}) = \frac{25}{16} > 1$, it follows from Lemma 8.8 that there exists a cyclic trade of length two that produces a partition Pareto bigger than P. (This trade will be between Player 1 and Player 2.) However, since $\text{pr}_{12}\text{pr}_{23}\text{pr}_{31} = \text{pr}_{32}\text{pr}_{21}\text{pr}_{13} = (\frac{5}{4})(\frac{1}{4})(\frac{1}{8}) = \frac{5}{128} < 1$, it follows that there are no cyclic sequences of length three having product greater than one and therefore, by Lemma 8.8, there is no cyclic trade of length three that produces a partition Pareto bigger than P.

The RNS provides some additional perspective. Let f_1, f_2, and f_3 be the density functions of $m_1, m_2,$ and m_3, respectively, with respect to $\mu = m_1 + m_2 + m_3$. As described earlier, for distinct $i, j = 1, 2, 3$, $f_i(a) = \frac{\alpha_{ij}}{\alpha_{1j}+\alpha_{2j}+\alpha_{3j}}$

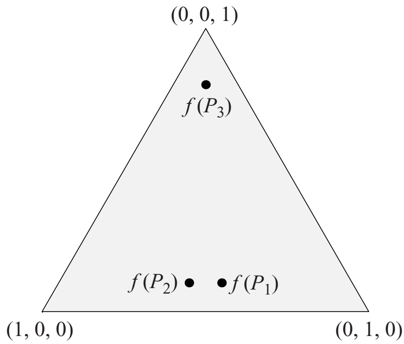


Figure 13.4

for all $a \in P_j$. Hence, we have:

$$\begin{array}{lll} f_1(a) = .4 \text{ for } a \in P_1, & f_1(a) = .5 \text{ for } a \in P_2, & f_1(a) = .1 \text{ for } a \in P_3 \\ f_2(a) = .5 \text{ for } a \in P_1, & f_2(a) = .4 \text{ for } a \in P_2, & f_2(a) = .1 \text{ for } a \in P_3 \\ f_3(a) = .1 \text{ for } a \in P_1, & f_3(a) = .1 \text{ for } a \in P_2, & f_3(a) = .8 \text{ for } a \in P_3 \end{array}$$

We note that the preceding numbers turned out to be the same as the numbers in the table of the α_{ij} since, in the α_{ij} table, each column summed to one. In general, this need not be so. (The columns in the f_{ij} table will always sum to one, since they are simply the columns from the α_{ij} table, scaled by the appropriate constant.)

This table tells us that all points in P_1 correspond to the point (.4, .5, .1) in the simplex, all points in P_2 correspond to the point (.5, .4, .1) in the simplex, and all points in P_3 correspond to the point (.1, .1, .8) in the simplex. Hence, the RNS consists of these three points. This is illustrated in Figure 13.4.

It is clear from the figure that Player 1 and Player 2 can each gain from a trade with each other, since $f(P_2)$ is closer to Player 1's vertex than is $f(P_1)$, and $f(P_1)$ is closer to Player 2's vertex than is $f(P_2)$. In fact, in this case, a trade of all of P_1 for all of P_2 would benefit both players. However, Player 3 does not value P_1 or P_2 highly, and neither Player 1 nor Player 2 values P_3 highly. Thus, Player 3 is not anxious to give up any of P_3 in return for some of P_1 or P_2, and neither Player 1 nor Player 2 is anxious to give up any of P_1 or P_2, respectively, in return for some of P_3. Consequently, any trade involving Player 3 will make some player unhappy, and therefore there is no cyclic trade of length three that produces a partition Pareto bigger than P.

Example 13.23 A partition P for which there is a cyclic trade of length three that produces a partition Pareto bigger than P, but no cyclic trade of length two that produces a partition Pareto bigger that P.

We again let $C = [0, 3)$ and we define α_{ij} for $i, j = 1, 2, 3$ as follows:

$$\begin{array}{lll} \alpha_{11} = .3 & \alpha_{12} = .1 & \alpha_{13} = .6 \\ \alpha_{21} = .6 & \alpha_{22} = .3 & \alpha_{23} = .1 \\ \alpha_{31} = .1 & \alpha_{32} = .6 & \alpha_{33} = .3 \end{array}$$

Let m_1, m_2, and m_3 be defined from these α_{ij}, as described before the previous example, and let $P = \langle P_1, P_2, P_3 \rangle = \langle [0, 1), [1, 2), [2, 3) \rangle$. Then (as in the previous example) for distinct $i, j = 1, 2, 3$, $\text{pr}_{ij} = \frac{\alpha_{ji}}{\alpha_{ii}}$. Therefore, we have:

$$\begin{array}{lll} & \text{pr}_{12} = 2 & \text{pr}_{13} = \frac{1}{3} \\ \text{pr}_{21} = \frac{1}{3} & & \text{pr}_{23} = 2 \\ \text{pr}_{31} = 2 & \text{pr}_{32} = \frac{1}{3} & \end{array}$$

We note that $\text{pr}_{12}\text{pr}_{23}\text{pr}_{31} = (2)(2)(2) = 8 > 1$ and, hence, by Lemma 8.8, there exists a cyclic trade of length three that produces a partition Pareto bigger than P. Since $\text{pr}_{12}\text{pr}_{21} = \text{pr}_{13}\text{pr}_{31} = \text{pr}_{23}\text{pr}_{32} = \frac{2}{3} < 1$, there is no cyclic trade of length two that produces a partition Pareto bigger than P.

We look again to the RNS for additional perspective. Let f_1, f_2, and f_3 be the density functions of m_1, m_2, and m_3, respectively, with respect to $\mu = m_1 + m_2 + m_3$. Then we have:

$$\begin{array}{lll} f_1(a) = .3 \text{ for } a \in P_1, & f_1(a) = .1 \text{ for } a \in P_2, & f_1(a) = .6 \text{ for } a \in P_3 \\ f_2(a) = .6 \text{ for } a \in P_1, & f_2(a) = .3 \text{ for } a \in P_2, & f_2(a) = .1 \text{ for } a \in P_3 \\ f_3(a) = .1 \text{ for } a \in P_1, & f_3(a) = .6 \text{ for } a \in P_2, & f_3(a) = .3 \text{ for } a \in P_3 \end{array}$$

This table tells us that all points in P_1 correspond to the point (.3, .6, .1) in the simplex, all points in P_2 correspond to the point (.1, .3, .6) in the simplex, and all points in P_3 correspond to the point (.6, .1, .3) in the simplex. Hence, the RNS consists of these three points. This is illustrated in Figure 13.5.

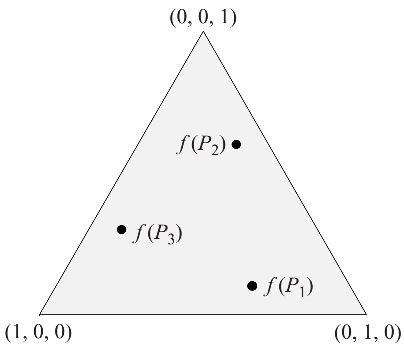


Figure 13.5

The picture makes clear that there is a cyclic trade from Player 1 to Player 2 to Player 3 to Player 1 that benefits all players, since each player trades for a set that is closer to that player's vertex than the set that is given up. (We are blurring the distinction here between subsets of C and the corresponding points in the RNS.) In this case, a trade of all of P_1, all of P_2, and all of P_3 to Player 2, Player 3, and Player 1, respectively, benefits all three players. However, any trade between two players hurts at least one of the players. One way to see this is to notice that, for example, it is possible to draw a line through Player 3's vertex such that $f(P_1)$ is on Player 1's side of the line and $f(P_2)$ is on Player 2's side on the line. Hence, any non-trivial trade between Player 1 and Player 2 must hurt one or both players. The situation is the same for the other two pairs of players. This is in contrast to the situation in Figure 13.4, where it is not possible to draw such a line between $f(P_1)$ and $f(P_2)$.

The cake, measures, and partition in Example 13.23 were the same as in Examples 6.3, 8.10, and 10.10, presented in a slightly different form.

We close this section by briefly discussing the chores version of the ideas considered in this section. By Theorem 8.11, if a partition P is not Pareto minimal, then there is a positive cyclic trade that yields a partition Pareto smaller than P. Hence, for such a P, we can consider the length of a cyclic trade that produces a partition Pareto smaller than P. However, there are examples analogous to those presented in this section showing that this classification does not yield a hierarchy.

13D. Convexity

In this section, we consider the convexity of certain subsets of the simplex, S, and of the set of all partitions, Part, that arise naturally in our present context. The notion of a convex subset of S is clear. There is also a natural notion of convexity for Part.

Definition 13.24

a. Suppose that $P^1, P^2, \ldots, P^t \in$ Part where, for each $k = 1, 2, \ldots, t$, we set $P^k = \langle P_1^k, P_2^k, \ldots, P_n^k \rangle$. A partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a *convex combination* of $P^1, P^2, \ldots, P^t$ if and only if, for every $i = 1, 2, \ldots, n$, $P_i \subseteq \bigcup_{k=1}^{t} P_i^k$.
b. A subset of Part is *convex* if and only if it contains all convex combinations of its elements.

Part a of the definition is a natural generalization of the usual notion of convex combination in the sense that we consider P to be a convex combination of $P^1, P^2, \ldots, P^t$ if and only if P's value in each coordinate lies within the extremes mapped out by $P^1, P^2, \ldots, P^t$. Notice that if P is a convex combination of $P^1, P^2, \ldots, P^t$ then, with the notation given in part a of the definition, $\bigcap_{k=1}^{t} P_i^k \subseteq P_i$ for every $i = 1, 2, \ldots, n$.

As in Chapter 12, for $\omega \in S^+$ we let ω^* denote the set of all partitions that are w-associated with ω, and for any partition P we let P^* denote the set of all $\omega \in S^+$ with which P is w-associated. Then $\omega \in P^*$ if and only if $P \in \omega^*$.

Theorem 13.25

a. For any $\omega \in S^+$, ω^* is a convex subset of Part.
b. For any $P \in \text{Part}^+$, P^* is a convex subset of S.

Proof: For part a, suppose that $\omega \in S^+$, $P^1, P^2, \ldots, P^t \in \omega^*$, and P is a convex combination of $P^1, P^2, \ldots, P^t$. We must show that $P \in \omega^*$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and let $P = \langle P_1, P_2, \ldots, P_n \rangle$. As before, for each $k = 1, 2, \ldots, t$, set $P^k = \langle P_1^k, P_2^k, \ldots, P_n^k \rangle$. Fix distinct $i, j = 1, 2, \ldots, n$ and any positive-measure $A \subseteq P_i$. It suffices to show that $\frac{m_i(A)}{m_j(A)} \geq \frac{\omega_i}{\omega_j}$.

Since P is a convex combination of $P^1, P^2, \ldots, P^t$, we can partition A into disjoint sets $A^1, A^2, \ldots, A^t$ (some of which may be empty) such that, for each $k = 1, 2, \ldots, t$, $A^k \subseteq P_i^k$. Then, for each such k, since $P^k \in \omega^*$, it follows that $\frac{m_i(A^k)}{m_j(A^k)} \geq \frac{\omega_i}{\omega_j}$ and, hence, $m_i(A^k) \geq (\frac{\omega_i}{\omega_j}) m_j(A^k)$. Then we have

$$m_i(A) = m_i\left(\bigcup_{k=1}^{t} A^k\right) = \sum_{k=1}^{t} m_i(A^k) \geq \left(\frac{\omega_i}{\omega_j}\right) \sum_{k=1}^{t} m_j(A^k)$$
$$= \left(\frac{\omega_i}{\omega_j}\right) m_j\left(\bigcup_{k=1}^{t} A^k\right) = \left(\frac{\omega_i}{\omega_j}\right) m_j(A).$$

This establishes that $\frac{m_i(A)}{m_j(A)} \geq \frac{\omega_i}{\omega_j}$, as desired.

For part b, suppose that $P \in \text{Part}^+, \omega^1, \omega^2, \ldots, \omega^t \in P^*$, and ω is a convex combination of $\omega^1, \omega^2, \ldots, \omega^t$. We must show that $\omega \in P^*$. Let $P = \langle P_1, P_2, \ldots, P_n \rangle$ and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. For each $k = 1, 2, \ldots, t$, set $\omega^k = (\omega_1^k, \omega_2^k, \ldots, \omega_n^k)$. Fix distinct $i, j = 1, 2, \ldots, n$ and any positive-measure $A \subseteq P_i$. It suffices to show that $\frac{m_i(A)}{m_j(A)} \geq \frac{\omega_i}{\omega_j}$.

Since $\omega^k \in P^*$ for each $k = 1, 2, \ldots, t$, it follows that, for each such k, $\frac{m_i(A)}{m_j(A)} \geq \frac{\omega_i^k}{\omega_j^k}$. Also, since ω is a convex combination of $\omega^1, \omega^2, \ldots, \omega^t$, it is not hard to see that $\frac{\omega_i}{\omega_j} \leq \max\{\frac{\omega_i^k}{\omega_j^k} : k = 1, 2, \ldots, t\}$. This establishes that $\frac{m_i(A)}{m_j(A)} \geq \frac{\omega_i}{\omega_j}$, as desired, and hence completes the proof of the theorem. □

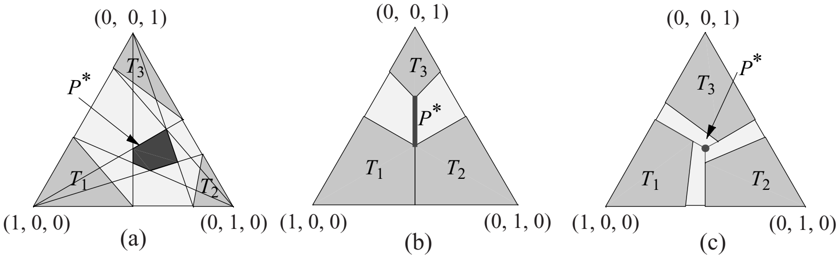


Figure 13.6

Figure 13.6 illustrates three possibilities for the convex set P^* when there are three players. In each of the figures, T_1, T_2, and T_3 are the parts of the RNS that correspond to P_1, P_2, and P_3, respectively, from some partition $P = \langle P_1, P_2, P_3 \rangle$. In Figures 13.6a, 13.6b, and 13.6c, P^* consists of a two-dimensional region, a line segment, and a single point, respectively, on the simplex. (Figure 13.6 is the same as Figure 12.6. We used Figure 12.6 in our discussion of the notion of separability.)

13E. The Situation Without Absolute Continuity

In this section, we reconsider the ideas presented in previous sections of this chapter, without assuming that the measures are absolutely continuous with respect to each other. As in previous chapters, we adopt the convention that the expressions "for almost every," "has positive measure," or "has measure zero" refer to the measure $\mu = m_1 + m_2 + \cdots + m_n$ unless otherwise stated.

We begin with Section 13A, where we studied the relationship between partition ratios and w-association. The proof of Theorem 13.1 did not use absolute continuity. (However, some adjustments in the proof are needed in the absence of absolute continuity. Wording such as "for almost every," "has positive measure," or "has measure zero" must be changed to refer to the measure $\mu = m_1 + m_2 + \cdots + m_n$, and inequalities such as $\frac{f_j(a)}{f_i(a)} \le \frac{\omega_j}{\omega_i}$ may need to be interpreted using the arithmetic of infinities described in Section 10C.) Hence, this result is true without the assumption of absolute continuity, but it says less than we wish. We want this result to say something about how to describe Pareto maximal partitions. As we saw in Section 10C, problems arise in attempting to characterize Pareto maximality using the notion of w-association, when absolute continuity fails. This was the reason for our use of partition sequence pairs in the absence of absolute continuity. We categorized Pareto maximality in terms of a partition being w-associated with some partition sequence pair (see Definitions 7.11 and 10.26, and Theorem 10.28). We will use partition sequence pairs to obtain an appropriate version of Theorem 13.1 for our present context.

Assume that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a Pareto maximal partition. We studied partition ratios without the assumption of absolute continuity in Section 8C. We found that we needed to consider two types of infinite partition ratios, ∞^* and ∞^{**}, and their arithmetic (see Notation 8.21 and Definition 8.22). Using these partition ratios, we were able to characterize Pareto maximality (see Theorem 8.24).

Theorem 13.26 connects partition ratios and w-association with a partition sequence pair. This is the appropriate adjustment of Theorem 13.1 to our present context. We recall that, for a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ and distinct $i, j = 1, 2, \ldots, n$, the partition ratio pr_{ij} is undefined if and only if $m_i(P_i) = 0$ and $m_j(P_i) = 0$, and is equal to ∞^{**} if and only if, for some $A \subseteq P_i$, $m_j(A) \neq 0$ and $m_i(A) = 0$.

Theorem 13.26 *Suppose that $P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of C and (ω, γ) is a partition sequence pair where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$. Then P is w-associated with (ω, γ) if and only if no partition ratio is equal to ∞^{**} and the following two conditions hold.*

a. *For every $k = 1, 2, \ldots, t$ and distinct $j, j' \in \gamma_k$, $\mathrm{pr}_{jj'} \leq \frac{\omega_{j'}}{\omega_j}$ or $\mathrm{pr}_{jj'}$ is undefined.*
b. *Either*
   i. *for every $k, k' = 1, 2, \ldots, t$ with $k < k'$, and every $j \in \gamma_k$ and $j' \in \gamma_{k'}$, either $\mathrm{pr}_{jj'} = 0$ or $\mathrm{pr}_{jj'}$ is undefined, or*
   ii. *for every $k, k' = 1, 2, \ldots, t$ with $k > k'$, and every $j \in \gamma_k$ and $j' \in \gamma_{k'}$, either $\mathrm{pr}_{jj'} = 0$ or $\mathrm{pr}_{jj'}$ is undefined.*

We need to recall some notation from Chapter 10. Suppose that $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$ is as in the statement of the theorem. For each $k = 1, 2, \ldots, t$, set $\mu^{\gamma_k} = \sum_{i \in \gamma_k} m_i$. Also, Ψ denotes the set of all partition sequence pairs.

Proof of Theorem 13.26 Let P, ω, and γ be as in the statement of the theorem. Recalling that the function $\mathrm{RD}_\Psi : \Psi \to \Psi$ (see Definition 10.27 and the discussion following the definition) is a bijection, we may fix a partition sequence pair (α, γ) with $\mathrm{RD}_\Psi(\alpha, \gamma) = (\omega, \gamma)$. Then Lemma 10.29 implies that P a-maximizes (α, γ) and is non-wasteful if and only if P is w-associated with (ω, γ). (For the definition of a-maximization, see Definition 7.12. For the definition of non-wasteful, see Definition 6.5.) Hence, in order to prove the theorem, it suffices to show that P a-maximizes (α, γ) and is non-wasteful if and only if no partition ratio is equal to ∞^{**} and conditions a and b of the theorem are satisfied. Since non-wastefulness is equivalent to the assertion that no partition ratio is equal to ∞^{**}, we must show that P a-maximizes (α, γ) if and only if conditions a and b of the theorem are satisfied. Set $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

We claim that P and (α, γ) satisfy condition a of Definition 7.12 if and only if P and (ω, γ) satisfy condition a of the theorem. We establish this as follows:

P and (α, γ) satisfy condition a of Definition 7.12
if and only if
for every $k = 1, 2, \ldots, t$, the partition $\langle P_i : i \in \gamma_k \rangle$ of $\bigcup_{i \in \gamma_k} P_i$ maximizes the convex combination of the measures $\langle m_i : i \in \gamma_k \rangle$ corresponding to $(\alpha_i : i \in \gamma_k)$
if and only if (Theorem 10.6)
for every $k = 1, 2, \ldots, t$, $\langle P_i : i \in \gamma_k \rangle$ is w-associated with $\text{RD}(\alpha_i : i \in \gamma_k)$
if and only if (Definition 10.27)
for every $k = 1, 2, \ldots, t$, $\langle P_i : i \in \gamma_k \rangle$ is w-associated with $(\omega_i : i \in \gamma_k)$
if and only if (Theorem 13.1)
for every $k = 1, 2, \ldots, t$ and distinct $j, j' \in \gamma_k$, $\text{pr}_{jj'} \leq \frac{\omega_{j'}}{\omega_j}$ or $\text{pr}_{jj'}$ is undefined
if and only if
P and (ω, γ) satisfy condition a of the theorem.

Next, we claim that P and γ satisfy condition bi of Definition 7.12 if and only if they satisfy condition bi of the theorem. (Notice that the "γ" in "(α, γ)" is the same as the "γ" in "(ω, γ).") We establish this as follows:

P and γ satisfy condition bi of Definition 7.12
if and only if
for every $k, k' = 1, 2, \ldots, t$ with $k < k'$, if $j \in \gamma_k$ and $j' \in \gamma_{k'}$, then $m_{j'}(P_j) = 0$
if and only if
for every $k, k' = 1, 2, \ldots, t$ with $k < k'$, if $j \in \gamma_k$ and $j' \in \gamma_{k'}$, then $m_{j'}(A) = 0$ for every $A \subseteq P_j$
if and only if
for every $k, k' = 1, 2, \ldots, t$ with $k < k'$, and every $j \in \gamma_k$ and $j' \in \gamma_{k'}$, $\sup\{\frac{m_{j'}(A)}{m_j(A)} : A \subseteq P_j \text{ and either } m_j(A) \neq 0 \text{ or } m_{j'}(A) \neq 0\} = 0$ or else this quantity is undefined
if and only if
for every $k, k' = 1, 2, \ldots, t$ with $k < k'$, and every $j \in \gamma_k$ and $j' \in \gamma_{k'}$, either $\text{pr}_{jj'} = 0$ or $\text{pr}_{jj'}$ is undefined
if and only if
P and γ satisfy condition bi of the theorem.

We note that in the fourth and fifth of the preceding assertions, the given quantity is equal to zero if $m_j(P_j) > 0$ and is undefined if $m_j(P_j) = 0$. The proof that P and γ satisfy condition bii of Definition 7.12 if and only if they satisfy condition bii of the theorem is similar and we omit it.

This competes the proof of the theorem. □

By Theorem 10.28, the conditions of the theorem hold if and only if P is Pareto maximal.

We can view Theorem 13.26 as a general result that holds for any partition, regardless of whether the measures are absolutely continuous with respect to each other, and Theorem 13.1 as a special case of this result that holds when absolute continuity holds. (This is analogous to what we did for Theorems 7.4, 7.13, and 7.18 and also for Theorems 10.9 and 10.28. See the discussions following the proofs of Theorems 7.18 and 10.28.) To see this, suppose that the measures are absolutely continuous with respect to each other, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. We will show that, with these assumptions, Theorem 13.1 follows from Theorem 13.26. We first note that, by absolute continuity, no partition ratio is equal to ∞^{**}. Then,

P is w-associated with ω

if and only if

P is w-associated with the partition sequence pair (ω, γ) where $\gamma = \langle\{1, 2, \ldots, n\}\rangle$ (i.e., γ is the trivial partition of $\{1, 2, \ldots, n\}$ into one piece)

if and only if (Theorem 13.26)

for distinct $i, j = 1, 2, \ldots, n$, $\text{pr}_{ij} \leq \frac{\omega_j}{\omega_i}$, or pr_{ij} is undefined.

Next, we consider adjustments of the results of Section 13B. We need no adjustment of the definition of generalized transfer (Definition 13.4). However, we do need to adjust the definition of positive generalized transfer. In the absence of absolute continuity, what we care about is that the generalized transfer is nontrivial, in the sense that either Player i or Player j (or both) sees his or her portion of cake change in size.

Definition 13.27 The generalized transfer $\text{Tr}(\langle i, k_1, k_2, \ldots, k_t, j\rangle | \langle Q_{k_0}, Q_{k_1}, \ldots, Q_{k_t}\rangle)$ is a *positive generalized transfer* if either $m_i(Q_{k_0}) > 0$ or $m_j(Q_{k_t}) > 0$ (or both).

In going from Definition 13.4 to 13.27, we have not changed the name "positive generalized transfer," since these definitions are consistent. In other words, if absolute continuity holds, then a generalized transfer is positive according to Definition 13.4 if and only if it is positive according to Definition 13.27. (If absolute continuity fails, then Definition 13.4 cannot be used, since the phrase "of positive measure" is ambiguous.)

We need no adjustment in the definition of efficiency (Definition 13.5), but we do note that, in contrast with the absolute continuity context (where efficiency is always equal to a positive number), efficiency can be zero, positive, or infinite. (Using the notation from Definition 13.5, the efficiency of a positive generalized transfer from Player i to Player j is zero if $\Delta_j = 0$ and $\Delta_i > 0$ and is infinite if $\Delta_j > 0$ and $\Delta_i = 0$. By the definition of "positive generalized transfer," we cannot have $\Delta_i = \Delta_j = 0$.)

Next, we consider the function $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$ given by Definition 13.6. In the absolute continuity context, this function can be positive or (if the relevant set of efficiencies $\frac{\Delta_j}{\Delta_i}$ is unbounded) infinite. In our present context, $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$ can be zero, positive, or infinite, and it can be infinite in two different ways. It can be infinite as in the absolute continuity context, or a single term in the set over which the supremum is taken can be infinite. This situation is very much like the two different ways that a partition ratio can be infinite, as given by Notation 8.21. We shall need to distinguish between these two cases, and we do so in a manner analogous to what we did in Notation 8.21.

Notation 13.28 Fix distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$ and suppose that $P = \langle P_1, P_2, \ldots, P_n\rangle$ is a partition.

a. We write $\mathit{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \infty^*$ if and only if there does not exist a positive generalized transfer from Player i to Player j using intermediate players $k_1, k_2, \ldots, k_t$, in that order, having infinite efficiency (i.e., for no generalized transfer do we have $\Delta_j > 0$ and $\Delta_i = 0$) but $\{\frac{\Delta_j}{\Delta_i} : \frac{\Delta_j}{\Delta_i}$ is the efficiency of a positive generalized transfer from Player i to Player j using intermediate players $k_1, k_2, \ldots, k_t$, in that order$\}$ is unbounded.
b. We write $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \infty^{**}$ if and only if there is a positive generalized transfer $\text{Tr}(\langle i, k_1, k_2, \ldots, k_t, j\rangle | \langle Q_{k_0}, Q_{k_1}, \ldots, Q_{k_t}\rangle)$ having infinite efficiency (i.e., there exists a generalized transfer with $\Delta_j > 0$ and $\Delta_i = 0$).

Next we wish to relate the efficiency of generalized transfers to partition ratios, in the absence of absolute continuity. Both Lemma 13.7 (which states that, for distinct $i, j = 1, 2, \ldots, n$, $\text{Ef}(\langle i, j\rangle) = \text{pr}_{ij}$) and Theorem 13.8 (which states that, for distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$, $\text{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \text{pr}_{ik_1}\text{pr}_{k_1k_2}\ldots\text{pr}_{k_{t-1}k_t}\text{pr}_{k_tj}$) are true in our present context. However, each of these results says more than it may appear to say.

We recall that, in the presence of absolute continuity, $\mathrm{Ef}(\langle i, j\rangle)$ and pr_{ij} are either both equal to some positive number or else are both infinite, and similarly $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$ and $\mathrm{pr}_{ik_1}\mathrm{pr}_{k_1k_2}\ldots\mathrm{pr}_{k_{t-1}k_t}\mathrm{pr}_{k_tj}$ are either both equal to some positive number or else are both infinite. In our present context, these terms can be zero, and there are two different ways for each expression to be infinite. The lemma and the theorem include both of these cases. In other words, for distinct $i, j = 1, 2, \ldots, n$, $\mathrm{Ef}(\langle i, j\rangle) = \infty^*$ if and only if $\mathrm{pr}_{ij} = \infty^*$, and $\mathrm{Ef}(\langle i, j\rangle) = \infty^{**}$ if and only if $\mathrm{pr}_{ij} = \infty^{**}$. Similarly, for distinct $i, k_1, k_2, \ldots, k_t, j = 1, 2, \ldots, n$, $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \infty^*$ if and only if $\mathrm{pr}_{ik_1}\mathrm{pr}_{k_1k_2}\ldots\mathrm{pr}_{k_{t-1}k_t}\mathrm{pr}_{k_tj} = \infty^*$, and $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \infty^{**}$ if and only if $\mathrm{pr}_{ik_1}\mathrm{pr}_{k_1k_2}\ldots\mathrm{pr}_{k_{t-1}k_t}\mathrm{pr}_{k_tj} = \infty^{**}$. As was the case when absolute continuity was assumed, the proof that Lemma 13.7 holds in the absence of absolute continuity follows easily from the relevant definitions. Concerning the proof of Theorem 13.8 without absolute continuity, it is straightforward to show that $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle) = \infty^{**}$ if and only if $\mathrm{pr}_{ik_1}\mathrm{pr}_{k_1k_2}\ldots\mathrm{pr}_{k_{t-1}k_t}\mathrm{pr}_{k_tj} = \infty^{**}$. If neither $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$ nor $\mathrm{pr}_{ik_1}\mathrm{pr}_{k_1k_2}\ldots\mathrm{pr}_{k_{t-1}k_t}\mathrm{pr}_{k_tj}$ is equal to ∞^{**}, then the proof is the same as in Section 13B.

Definition 13.9 and the statement and proof of parts a and b of Theorem 13.10 carry over to our present context. However, part c of the theorem may not hold if absolute continuity fails. Consider Figure 13.7. It follows from Theorem 11.1 that there is a cake C and measures m_1 and m_2 such that the given figure is the corresponding IPS. This IPS includes a vertical line segment going up from the point $(1, 0)$ and a horizontal line segment going to the right

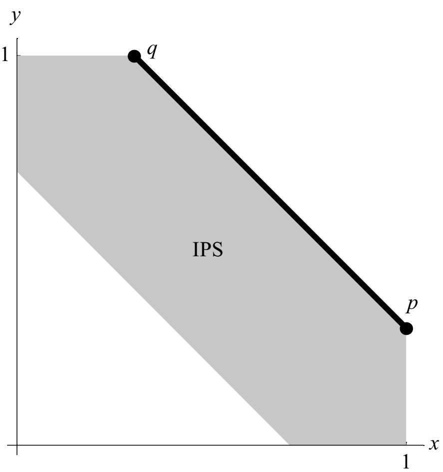


Figure 13.7

from (0, 1). It follows that neither measure is absolutely continuous with respect to the other (and so the measures are certainly not equal). In the figure, the outer Pareto boundary (which we have darkened) consists of all points on the outer boundary of the IPS that are between p and q, including p and q. We have drawn the figure so that the line segment from (1, 0) to p and the line segment from (0, 1) to q have the same length, and the outer boundary between p and q is a line segment. Then it is easy to see that the family of parallel lines $(\frac{1}{2})x + (\frac{1}{2})y = c$ makes first contact with the IPS at all points along this line segment. This implies that all Pareto maximal partitions maximize total utility.

The counter-example to part c of Theorem 13.10 in the absence of absolute continuity, given in the previous paragraph, used Theorem 11.1, and we proved this result only for the two-player context. However, we can generalize the preceding idea, without relying on Theorem 11.1, to show that, in the absence of absolute continuity, part c of Theorem 13.10 may fail for any number of players. The IPS of Figure 13.7 arises from a cake C that can be thought of as the disjoint union of A_1, A_2, and A_3, where $m_1(A_1) = m_2(A_2) > 0$, $m_1(A_2) = m_2(A_1) = 0$, and $m_1(B) = m_2(B)$ for every $B \subseteq A_3$. This idea is easy to generalize. We illustrate this for three players in the following example.

Example 13.29 Let $C = [0, 4)$, let m_L be Lebesgue measure on C, and define measures m_1, m_2, and m_3 on C as follows: for any $A \subseteq C$,

$$\begin{aligned} m_1(A) &= .5[m_L(A \cap [0, 1)) + m_L(A \cap [3, 4))] \\ m_2(A) &= .5[m_L(A \cap [1, 2)) + m_L(A \cap [3, 4))] \\ m_3(A) &= .5[m_L(A \cap [2, 3)) + m_L(A \cap [3, 4))] \end{aligned}$$

It is easy to check that m_1, m_2, and m_3 are (countably additive, non-atomic, probability) measures on C and that a partition is Pareto maximal if and only if it gives all of [0, 1) to Player 1, all of [1, 2) to Player 2, and all of [2, 3) to Player 3. The corresponding RNS consists of four points and is as shown in Figure 13.8a. The piece of cake [0, 1) corresponds to the point (1, 0, 0) of the RNS, since each element of this piece has positive value to Player 1 and has value zero to Player 2 and to Player 3. Similarly, piece [1, 2) corresponds to point (0, 1, 0), and piece [2, 3) corresponds to point (0, 0, 1). Each element of piece [3, 4) is equally valued by all three players, so this piece of cake corresponds to the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in the figure.

Using ideas developed in Chapter 12, we find that the IPS consists of a collection of flat regions and is as displayed in Figure 13.8b. In particular, the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of the RNS corresponds to the flat region on the outer boundary

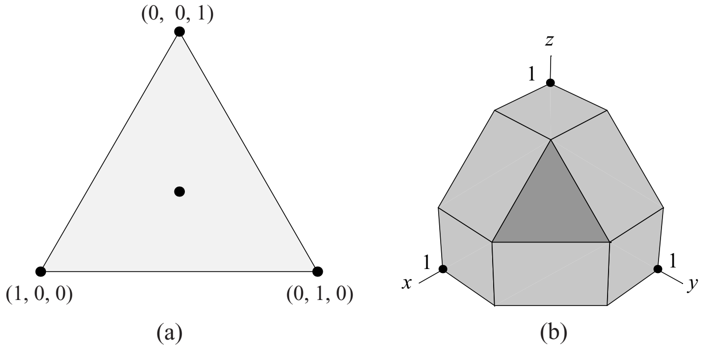


Figure 13.8

of the IPS consisting of the triangle with vertices (1, .5, .5), (.5, 1, .5), and (.5, .5, 1), and its interior. This flat region is the outer Pareto boundary of the IPS and we have darkened this region in the figure.

This IPS is the three-player analog of the situation just described for two players. The set of points of first contact with the IPS of the family of parallel planes $(\frac{1}{3})x + (\frac{1}{3})y + (\frac{1}{3})z = c$ is precisely the darkened region. Hence, all Pareto maximal partitions maximize total utility, even though the measures are not identical. This example generalizes in a natural way to more than three players.

Theorem 13.11 states that a partition P maximizes total utility if and only if, for all distinct $i, j = 1, 2, \ldots, n$, $\mathrm{pr}_{ij} \leq 1$. The proof used Theorems 10.6 and 13.1 and (as discussed in Section 10C and at the beginning of this section, respectively) these results hold without the assumption of absolute continuity. Hence, Theorem 13.11 holds without this assumption. We note that we may now have partition ratios that are equal to zero, ∞^*, or ∞^{**}. Of course, "$\infty^* \leq 1$" and "$\infty^{**} \leq 1$" are both false.

The definitions of the partition ratio functions Pr_{ij} and the function $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$ (see Definition 13.12) and the theorem connecting these notions (Theorem 13.13) stand exactly as before. We do note that we may have $\mathrm{Pr}_{ij} = 0$ or $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa) = 0$ but, in contrast with the partition ratios pr_{ij} and the function $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle)$, neither $\mathrm{Pr}_{ij}(\kappa)$ nor $\mathrm{Ef}(\langle i, k_1, k_2, \ldots, k_t, j\rangle, \kappa)$ can be infinite.

Absolute continuity was not used in our work in Section 13C. Hence, the discussion and examples in that section are correct without the assumption of absolute continuity. We simply note that, if desired, we could alter Examples 13.22 and 13.23 slightly so that the three points of the RNS in each of these examples are on the boundary of the RNS. Then, instead of having the

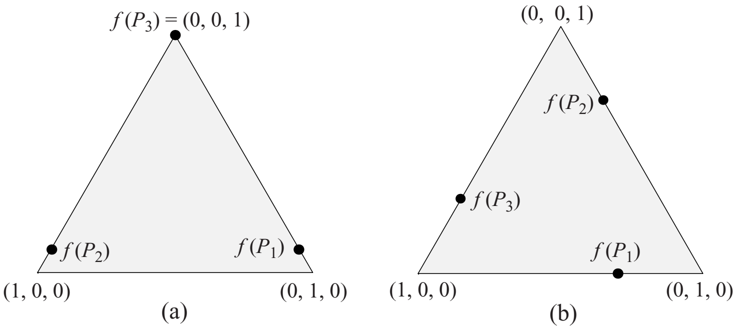


Figure 13.9

RNSs illustrated in Figures 13.4 and 13.5, the associated RNSs would be as in Figures 13.9a and 13.9b, respectively.

Absolute continuity was not used in our study of convexity in Section 13D. In particular, Theorem 13.25 holds without the assumption of absolute continuity. Some minor adjustments in the proof are necessary (such as reinterpreting what "positive measure" means and considering that terms such as "$\frac{m_i(A)}{m_j(A)}$" can be zero or infinite). However, these adjustments are straightforward and we omit them.

We close this chapter by briefly discussing the chores versions of the ideas already considered in this section. We begin by noting that there does not appear to be a nice chores version of Theorem 13.26 (which connects the notions of partition ratio and w-association). This result is only relevant in the context of Pareto maximality. If instead we consider Pareto minimality then, by Theorem 13.30, chores w-association only applies to a certain subset of C (the "B" in the theorem). If we wish to relate the notions of chores partition rations and chores w-association, we can only do so on this subset. On this subset, the measures are absolutely continuous with respect to each other. Hence, such a result would simply be a direct application of Theorem 13.3, and we omit it.

All of the remaining ideas in this section carry over in a natural way to the chores context. In particular, Example 13.29, which shows that part c of Theorem 13.10 does not hold in the absence of absolute continuity, also shows that part c of Theorem 13.18 does not hold in the absence of absolute continuity.

14

Strong Pareto Optimality

In this chapter, we study a natural strengthening of Pareto maximality and Pareto minimality. After introducing this notion in Section 14A, we present various characterizations in Section 14B. In Sections 14C and 14D, we consider existence questions in the two-player context and in the general n-player context, respectively. In Sections 14A, 14B, 14C, and 14D, we assume that the measures are absolutely continuous with respect to each other. In Section 14E, we consider what happens when absolute continuity fails. In Section 14F, we also do not assume that the measures are absolutely continuous with respect to each other and we consider connections with the main theorem of Section 12E.

14A. Introduction

One way to describe Pareto optimality is to say that a partition P is Pareto optimal if and only if no collection of transfers of cake among the players produces a partition that makes every player at least as happy and makes at least one player strictly happier. We strengthen this by insisting that any non-trivial (in a sense to be made precise) collection of transfers produces a partition that makes at least one player less happy.

Notice that if we start with a partition P and transfer various pieces of cake between various players, and each transferred piece has measure zero, then certainly the resulting partition makes no player less (or more) happy. Hence, we shall only consider collections of transfers that include at least one transfer of positive measure. We call a collection of transfers a *non-trivial collection of transfers* if and only if at least one transfer in the collection is of positive measure.

Definition 14.1 Let P be a partition. P is *strongly Pareto maximal* if and only if any non-trivial collection of transfers produces a partition in which at least one player receives a piece that this player believes to be smaller. P is *strongly Pareto minimal* if and only if any non-trivial collection of transfers produces a partition in which at least one player receives a piece that this player believes to be bigger.

Clearly strong Pareto maximality implies Pareto maximality, and strong Pareto minimality implies Pareto minimality. As we shall see, the converses of these statements do not hold, in general.

Theorem 14.4 in the next section gives five characterizations of strong Pareto maximality. One uses partition ratios, one uses the RNS and Weller's characterization of Pareto maximality, one involves the effect of a non-trivial collection of transfers, one involves the relationship between p-classes and s-classes, and one involves the shape of the IPS. Corollary 14.5 characterizes Pareto maximality when strong Pareto maximality fails. Theorem 14.8 and Corollary 14.9 give the analogous results for chores. In Section 14C, we consider existence questions for the two-player context. More specifically, we consider the possible numbers of strongly Pareto maximal partitions, the possible numbers of Pareto maximal partitions that are not strongly Pareto maximal, the relationship between these numbers, and the analogous chores results. In Section 14D, we consider these ideas in the general n-player context. In Section 14E, we consider the situation without absolute continuity. In Section 14F, we revisit Section 12E, where we showed that there exists a partition that is Pareto maximal and envy-free, and we studied the possible existence of a partition that is Pareto maximal and strongly envy-free. We shall consider the existence of such partitions that are also strongly Pareto maximal.

14B. The Characterization

We need to define two preliminary notions. The first is related to partition ratios and the second is related to w-association.

Recall that, for any partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ and distinct $i, j = 1, 2, \ldots, n$, the corresponding partition ratio p_{ij} is given by $\mathrm{pr}_{ij} = \sup\{\frac{m_j(A)}{m_i(A)} : A \subseteq P_i$ and A has positive measure$\}$.

Definition 14.2 For distinct $i, j = 1, 2, \ldots, n$, we shall say that the pr_{ij} *supremum is achieved* if there exists a positive-measure $A \subseteq P_i$ such that $\mathrm{pr}_{ij} = \frac{m_j(A)}{m_i(A)}$.

Definition 14.3 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ and a point $\omega \in S^+$. We shall say that P satisfies the *cyclic boundary condition with respect to* ω if and

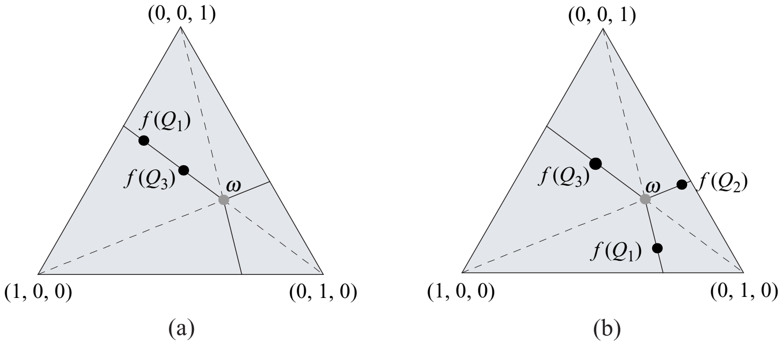


Figure 14.1

only if for no sequence $\langle k_1, k_2, \ldots, k_t\rangle$ of distinct elements of $\{1, 2, \ldots, n\}$ does there exist a sequence $\langle Q_{k_1}, Q_{k_2}, \ldots, Q_{k_t}\rangle$ of subsets of C of positive measure such that, for each $s = 1, 2, \ldots, t$, $Q_{k_s} \subseteq P_{k_s}$ and $f(Q_{k_s})$ is on the k_s,k_{s+1} boundary associated with ω, where we set $k_{t+1} = k_1$. (For the definition of the function f, see Definition 9.2. For the definition of the i, j boundary associated with ω, see Definition 12.10.)

Two examples of this definition (actually, for the failure of this definition) for three players are illustrated in Figure 14.1. In each figure, we assume that $P = \langle P_1, P_2, P_3\rangle$ is a partition. In Figure 14.1a, we assume that $Q_1 \subseteq P_1$, $Q_3 \subseteq P_3$, and Q_1 and Q_3 each have positive measure. Since $f(Q_1)$ and $f(Q_3)$ are each on the 1,3 boundary associated with ω, we see that P does not satisfy the cyclic boundary condition with respect to ω. In Figure 14.1b, we assume that $Q_1 \subseteq P_1$, $Q_2 \subseteq P_2$, and $Q_3 \subseteq P_3$. Since $f(Q_1)$ is on the 1,2 boundary, $f(Q_2)$ is on the 2,3 boundary, and $f(Q_3)$ is on the 3,1 boundary associated with ω, it follows that P does not satisfy the cyclic boundary condition with respect to ω.

In our characterization theorem that follows, we shall only consider partitions $P \in \text{Part}^+$, i.e., partitions that give a positive-measure piece of cake to each player. Our reasons are as outlined in Chapter 10. (See the discussion following Example 10.12.) We note that this is not a serious restriction because of the following observation:

> Fix any $m = 1, 2, \ldots, n$. Partition $P = \langle P_1, P_2, \ldots, P_m\rangle$ is a strongly Pareto maximal partition of the cake among Player 1, Player 2, …, Player m if and only if partition $P' = \langle P_1, P_2, \ldots, P_m, \emptyset, \emptyset, \ldots, \emptyset\rangle$ is a strongly Pareto maximal partition of the cake among Player 1, Player 2, …, Player n. An analogous statement holds for strong Pareto minimality.

Our characterization theorem is the following.

Theorem 14.4 *Fix a partition* $P = \langle P_1, P_2, \ldots, P_n \rangle \in \mathrm{Part}^+$. *The following are equivalent:*

a. P *is strongly Pareto maximal.*
b. For every $\varphi \in \mathrm{CS}$, *either*
 i. $\mathrm{CP}(\varphi) < 1$ *or*
 ii. $\mathrm{CP}(\varphi) = 1$ *and at least one of the suprema in the definitions of the relevant* pr_{ij} *is not achieved.*
c. For some $\omega \in S^+$, P *is w-associated with* ω *and satisfies the cyclic boundary condition with respect to* ω.
d. P *is Pareto maximal and no non-trivial collection of transfers produces a partition that is p-equivalent to* P.
e. P *is Pareto maximal and its p-class consists of a single s-class (or, equivalently,* P *is Pareto maximal and* $[P]_p = [P]_s$*)).*
f. $m(P)$ *lies on the outer Pareto boundary of the* IPS *but does not lie in the interior of a line segment contained in the* IPS.

(For the definitions of CS and CP, see Definition 8.7. For the definition of CT, which we shall use in the proof, see Definition 8.1.)

Proof: We shall show that condition a implies condition b, condition b implies condition c, condition c implies condition d, condition d implies condition e, conditions e and f are equivalent, and condition e implies condition a.

To show that condition a implies condition b, suppose that condition b fails. Then for some $\varphi = \langle \mathrm{pr}_{k_t k_1}, \mathrm{pr}_{k_1 k_2}, \ldots, \mathrm{pr}_{k_{t-2} k_{t-1}}, \mathrm{pr}_{k_{t-1} k_t} \rangle \in \mathrm{CS}$ either $\mathrm{CP}(\varphi) > 1$ or else $\mathrm{CP}(\varphi) = 1$ and all of the suprema of the relevant pr_{ij} are achieved. If $\mathrm{CP}(\varphi) > 1$, then it follows from Theorem 8.9 that P is not Pareto maximal and, hence, is not strongly Pareto maximal. Thus condition a fails.

Suppose that $\mathrm{CP}(\varphi) = 1$ and all of the suprema of the relevant pr_{ij} are achieved. Then, for each $s = 1, 2, \ldots, t$, there exists a positive-measure $Q_{k_s} \subseteq P_{k_s}$ such that $\frac{m_{k_{s+1}}(Q_{k_s})}{m_{k_s}(Q_{k_s})} = \mathrm{pr}_{k_s k_{s+1}}$, where we set $k_{t+1} = k_1$. Then we have

$$\frac{m_{k_2}(Q_{k_1})}{m_{k_1}(Q_{k_1})}\frac{m_{k_3}(Q_{k_2})}{m_{k_2}(Q_{k_2})}\cdots\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_{k_1}(Q_{k_t})}{m_{k_t}(Q_{k_t})} = 1.$$

Rearranging terms, we have

$$\frac{m_{k_1}(Q_{k_t})}{m_{k_1}(Q_{k_1})}\frac{m_{k_2}(Q_{k_1})}{m_{k_2}(Q_{k_2})}\cdots\frac{m_{k_{t-1}}(Q_{k_{t-2}})}{m_{k_{t-1}}(Q_{k_{t-1}})}\frac{m_{k_t}(Q_{k_{t-1}})}{m_{k_t}(Q_{k_t})} = 1.$$

By applying Lemma 8.3 to the product on the left, we obtain a positive-measure $R_{k_s} \subseteq Q_{k_s}$ for each $s = 1, 2, \ldots, t$, such that the following equality holds and all but possibly the last of the fractions on the left of the equality are equal to one:

$$\frac{m_{k_1}(R_{k_t})}{m_{k_1}(R_{k_1})}\frac{m_{k_2}(R_{k_1})}{m_{k_2}(R_{k_2})}\cdots\frac{m_{k_{t-1}}(R_{k_{t-2}})}{m_{k_{t-1}}(R_{k_{t-1}})}\frac{m_{k_t}(R_{k_{t-1}})}{m_{k_t}(R_{k_t})}=1$$

(We do not use part d of the lemma here.) This implies that each of these fractions must equal one. It follows that $\mathrm{CT}(\langle k_1, k_2, \ldots, k_t\rangle \mid \langle R_{k_1}, R_{k_2}, \ldots, R_{k_t}\rangle)$ is a cyclic trade witnessing that $P = \langle P_1, P_2, \ldots, P_n\rangle$ is not strongly Pareto maximal and, hence, condition a fails. This establishes that condition a implies condition b.

To show that condition b implies condition c, we assume that condition b holds. Then, for any $\varphi \in \mathrm{CS}$, $\mathrm{CP}(\varphi) \leq 1$ and it follows from Theorem 8.9 that P is Pareto maximal. Theorem 10.9 then tells us that, for some $\omega \in S^+$, P is w-associated with ω. We wish to show that P satisfies the cyclic boundary condition with respect to ω.

Suppose, by way of contradiction, that P does not satisfy the cyclic boundary condition with respect to ω, and let $\langle k_1, k_2, \ldots, k_t\rangle$ and $\langle Q_{k_1}, Q_{k_2}, \ldots, Q_{k_t}\rangle$ be as in Definition 14.3. Then, for each $s = 1, 2, \ldots, t$, $Q_{k_s} \subseteq P_{k_s}$ and $f(Q_{k_s})$ is on the k_s, k_{s+1} boundary associated with ω. Setting $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, this implies that $\frac{m_{k_{s+1}}(Q_{k_s})}{m_{k_s}(Q_{k_s})} = \frac{\omega_{k_{s+1}}}{\omega_{k_s}}$. Since $Q_{k_s} \subseteq P_{k_s}$, it follows from the definition of partition ratio that $\frac{m_{k_{s+1}}(Q_{k_s})}{m_{k_s}(Q_{k_s})} \leq p_{k_s k_{s+1}}$, and Theorem 13.1 tells us that $p_{k_s k_{s+1}} \leq \frac{\omega_{k_{s+1}}}{\omega_{k_s}}$. Hence, we have $\frac{\omega_{k_{s+1}}}{\omega_{k_s}} = \frac{m_{k_{s+1}}(Q_{k_s})}{m_{k_s}(Q_{k_s})} \leq p_{k_s k_{s+1}} \leq \frac{\omega_{k_{s+1}}}{\omega_{k_s}}$, and consequently $\frac{m_{k_{s+1}}(Q_{k_s})}{m_{k_s}(Q_{k_s})} = p_{k_s k_{s+1}} = \frac{\omega_{k_{s+1}}}{\omega_{k_s}}$. Since, for each $s = 1, 2, \ldots, t$, $Q_{k_s} \subseteq P_{k_s}$, this tells us that each $p_{k_s k_{s+1}}$ supremum is achieved. Letting $\varphi = \langle \mathrm{pr}_{k_t k_1}, \mathrm{pr}_{k_1 k_2}, \ldots, \mathrm{pr}_{k_{t-2} k_{t-1}}, \mathrm{pr}_{k_{t-1} k_t}\rangle$, we have

$$\begin{aligned}\mathrm{CP}(\varphi) &= \mathrm{pr}_{k_t k_1}\mathrm{pr}_{k_1 k_2}\cdots\mathrm{pr}_{k_{t-2}k_{t-1}}\mathrm{pr}_{k_{t-1}k_t}\\ &= \left(\frac{\omega_{k_1}}{\omega_{k_t}}\right)\left(\frac{\omega_{k_2}}{\omega_{k_1}}\right)\cdots\left(\frac{\omega_{k_{t-1}}}{\omega_{k_{t-2}}}\right)\left(\frac{\omega_{k_t}}{\omega_{k_{t-1}}}\right)=1\end{aligned}$$

This contradicts condition b and, thus, establishes that condition b implies condition c.

To show that condition c implies condition d, we assume that condition d fails. If P is not Pareto maximal, then if follows from Theorem 10.9 that P is not w-associated with any $\omega \in S^+$ and, hence, condition c fails.

Assume then that P is Pareto maximal and some non-trivial collection of transfers produces a partition R that is p-equivalent to P. Since P is Pareto maximal, Theorem 10.9 implies that P is w-associated with some $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. Then R is also w-associated with ω. (This follows easily from Theorem 10.6. See the paragraph following the proof of the theorem.)

In the transition from partition P to partition R, a player receives some cake of positive measure if and only if this player gives up some cake of positive measure. This implies that this collection of transfers includes at least one cycle. In other words, there is a sequence $\langle k_1, k_2, \ldots, k_t \rangle$ of distinct elements of $\{1, 2, \ldots, n\}$ such that, for each $s = 1, 2, \ldots, t$, this collection of transfers includes a transfer of a positive-measure piece of cake Q_{k_s} from Player k_s to Player k_{s+1} (where, as before, we set $k_{t+1} = k_1$). We claim that the sequences $\langle k_1, k_2, \ldots, k_t \rangle$ and $\langle Q_{k_1}, Q_{k_2}, \ldots, Q_{k_t} \rangle$ witness that P does not satisfy the cyclic boundary condition with respect to ω. Clearly, for each $s = 1, 2, \ldots, t$, $Q_{k_s} \subseteq P_{k_s}$ since Q_{k_s} is transferred from Player k_s. We must show that, for each such s, $f(Q_{k_s})$ is on the k_s,k_{s+1} boundary associated with ω. In other words (see Definition 12.10), we must show that for almost every $a \in Q_{k_s}$

i. $\frac{f_{k_s}(a)}{f_{k_{s+1}}(a)} = \frac{\omega_{k_s}}{\omega_{k_{s+1}}}$ and
ii. for any $i = 1, 2, \ldots, n$ with $i \neq k_s$ and $i \neq k_{s+1}$, $\frac{f_{k_s}(a)}{f_i(a)} \geq \frac{\omega_{k_s}}{\omega_i}$ and $\frac{f_{k_{s+1}}(a)}{f_i(a)} \geq \frac{\omega_{k_{s+1}}}{\omega_i}$.

Fix some $s = 1, 2, \ldots, t$. For condition i, we note that, since P is w-associated with ω and $Q_{k_s} \subseteq P_{k_s}$, it follows that $\frac{f_{k_s}(a)}{f_{k_{s+1}}(a)} \geq \frac{\omega_{k_s}}{\omega_{k_{s+1}}}$ for almost every $a \in Q_{k_s}$. Similarly, since R is w-associated with ω and $Q_{k_s} \subseteq R_{k_{s+1}}$, it follows that $\frac{f_{k_{s+1}}(a)}{f_{k_s}(a)} \geq \frac{\omega_{k_{s+1}}}{\omega_{k_s}}$ for almost every $a \in Q_{k_s}$. Hence, for almost every such a, $\frac{f_{k_s}(a)}{f_{k_{s+1}}(a)} \geq \frac{\omega_{k_s}}{\omega_{k_{s+1}}} \geq \frac{f_{k_s}(a)}{f_{k_{s+1}}(a)}$ and, therefore, $\frac{f_{k_s}(a)}{f_{k_{s+1}}(a)} = \frac{\omega_{k_s}}{\omega_{k_{s+1}}}$. This establishes condition i. The inequalities in condition ii follow immediately from the fact that P and R are each w-associated with ω.

We have shown that P does not satisfy the cyclic boundary condition with respect to ω and, hence, condition c fails. This establishes that condition c implies condition d.

To show that condition d implies condition e, assume that condition e fails. If P is not Pareto maximal, then obviously condition d fails. Assume then that P is Pareto maximal and its p-class consists of at least two s-classes. Let Q be a partition that belongs to P's p-class but not to P's s-class. Then P and Q are p-equivalent and the transition from P to Q involves non-trivial transfers. This implies that condition d fails.

To show that conditions e and f are equivalent, we first recall that P is Pareto maximal if and only if $m(P)$ lies on the outer Pareto boundary of the IPS. And (the equivalence class version of) Theorem 4.4 tells us that the p-class of P is made up of a single s-class if and only if $m(P)$ does not lie in the interior of a line segment contained in the IPS. This establishes that conditions e and f are equivalent.

To show that condition e implies condition a, assume that condition a fails. Then there is a non-trivial collection of transfers that produces a partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ where $m_i(Q_i) \geq m_i(P_i)$ for each $i = 1, 2, \ldots, n$. If, for some such i, $m_i(Q_i) > m_i(P_i)$, then P is not Pareto maximal and hence condition e fails. Suppose then that $m_i(Q_i) = m_i(P_i)$ for each $i = 1, 2, \ldots, n$. Then P and Q are p-equivalent. Since the collection of transfers that produces Q from P is non-trivial, it follows that P and Q are not s-equivalent. These two facts imply that P's p-class consists of more than one s-class and, hence, condition e fails. This establishes that condition e implies condition a and completes the proof of the theorem. □

Corollary 14.5 *Fix a partition* $P = \langle P_1, P_2, \ldots, P_n \rangle \in \text{Part}^+$. *The following are equivalent:*

a. P is Pareto maximal but not strongly Pareto maximal.
b. The following two conditions hold:
   i. For all $\varphi \in$ CS, $\text{CP}(\varphi) \leq 1$.
   ii. There exists $\varphi \in$ CS such that $\text{CP}(\varphi) = 1$ and all of the suprema of the relevant pr_{ij} are achieved.
c. For some $\omega \in S^+$, P is w-associated with ω and does not satisfy the cyclic boundary condition with respect to ω.
d. P is Pareto maximal and some non-trivial collection of transfers produces a partition that is p-equivalent to P.
e. P is Pareto maximal and its p-class is the union of at least two s-classes (or, equivalently, P is Pareto maximal and $[P]_p \neq [P]_s$).
f. $m(P)$ lies in the interior of a line segment on the outer Pareto boundary of the IPS.

Proof: The proofs that condition a implies condition b, condition b implies condition c, condition c implies condition d, condition d implies condition e, conditions e and f are equivalent, and condition e implies condition a are each either an immediate consequence of the theorem or else require only a straightforward additional argument. We omit the details. □

We note that by (the equivalence class version of) Theorem 4.4, the "at least two" in part e of the corollary can be replaced by "infinitely many."

By Theorem 12.14, some two players are in relative agreement on some set of positive measure (or, equivalently, the RNS is concentrated) if and only if there is a line segment on the outer Pareto boundary of the IPS. (For the definitions of concentrated and of relative agreement see Definitions 12.9 and 12.11, respectively.) It follows from the equivalence of part a and f of the

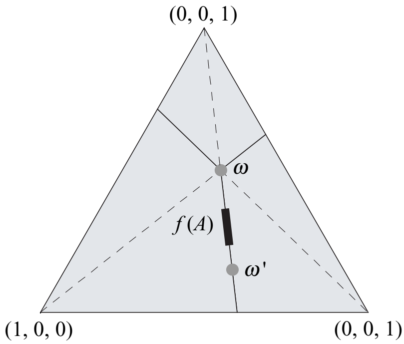


Figure 14.2

corollary that this occurs if and only if there is a Pareto maximal partition that is not strongly Pareto maximal. We shall use this idea in the proof of Theorem 14.14.

The idea in the previous paragraph connects nicely with the notion of w-association. We first consider the case of three players and then generalize to more than three players. Consider Figure 14.2. Assume that Player 1 and Player 2 are in relative agreement on some positive-measure set A. Then $f(A)$ lies on a line and this line contains Player 3's vertex, as indicated in the figure. By the preceding paragraph, there exists a partition that is Pareto maximal but not strongly Pareto maximal. It is not hard to see that there is such a partition that is w-associated with ω, where ω is as indicated in the figure. In fact, *any* partition that is w-associated with ω and does not give almost all of A to Player 1 or almost all of A to Player 2 is Pareto maximal but not strongly Pareto maximal. (Given A and $f(A)$ as in the figure, it is not the case that any ω on the line containing $f(A)$ will work. The point ω must be chosen so that it is not the case that almost all of A goes to Player 3. So, for example, the ω' shown in the figure will not work.) Thus we see that in the case of three players there exists a partition that is Pareto maximal but not strongly Pareto maximal if and only if the set of points of C associated with some line containing one of the vertices of S has positive measure. Or, more specifically: for any $\omega \in S^+$, there exists a partition that is Pareto maximal but not strongly Pareto maximal and is w-associated with ω if and only if, for some distinct $i, j = 1, 2, 3$, the i, j boundary associated with ω corresponds to a piece of cake of positive measure.

Noting that a line is a one-dimensional subset of $\mathbf{R}^3$, we see how to generalize the ideas in the preceding paragraph to the context of n players for $n > 3$. For any such n, there exists a partition that is Pareto maximal but not strongly Pareto maximal if and only if the set of points of C associated with some

$(n-2)$-dimensional subset of $\mathbf{R}^n$ containing all but two of the vertices of S has positive measure. Or, more specifically: for any $\omega \in S^+$, there exists a partition that is Pareto maximal but not strongly Pareto maximal and is w-associated with ω if and only if, for some $i, j = 1, 2, \ldots n$, the i, j boundary associated with ω corresponds to a piece of cake of positive measure.

We close this section by stating the chores versions of the two main results of this section, Theorem 14.4 and Corollary 14.5. Recall that a chores partition ratio, qr_{ij} (see Definition 8.12), is an infimum. The definition of "a qr_{ij} infimum is achieved" is the natural adjustment of "a pr_{ij} supremum is achieved," given in Definition 14.2. We also recall that CCS denotes the set of all chores cyclic sequences and that if $\varphi = \langle \mathrm{qr}_{i_t i_1}, \mathrm{qr}_{i_1 i_2}, \ldots, \mathrm{qr}_{i_{t-2} i_{t-1}}, \mathrm{qr}_{i_{t-1} i_t} \rangle \in$ CCS, then the chores cyclic product of φ, denoted by CCP(φ), is the product $\mathrm{qr}_{i_t i_1} \mathrm{qr}_{i_1 i_2} \cdots \mathrm{qr}_{i_{t-2} i_{t-1}} \mathrm{qr}_{i_{t-1} i_t}$. We shall need to revise the definition of cyclic boundary condition with respect to ω (Definition 14.3) for our present context, and to do so, we shall first need to revise the definition of the i, j boundary associated with ω (Definition 12.10).

Definition 14.6 Fix $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^+$. For distinct $i, j = 1, 2, \ldots, n$, a point $p = (p_1, p_2, \ldots, p_n)$ is on the *chores i, j boundary associated with ω* if and only if

a. $\frac{p_i}{p_j} = \frac{\omega_i}{\omega_j}$ and
b. for any $k = 1, 2, \ldots, n$ with $k \neq i$ and $k \neq j$, $\frac{p_i}{p_k} \leq \frac{\omega_i}{\omega_k}$ and $\frac{p_j}{p_k} \leq \frac{\omega_j}{\omega_k}$.

Definition 14.7 Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ and a point $\omega \in S^+$. We shall say that P satisfies the *chores cyclic boundary condition with respect to ω* if and only if for no sequence $\langle k_1, k_2, \ldots, k_t \rangle$ of distinct elements of $\{1, 2, \ldots, n\}$ does there exist a sequence $\langle Q_{k_1}, Q_{k_2}, \ldots, Q_{k_t} \rangle$ of subsets of C of positive measure such that, for each $s = 1, 2, \ldots, t$, $Q_{k_s} \subseteq P_{k_s}$ and $f(Q_{k_s})$ is on the chores k_s, k_{s+1} boundary associated with ω, where we set $k_{t+1} = k_1$.

The natural adjustments of Theorem 14.4 and Corollary 14.5 to the chores context are the following. The proofs are analogous and we omit them.

Theorem 14.8 *Fix a partition $P = \langle P_1, P_2, \ldots, P_n \rangle \in$ Part$^+$. The following are equivalent:*

a. P is strongly Pareto minimal.
b. For every $\varphi \in$ CCS, either
  i. CCP(φ) > 1 *or*
  ii. CCP(φ) $= 1$ *and at least one of the infima in the definitions of the relevant* qr_{ij} *is not achieved.*

c. *For some* $\omega \in S^+$, *P is chores w-associated with ω and satisfies the chores cyclic boundary condition with respect to ω.*
d. *P is Pareto minimal and no non-trivial collection of transfers produces a partition that is p-equivalent to P.*
e. *P is Pareto minimal and its p-class consists of a single s-class (or, equivalently, P is Pareto minimal and $[P]_p = [P]_s$).*
f. *$m(P)$ lies on the inner Pareto boundary of the* IPS *but does not lie in the interior of a line segment contained in the* IPS.

Corollary 14.9 *Fix a partition* $P = \langle P_1, P_2, \ldots, P_n \rangle \in \text{Part}^+$. *The following are equivalent:*

a. *P is Pareto minimal but not strongly Pareto minimal.*
b. *The following two conditions hold:*
   i. *For all* $\varphi \in \text{CCS}$, $\text{CCP}(\varphi) \geq 1$.
   ii. *There exists* $\varphi \in \text{CCS}$ *such that* $\text{CCP}(\varphi) = 1$ *and all of the infima of the relevant* qr_{ij} *are achieved.*
c. *For some* $\omega \in S^+$, *P is chores w-associated with ω and does not satisfy the chores cyclic boundary condition with respect to ω.*
d. *P is Pareto minimal and some non-trivial collection of transfers produces a partition that is p-equivalent to P.*
e. *P is Pareto minimal and its p-class is the union of at least two s-classes (or, equivalently, P is Pareto minimal and $[P]_p \neq [P]_s$).*
f. *$m(P)$ lies in the interior of a line segment on the inner Pareto boundary of the* IPS.

14C. Existence Questions in the Two-Player Context

If two partitions are p-equivalent and one is strongly Pareto maximal, then so is the other. (This follows easily from the equivalence of condition a with either condition e or condition f of Theorem 14.4.) Thus, we can refer to strongly Pareto maximal p-classes and non-strongly Pareto maximal p-classes, and therefore we can refer to strongly Pareto maximal points and non-strongly Pareto maximal points of the IPS. Similarly, we can refer to strongly Pareto minimal points and non-strongly Pareto minimal points of the IPS. In this section, we consider existence questions. More specifically, we consider the following three questions:

a. What are the possible numbers of strongly Pareto maximal points?
b. What are the possible numbers of Pareto maximal points that are not strongly Pareto maximal?
c. What is the relationship between the answers to questions a and b?

Of course, these "numbers" can be infinite. In what follows, we shall distinguish between countably infinite and uncountably infinite.

By the equivalence of conditions a and f of Theorem 14.4, a point of the IPS is strongly Pareto maximal if and only if it lies on the outer Pareto boundary of the IPS but does not lie in the interior of a line segment contained in the IPS. We shall use this result, together with our work on the possible shapes of the IPS in Chapter 11, to answer the three preceding questions for the two-player context. In the next section, we consider the general n-player context.

Absolute continuity implies that the partitions $\langle C, \emptyset\rangle$ and $\langle \emptyset, C\rangle$ are each strongly Pareto maximal. Obviously, these two partitions are not p-equivalent. Hence, they correspond to distinct p-classes and different points in the IPS (namely $(1, 0)$ and $(0, 1)$). It follows that for every cake C and corresponding measures m_1 and m_2, there are at least two strongly Pareto maximal p-classes and the corresponding IPS has a least two strongly Pareto maximal points.

We are now ready to completely answer the preceding questions, for the two-player context.

Theorem 14.10 *For each of the following conditions, there exists a cake C and measures m_1 and m_2 on* C *such that the given condition is satisfied:*

	Number of Strongly Pareto Maximal Points	*Number of Pareto Maximal Points That Are Not Strongly Pareto Maximal*
a.	*Any finite $k \geq 2$*	*Uncountably infinite*
b.	*Countably infinite*	*Uncountably infinite*
c.	*Uncountably infinite*	0
d.	*Uncountably infinite*	*Uncountably infinite*

Also, this list is complete in the sense that any other combination of numbers for the two given types of points is impossible.

Before proving Theorem 14.10, we re-examine Theorem 11.1, which characterized the possible shapes of the IPS in the two-player context. We recall that measures m_1 and m_2 are absolutely continuous with respect to each other if and only if there are no points in the corresponding IPS that are on the line segment between $(1, 0)$ and $(1, 1)$ or on the line segment between $(0, 1)$ and $(1, 1)$, other than $(1, 0)$ and $(0, 1)$. Then, since we only consider measures in this section that are absolutely continuous with respect to each other, we may revise Theorem 11.1 as follows.

Theorem 14.11 *Let G be a subset of $\mathbf{R}^2$. There exists a cake C and measures m_1 and m_2 on C such that G is the* IPS *associated with the cake C and the measures m_1 and m_2 if and only if G*

a. is a subset of $[0, 1]^2$,
b. contains the points $(1, 0)$ *and* $(0, 1)$,
c. is closed,
d. is convex,
e. is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$, *and*
f. contains no points on the line segment between $(1, 0)$ *and* $(1, 1)$ *or on the line segment between* $(0, 1)$ *and* $(1, 1)$, *other than* $(1, 0)$ *and* $(0, 1)$.

Proof of Theorem 14.10: We recall (see Theorem 3.9) that in the two-player context, when the measures are absolutely continuous with respect to each other, the outer boundary of the IPS is equal to the outer Pareto boundary of the IPS. Clearly the number of points on the outer boundary is uncountably infinite.

By the equivalence of conditions a and f of Theorem 14.4, a Pareto maximal point is strongly Pareto maximal if and only if it does not lie in the interior of a line segment contained in the IPS. Thus, to establish that situations a through d are possible, we need only exhibit a set $G \subseteq \mathbf{R}^2$ such that

- G satisfies the six conditions of Theorem 14.11,
- the number of points on the outer boundary of G that do not lie in the interior of a line segment contained in the IPS is equal to the desired number of strongly Pareto maximal points, and
- the number of points on the outer boundary of G that lie in the interior of a line segment contained in the IPS is equal to the desired number of Pareto maximal points that are not strongly Pareto maximal.

Such sets G are easy to exhibit. Some examples are shown in Figure 14.3. In each of these figures, we have displayed an IPS satisfying the six conditions of Theorem 14.11 and have darkened the outer boundary. Figures 14.3ai, 14.3aii, and 14.3aiii correspond to situation a of the theorem with $k = 3$, $k = 4$, and $k = 7$, respectively. In each of these figures, the outer boundary consists of a finite set of line segments. The corner points where two such line segments meet, together with the points $(1, 0)$ and $(0, 1)$, are the only points on the outer boundary of the IPS that do not lie in the interior of a line segment contained in the IPS. Since the total number of Pareto maximal points is uncountably infinite and the number of strongly Pareto maximal points is finite, it follows that the number of Pareto maximal points that are not strongly Pareto maximal is uncountably infinite.

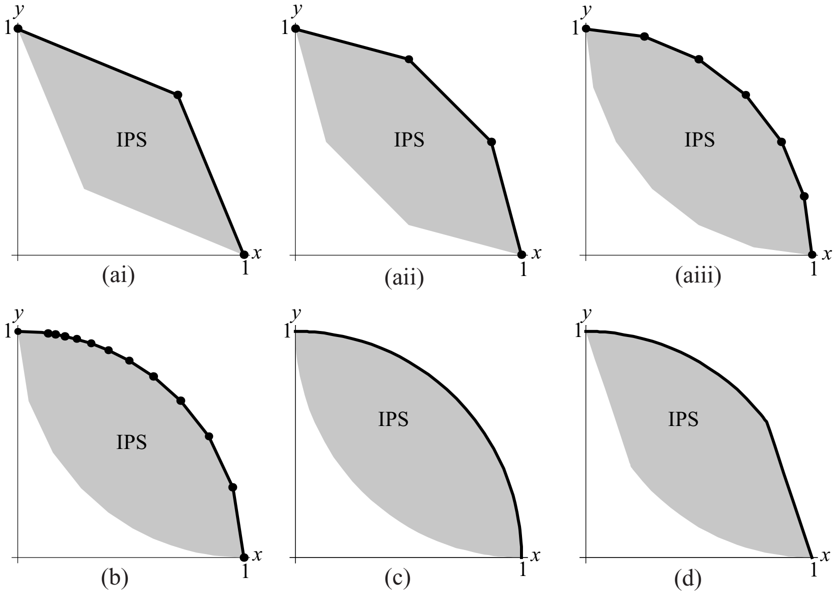


Figure 14.3

Figure 14.3b corresponds to situation b of the theorem. The number of corner points on the outer boundary of this IPS is countably infinite. (We have highlighted only a finite number of these corner points. As we proceed from the point (1, 0) to the point (0, 1), the line segments shorten in length and change their slope.) Thus, the number of strongly Pareto maximal points is countably infinite, and it follows that the number of Pareto maximal points that are not strongly Pareto maximal is uncountably infinite.

Figure 14.3c corresponds to situation c of the theorem. The outer boundary of this IPS consists of a (non-straight line) curve and hence contains no line segments. Then certainly no point along this outer boundary lies in the interior of a line segment contained in the IPS. Consequently, there are no Pareto maximal points that are not strongly Pareto maximal, and the number of strongly Pareto maximal points is uncountably infinite.

Figure 14.3d corresponds to situation d of the theorem. The outer boundary of this IPS consists of two pieces. The (non-straight line) curve making up the upper-left piece guarantees that the number of strongly Pareto maximal points is uncountably infinite, as in situation c, while the line segment making up the lower-right piece guarantees that the number of Pareto maximal points that are not strongly Pareto maximal is also uncountably infinite.

It remains for us to show that any combination of numbers not given by situations a through d is impossible. We first consider the possible numbers of strongly Pareto maximal points, then the possible number of Pareto maximal points that are not strongly Pareto maximal, and lastly the combination of these numbers.

The only omissions in the "Number of Strongly Pareto Maximal Points" list are "0" and "1." But we have previously shown that there must be at least two strongly Pareto maximal points (namely the points (1,0) and (0,1)), and so "0" and "1" are impossible.

Next, we must show that "uncountably infinite" and "0" are the only possibilities for the number of Pareto maximal points that are not strongly Pareto maximal. This is so since the outer boundary of the IPS either contains a line segment or else it does not. It follows from the equivalence of conditions a and f of Corollary 14.5 that, in the first case, there are uncountable many such points and, in the second case, there are none.

Finally, we consider the relationship between the number of strongly Pareto maximal points and the number of Pareto maximal points that are not strongly Pareto maximal. Since the number of Pareto maximal points is uncountably infinite, it follows that if the number of strongly Pareto maximal points is finite or countably infinite, then the number of Pareto maximal points that are not strongly Pareto maximal must be uncountably infinite, as stated in the theorem. On the other hand, if the number of strongly Pareto maximal points is uncountably infinite, then the theorem tells us that both "0" and "uncountably infinite" are possibilities for the number of Pareto maximal points that are not strongly Pareto maximal. Thus, the theorem gives a complete list of all such possibilities. □

We observe that the theorem, combined with the equivalence of parts a and e of Theorem 14.4, the equivalence of parts a and e of Corollary 14.5, and the (equivalence class version of) Theorem 2.6, implies that there can be Pareto maximal partitions whose p-class is the union of infinitely many s-classes, and there can be Pareto maximal partitions whose p-class is a single s-class. We discussed this possibility at the end of Section 5B. Theorem 14.13 will shed additional light on this issue.

The chores version of Theorem 14.10 is the following. The proof is analogous and we omit it.

Theorem 14.12 *For each of the following conditions, there exists a cake C and measures m_1 and m_2 on C such that the given condition is satisfied:*

	Number of Strongly Pareto Minimal Points	Number of Pareto Minimal Points That Are Not Strongly Pareto Minimal
a.	*Any finite* $k \geq 2$	*Uncountably infinite*
b.	*Countably infinite*	*Uncountably infinite*
c.	*Uncountably infinite*	0
d.	*Uncountably infinite*	*Uncountably infinite*

Also, this list is complete in the sense that any other combination of numbers for the two given types of points is impossible.

We conclude this section by considering a topic that is not an existence result. It is closely related to ideas considered in this section, so we include it here. We have seen that every p-class is the union of s-classes and, by the equivalence of parts a and e of Theorem 14.4 and of Theorem 14.8, a Pareto maximal or a Pareto minimal p-class is strongly Pareto maximal or strongly Pareto minimal, respectively, if and only if it consists of a single s-class. This, together with (the equivalence class version of) Theorem 2.6, enables us to completely characterize which p-classes are single s-classes and which are unions of more than one s-class. We recall that each point in the IPS corresponds to a single p-class.

Theorem 14.13 *A p-class consists of a single s-class if and only it is either strongly Pareto maximal or strongly Pareto minimal.*

Proof: Fix some p-class and suppose that q is the corresponding point in the IPS.

For the forward direction, suppose that q is neither strongly Pareto maximal nor strongly Pareto minimal. We consider three cases.

Case 1: q is Pareto maximal. By the equivalence of conditions a and e of Theorem 14.4, the p-class corresponding to q is the union of at least two s-classes.

Case 2: q is Pareto minimal. By the equivalence of conditions a and e of Theorem 14.8, the p-class corresponding to q is the union of at least two s-classes.

Case 3: q is neither Pareto maximal nor Pareto minimal. Then q is not on the outer boundary or the inner boundary of the IPS. Hence, q is not on the boundary of the IPS and thus is an interior point of the IPS. Then certainly q lies in the interior of a line segment contained in the IPS and therefore (the equivalence class version of) Theorem 2.6 implies that the p-class corresponding to q is the union of at least two s-classes.

The reverse direction follows immediately from the equivalence of conditions a and e of Theorem 14.4 and the equivalence of conditions a and e of Theorem 14.8. □

It follows from (the equivalence class version of) Theorem 2.6 that if the conditions of the theorem fail, then the p-class corresponding to q is the union of infinitely many s-classes.

14D. Existence Questions in the General n-Player Context

In this section, we consider existence questions in the general n-player context. As in the previous section, we shall be concerned with the relationship between the number of strongly Pareto maximal points and the number of Pareto maximal points that are not strongly Pareto maximal. Unfortunately, we do not have a precise adjustment of Theorem 14.10. The proof of the this result relied on Theorem 11.1 and, as discussed in Chapter 11, the obvious generalization of this theorem to the n-player context is false, and we do not have a complete picture of the possible shapes of the IPS in this general context. Thus, we are not in a position to generalize Theorem 14.10 precisely. However, we shall come close to doing so in Theorem 14.14, although the proof is quite different.

We begin by noting that the proof of the last statement of Theorem 14.10 applies generally, not just in the two-player context. Thus, any combination of numbers not given by Theorem 14.10 for the two types of points is not possible. So the question is: of those combinations given by the theorem, which are possible in the general n-player context? Our answer is Theorem 14.14. It is almost the same as Theorem 14.10. Situations b, c, and d are identical but, because of our inability to use Theorem 11.1, situation a is weaker. For situation a, we have substituted "arbitrarily large finite" for "any finite $k \geq 2$." Since giving all of the cake to any one player results in a strongly Pareto maximal partition, and giving all of the cake to different players results in non-p-equivalent partitions, it follows that when there are n players there are at least n strongly Pareto maximal points in the IPS. However, we do not know, for example, whether, when $n = 10$, there exists a cake C and measures m_1, $m_2, \ldots, m_{10}$ on C such that there are exactly seventeen strongly Pareto maximal points.

Theorem 14.14 *Fix $n \geq 2$. For each of the following conditions, there exists a cake C and measures $m_1, m_2, \ldots, m_n$ on C such that the given condition is satisfied:*

	Number of Strongly Pareto Maximal Points	*Number of Pareto Maximal Points That Are Not Strongly Pareto Maximal*
a.	*Arbitrarily large finite*	*Uncountably infinite*
b.	*Countably infinite*	*Uncountably infinite*
c.	*Uncountably infinite*	0
d.	*Uncountably infinite*	*Uncountably infinite*

Also, this list is complete in the sense that any other combination of numbers for the two given types of points is impossible, with one exception. We do not know whether, in general, for a given (finite) number k, there exists a cake and corresponding measures such that there are exactly k strongly Pareto maximal points (and thus an uncountably infinite number of Pareto maximal points that are not strongly Pareto maximal).

The proof of Theorem 14.14 is somewhat harder than the proof of Theorem 14.10, since we have to actually construct each example, rather than being able to use Theorem 11.1 to obtain examples of IPSs having the desired properties.

Proof of Theorem 14.14: We have already observed that the proof of the completeness of the list (with the exception noted) is exactly as in the proof of Theorem 14.10. For situations a, b, and part of d, we shall use Theorem 14.4. The equivalence of parts a and f of this result tells us that it suffices to show that, for some cake C and measures $m_1, m_2, \ldots, m_n$ on C,

- the number of points on the outer Pareto boundary of the corresponding IPS that do not lie in the interior of a line segment contained in the IPS is equal to the desired number of strongly Pareto maximal points, and
- the number of points on the outer Pareto boundary of the corresponding IPS that lie in the interior of a line segment contained in the IPS is equal to the desired number of Pareto maximal points that are not strongly Pareto maximal.

By Theorem 5.18, we know that the number of Pareto maximal points is infinite. In particular, this result tells us that, for any $p \in \mathbf{R}^n$ with all non-negative coordinates and at least one positive coordinate, there is a positive number λ such that λp is a Pareto maximal point. There is obviously an uncountably infinite number of choices for p, and it is easy to see that each such choice results in a different Pareto maximal point. Hence, the number of Pareto maximal points is uncountably infinite. Thus, for situations a and b, it suffices to show that, for some cake C and measures $m_1, m_2, \ldots, m_n$ on C, the corresponding IPS has the desired number of strongly Pareto maximal points, since then the number of Pareto maximal points that are not strongly Pareto maximal will certainly be uncountably infinite.

For situation a, fix any $k > 0$. We must show that there is a cake C and measures $m_1, m_2, \ldots, m_n$ on C so that the number of points on the outer Pareto boundary of the IPS that are not interior points of line segments of the IPS is finite and at least k.

We first define a cake C and measures $m_1, m_2, \ldots, m_n$ on C so that the RNS consists of k points (each of which corresponds to a positive-measure piece of cake). We do so in a manner similar to the construction used in previous examples (see, for example, Examples 13.22 and 13.23). Let C be the interval $[0, k)$ and let m_L be Lebesgue measure on C. Fix $\alpha^1, \alpha^2, \ldots, \alpha^n \in S^+$, where S is the $(k-1)$-simplex and S^+ is its interior. (We shall discuss how to choose the α^i shortly.) For each $i = 1, 2, \ldots, n$, let $\alpha^i = (\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_k)$ and define m_i on C as follows: for any $A \subseteq C$,

$$m_i(A) = \alpha^i_1 m_L(A \cap [0, 1)) + \alpha^i_2 m_L(A \cap [1, 2)) + \cdots + \alpha^i_k m_L(A \cap [k-1, k))$$

It is straightforward to verify that each m_i is a (countably additive, non-atomic, probability) measure on C and, for each $j = 1, 2, \ldots, k$ and almost every $a \in [j-1, j)$, $f(a) = (\frac{1}{a^1_j + a^2_j + \cdots + a^n_j})(a^1_j, a^2_j, \ldots, a^n_j)$. (See the discussion preceding Example 13.22.) By redefining f on a set of measure zero if necessary, we may assume that, for *every* $a \in [j-1, j)$, $f(a) = (\frac{1}{a^1_j + a^2_j + \cdots + a^n_j})(a^1_j, a^2_j, \ldots, a^n_j)$. It is easy to see that we can choose the α^i so that the points $(\frac{1}{a^1_j + a^2_j + \cdots + a^n_j})(a^1_j, a^2_j, \ldots, a^n_j)$, for $j = 1, 2, \ldots, k$, are distinct. This implies that the RNS (which is the range of f) consists of k distinct points.

Using the ideas developed in Chapter 12, it is not hard to see that

- each of the k points defined in the previous paragraph corresponds to a "flat" region on the outer Pareto boundary of the IPS. In other words, each such point corresponds to a convex region on the outer Pareto boundary of maximal (i.e., $n - 1$) dimension.
- there are no curved regions on the outer Pareto boundary of the IPS.

In particular, the outer Pareto boundary of the IPS is the union of at least k flat regions. (There will, in general, be more than k such regions. We discussed this issue in Chapter 12, following the proof of Theorem 12.14, and illustrated it in Figure 12.5.) It follows that a point on the outer Pareto boundary of the IPS is not an interior point of a line segment contained in the IPS if and only if it is a jagged point of the IPS. Jagged points correspond to the open regions of the RNS that we discussed in Chapter 12. (See the discussion following the statement of Theorem 12.18 and see Figure 12.7.) Clearly, the number of such regions is finite and at least k. This establishes that situation a is possible.

The proof for situation b is similar. We must show that there is a cake C and measures $m_1, m_2, \ldots, m_n$ on C so that the number of points on the outer Pareto boundary of the corresponding IPS that are not interior points of line segments contained in the IPS is countably infinite. We begin by revising our preceding construction so that the resulting RNS consists of a countably infinite collection of points (with each such point corresponding to a piece of cake of positive measure).

Let C be the interval $[0, \infty)$ and let m_L be Lebesgue measure on C. Fix $\alpha^1, \alpha^2, \ldots \alpha^n$, where each α^i is an infinite sequence of positive real numbers that sum to one. (We shall discuss how to choose the α^i shortly.) For each $i = 1, 2, \ldots, n$, let $\alpha^i = (\alpha^i_1, \alpha^i_2, \alpha^i_3, \ldots)$ and define m_i on C as follows: for any $A \subseteq C$,

$$m_i(A) = \alpha^i_1 m_L(A \cap [0, 1)) + \alpha^i_2 m_L(A \cap [1, 2)) + \alpha^i_3 m_L(A \cap [2, 3)) + \cdots$$

It is straightforward to verify that each m_i is a (countably additive, non-atomic, probability) measure on C. For each j and almost every $a \in [j-1, j)$, $f(a) = (\frac{1}{a^1_j + a^2_j + \cdots + a^n_j})(a^1_j, a^2_j, \ldots, a^n_j)$ and, by redefining f on a set of measure zero, if necessary, we may assume that, for *every* $a \in [j-1, j)$, $f(a) = (\frac{1}{a^1_j + a^2_j + \cdots + a^n_j})(a^1_j, a^2_j, \ldots, a^n_j)$. As before, it is easy to see that we can choose the α^i so that the points $(\frac{1}{a^1_j + a^2_j + \cdots + a^n_j})(a^1_j, a^2_j, \ldots, a^n_j)$, for different j, are distinct. This implies that the RNS consists of a countably infinite collection of points.

We shall need to look at an infinite subcollection of the collection of points just defined. We want the collection of points in the RNS to contain none of its limit points. It is easy to find an infinite subcollection satisfying this property. (For example, if the previously defined collection does contain a limit point, consider some sequence of points from this collection that converges to the given limit point, and keep just the points in this sequence, not including the limit point. This collection will contain no limit points of itself.) Next we simply redefine the cake by throwing out any cake associated with points of the RNS that we have discarded and scale the measures accordingly. This may change

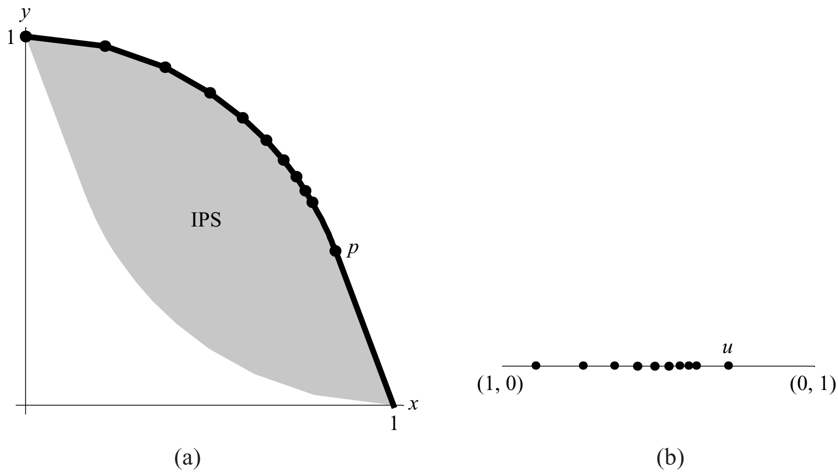


Figure 14.4

the location of the points that make up the RNS, but it will not change the fact that the RNS consists of a countably infinite collection of points that includes none of its limit points.

The proof for situation b continues to parallel that of situation a. We must show that the number of points on the outer Pareto boundary of the IPS that are not interior points of line segments contained in the IPS is countably infinite. As in our proof for situation a, we claim that a point on the outer Pareto boundary of the IPS is not an interior point of a line segment contained in the IPS if and only if it is a jagged point of the IPS. However, this fact is slightly harder to see now than in our previous setting.

It is certainly true that, just as the outer Pareto boundary of the IPS consisted of the union of at least k flat regions in our proof for situation a, in our present setting the outer Pareto boundary of the IPS consists of the union of a countably infinite collection of flat regions. However, this alone does not allow us to conclude that a point on the outer Pareto boundary of the IPS is a jagged point of the IPS if and only if it is not an interior point of a line segment contained in the IPS. Consider Figure 14.4, where we assume that there are two players. Figure 14.4a shows an IPS and Figure 14.4b shows the corresponding RNS. We assume that the RNS of Figure 14.4b includes a countable sequence of points that converges to the point u, which is also in the RNS. (In the figure, we have shown just nine points of this countable sequence.) As we have previously discussed (see Example 12.3 and Figures 12.1bi and 12.2bii), each such point in the RNS corresponds to a line segment on the outer Pareto boundary of the IPS. In particular, the first point in this sequence corresponds to the first line segment in the upper left of the outer Pareto boundary of the IPS in Figure 14.4a, the second point in the RNS sequence corresponds to the second line segment on the outer Pareto boundary of the IPS, etc. The limit of these line segments is the point p. The point u in the RNS corresponds to the line segment in the lower right of the outer Pareto boundary of the IPS, i.e., the line segment between the point p and the point $(1, 0)$. Every point in the RNS that is not equal to u corresponds to a line segment on the outer Pareto boundary of the IPS that has slope not equal to that of the line segment corresponding to u. It follows that p is not an interior point of a line segment contained in the IPS.

It is also not hard to see that p is not a corner point on the outer Pareto boundary of the IPS. (Recall that "corner point" is the two-player version of "edge point" and that a jagged point is a special case of an edge point.) One way to see this is to note that, since the sequence of points in the RNS approaches u, the corresponding sequence of slopes of line segments on the outer Pareto boundary of the IPS is a decreasing sequence (becoming more and more negative) that approaches the slope of the line segment corresponding to u. An equivalent perspective is provided by recalling Observation 12.5: a corner point on the outer Pareto boundary of the IPS corresponds to a gap in the RNS. Since there is no gap between u and the countable sequence of points in the RNS converging to u, it follows that p is not a corner point.

Thus, we see that a point on the outer Pareto boundary of the IPS that is not an interior point of a line segment contained in the IPS is not necessarily a jagged point of the IPS. Returning to the proof of situation b and the RNS (which consists of a countably infinite number of points) and IPS (whose outer Pareto boundary is the union of a countably infinite collection of flat regions) that we defined previously, we see that the situation described in the preceding two paragraphs does not arise in our present context, since we have made sure that no point in the RNS is the limit of points in the RNS. Hence, a point on the outer Pareto boundary of the IPS is not an interior point of a line segment contained in the IPS if and only if it is a jagged point of the IPS. We must show that the number of jagged points on the outer Pareto boundary of the IPS is countably infinite.

As we did for situation a, we shall again use the fact that jagged points of the IPS correspond to certain open regions of the RNS of the type discussed in Chapter 12. Since the countable collection of points that make up the RNS contains none of its limit points, it follows that there are countably many such regions and hence countably many jagged points. (The idea here is that if we consider Figure 12.7 and imagine countably many points making up the RNS instead of three points, then the lack of limit points implies that we will have the same sort of open regions as in the figure, just more of them. For an extreme but simple example of how limit points in the RNS can cause problems, suppose that the RNS consists of all points in S^+ with all rational coordinates. This is a countable set, but there are no open regions of the type under consideration and illustrated in Figure 12.7. Hence, the corresponding IPS has no jagged points.) This establishes that situation b is possible.

Next, we turn to situation c. Since we know that there are always uncountably many Pareto maximal points, it suffices to define a cake C and corresponding measures $m_1, m_2, \ldots, m_n$ so that every Pareto maximal point is strongly Pareto maximal.

We define the cake C in a rather different manner from that used in previous constructions. We shall define C to be a certain subset of the simplex (actually, the entire interior of the simplex) and then shall define the measures in such a way that each point of C is associated with itself. In other words, the function $f : C \to S$ will be the identity and the cake C will equal the associated RNS.

Our cake C is the interior of S, the $(n-1)$-simplex. Define a measure μ on C as follows: for $A \subseteq C$,

$$\mu(A) = n\left(\frac{\mathrm{m}_L(A)}{\mathrm{m}_L(\mathrm{C})}\right)$$

where m_L denotes $(n-1)$-dimensional Lebesgue measure on S. Then $\mu(C) = n$.

For each $i = 1, 2, \ldots, n$ define f_i on C as follows:

$$\text{for each } a \in C,\ f_i(a) = \text{the } i\text{th coordinate of } a$$

For each such a, $f_1(a) + f_2(a) + \cdots + f_n(a) = 1$, since $a \in S$. Symmetry considerations tell us that $\int_C f_1\,d\mu = \int_C f_2\,d\mu = \cdots = \int_C f_n\,d\mu$ and hence, since

$$\begin{aligned} &\int_C f_1\,d\mu + \int_C f_2\,d\mu + \cdots + \int_C f_n\,d\mu \\ &= \int_C (f_1 + f_2 + \cdots + f_n)\,d\mu \\ &= \int_C 1\,d\mu = \mu(C) = n \end{aligned}$$

it follows that $\int_C f_1\,d\mu = \int_C f_2\,d\mu = \cdots = \int_C f_n\,d\mu = 1$. Then, for each $i = 1, 2, \ldots, n$, we may define m_i on C as follows: for any $A \subseteq C$,

$$m_i(A) = \int_A f_i\,d\mu$$

Then each m_i is a (countably additive, non-atomic, probability) measure on C, and $f_1, f_2, \ldots, f_n$ are the density functions of $m_1, m_2, \ldots, m_n$, respectively, with respect to μ. It is clear from our construction that, in this case, the usual identification of points in C with points in S simply identifies points with themselves.

We claim that this RNS is not concentrated. Suppose, by way of contradiction, that for some $\omega \in S^+$ and $i, j = 1, 2, \ldots, n$ the RNS is i, j-concentrated with respect to ω. Let H denote the i, j boundary associated with ω, and note that H has dimension $n - 2$. Then the set of bits of cake corresponding to H has positive measure. Since in this case f is the identity function, it follows that H actually contains a piece of cake of positive measure. Call this piece of cake A. Then A has positive $(n - 1)$-dimensional Lebesgue measure. But this is a contradiction, since $A \subseteq H$, H has dimension $n - 2$, and $(n - 1)$-dimensional Lebesgue measure assigns measure zero to any such lower-dimensional object. Hence, the RNS is not concentrated.

Theorem 12.14 implies that there are no line segments on the outer Pareto boundary of the IPS. By the equivalence of parts a and f of Corollary 14.5, it follows that every Pareto maximal partition is strongly Pareto maximal. This establishes that situation c is possible.

For situation d, we need to define a cake C and measures $m_1, m_2, \ldots, m_n$ on C so that in the corresponding IPS there are uncountably many strongly Pareto maximal points and uncountable many Pareto maximal points that are not strongly Pareto maximal. Informally, the construction is as follows. The cake will consist of two pieces. One piece will be like the cake of situation c and will produce a part of the outer Pareto boundary of the IPS. As in situation c, this part will contain no line segments and, hence, will contain only strongly Pareto maximal points. The other piece of the cake will be a piece on which all n players' measures are equal to each other and, hence, will produce line segments on the outer Pareto boundary of the IPS. Thus, this part of the IPS will contain the required Pareto maximal points that are not strongly Pareto maximal.

To begin our construction, we let C_1 be the interior of the $(n - 1)$-simplex (and so C_1 is the same as C in our proof for situation c). C_2 can be defined somewhat arbitrarily. For definiteness, we let $C_2 = [0, 1]$, the unit interval on the real number line. Set $C = C_1 \cup C_2$.

Next, we wish to define measures $m_1, m_2, \ldots, m_n$ on C. For each $i = 1, 2, \ldots, n$, let m'_i be defined on C_1 precisely as m_i was defined on the cake C in our proof for situation c, and let m_L be Lebesgue measure on the unit interval. Then, for each such i, we define m_i on C as follows: for any $A \subseteq C$,

$$m_i(A) = \frac{1}{2}m'_i(A \cap C_1) + \frac{1}{2}m_L(A \cap C_2)$$

It is straightforward to verify that each m_i is a (countably additive, non-atomic, probability) measure on C. As in the proof for situation c, we see that all points of C_1 get mapped by f to themselves and, thus, the RNS associated with C_1 is C_1 itself. Since the measures are equal to each other on C_2, it follows that all of C_2 is mapped by f to the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ and, hence, all of C_2 is associated with this point.

Since $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is associated with a positive-measure piece of cake, this point corresponds to a flat region on the outer Pareto boundary of the IPS. Thus, there are line segments on the outer Pareto boundary of the IPS, and so the number of points on the outer Pareto boundary that are interior points of line segments is uncountably infinite. Therefore, the number of Pareto maximal points that are not strongly Pareto maximal is uncountably infinite.

We must show that the number of points that are strongly Pareto maximal is uncountably infinite. Fix $\omega \in S^+$ such that for no $i, j = 1, 2, \ldots, n$ does the i, j boundary associated with ω contain the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, and suppose that P is any partition of C that is w-associated with ω. It is easy to see that each such i, j boundary is associated with a piece of cake of measure zero. It follows that P satisfies the cyclic boundary condition with respect to ω (see Definition 14.3). The equivalence of parts a and c of Theorem 14.4 implies that P is strongly Pareto maximal.

There is certainly an uncountably infinite number of $\omega \in S^+$ satisfying that the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is not on any i, j boundary associated with ω. It is easy to see (using the definition of the m_i') that p-equivalent partitions cannot be w-associated with distinct ω. We conclude that there is an uncountably infinite collection of strongly Pareto maximal partitions that are pairwise non-p-equivalent and, hence, that the number of strongly Pareto maximal points in the IPS is uncountably infinite. This establishes that situation d is possible and thus completes the proof of the theorem. □

The cake and measures constructed for situation c of the theorem relates to issues we studied in Chapter 12. We showed that the RNS in this situation is not concentrated. It is not hard to see that this RNS is also not separable. Thus, besides containing no line segments, we also know by Theorem 12.18 that the outer Pareto boundary of the IPS contains no edge points. Theorems 12.12 and 12.16 imply that the relation M (see Definition 12.1) is one-one on the entire cake. We also note that since this RNS is not concentrated, Theorem 12.36 implies that there exists a partition of the corresponding cake that is strongly envy-free and Pareto maximal.

Next, we state the chores version of Theorem 14.14. The proof is analogous and we omit it.

Theorem 14.15 *Fix $n \geq 2$. For each of the following conditions, there exists a cake C and measures $m_1, m_2, \ldots, m_n$ on C such that the given condition is satisfied:*

	Number of Strongly Pareto Minimal Points	*Number of Pareto Minimal Points That Are Not Strongly Pareto Minimal*
a.	*Arbitrarily large finite*	*Uncountably infinite*
b.	*Countably infinite*	*Uncountably infinite*
c.	*Uncountably infinite*	0
d.	*Uncountably infinite*	*Uncountably infinite*

Also, this list is complete in the sense that any other combination of numbers for the two given types of points is impossible, with one exception. We do not know whether, in general, for a given (finite) number k there exists a cake and corresponding measures such that there are exactly k strongly Pareto minimal points (and thus an uncountably infinite number of Pareto minimal points that are not strongly Pareto minimal).

We close this section by noting a connection between the standard context and the chores context. As we discussed earlier in this chapter (see the discussion following the proof of Corollary 14.5), there exists a Pareto maximal point that is not strongly Pareto maximal if and only if the RNS is concentrated. (We used this in the proof for situation c of Theorem 14.14.) It is also true that there exists a Pareto minimal point that is not strongly Pareto minimal if and only if the RNS is concentrated. Hence, there exists a Pareto maximal point that is not strongly Pareto maximal if and only if there exists a Pareto minimal point that is not strongly Pareto minimal.

14E. The Situation Without Absolute Continuity

In this section, we reconsider the results of the previous sections of this chapter, no longer assuming that the measures are absolutely continuous with respect to each other. Theorem 14.4 gave various characterizations of strong Pareto maximality, Corollary 14.5 gave various characterizations of Pareto maximality with the failure of strong Pareto maximality, and Theorem 14.8 and Corollary 14.9 gave analogous results in the chores context. These results hold in the absence of absolute continuity. Some minor revisions are necessary in some definitions and terminology, and in the proof. (For example, "cyclic boundary condition with respect to ω" and "non-trivial collection of transfers" must be redefined to refer to specific measures instead of referring to "positive measure," and Lemma 8.17 and Theorem 8.24 must be used instead of Lemma 8.3 and Theorem 8.9, respectively.) We omit the details.

Next, we consider Theorem 14.10, which told us the possible numbers of strongly Pareto maximal points versus Pareto maximal points that are not strongly Pareto maximal, when there are two players. Obviously, each of the four situations named in the theorem is still possible if we no longer assume absolute continuity. (We are not assuming that absolute continuity fails; hence, the same examples work in our present setting.) However, this list is not complete. By Theorem 5.40, we see that it is possible to have only one Pareto maximal point. Hence, there is a situation that is possible in our present setting that was not possible when we assumed that the two measures were absolutely continuous with respect to each other.

Our adjusted version of Theorem 14.10 is the following.

Theorem 14.16 *For each of the following conditions, there exists a cake C and measures m_1 and m_2 on C such that the given condition is satisfied:*

	Number of Strongly Pareto Maximal Points	*Number of Pareto Maximal Points That Are Not Strongly Pareto Maximal*
a.	1	0
b.	*Any finite* $k \geq 2$	*Uncountably infinite*
c.	*Countably infinite*	*Uncountably infinite*
d.	*Uncountably infinite*	0
e.	*Uncountably infinite*	*Uncountably infinite*

Also, this list is complete in the sense that any other combination of numbers for the two given types of points is impossible.

Proof: For situation a, we must show that, for some cake C and measures m_1 and m_2 on C, the corresponding IPS has exactly one strongly Pareto maximal point and no Pareto maximal points that are not strongly Pareto maximal. Choose any cake C and measures m_1 and m_2 on C that concentrate on disjoint sets. By Theorem 5.40, the corresponding IPS has exactly one Pareto maximal point, and this point is $(1, 1)$. Since the IPS is a subset of the unit square, it is clear that $(1, 1)$ is not an interior point of a line segment contained in the IPS. It follows from the equivalence of parts a and f of Theorem 14.4 that $(1, 1)$ is a strongly Pareto maximal point. Hence, $(1, 1)$ is the only strongly Pareto maximal point and there are no Pareto maximal points that are not strongly Pareto maximal.

As discussed earlier, Theorem 14.10 implies that all of the other situations given in the theorem are possible. To see that the list is complete, we note that if the measures concentrate on disjoint sets then, by Theorem 5.40, there is exactly one Pareto maximal point, (1, 1), and as discussed in the preceding paragraph this point is strongly Pareto maximal. Thus, situation a is the only possibility in this case. If the measures do not concentrate on disjoint sets, then it is easy to see that the number of Pareto maximal points in uncountably infinite. (By Theorem 5.40, the number of Pareto maximal points is infinite. Since the outer Pareto boundary is certainly connected, it follows that the number of such points is uncountably infinite.) In this case, the proof that situations b, c, d, and e are the only possibilities is as in the proof of Theorem 14.10. □

The chores version of Theorem 14.16 is the following. The proof is analogous and we omit it.

Theorem 14.17 *For each of the following conditions, there exists a cake C and measures m_1 and m_2 on C such that the given condition is satisfied:*

	Number of Strongly Pareto Minimal Points	*Number of Pareto Minimal Points That Are Not Strongly Pareto Minimal*
a.	1	0
b.	*Any finite* $k \geq 2$	*Uncountably infinite*
c.	*Countably infinite*	*Uncountably infinite*
d.	*Uncountably infinite*	0
e.	*Uncountably infinite*	*Uncountably infinite*

Also, this list is complete in the sense that any other combination of numbers for the two given types of points is impossible.

Theorem 14.13 told us that a p-class consists of a single s-class if and only if it is either strongly Pareto maximal or strongly Pareto minimal. The reverse direction of this result holds regardless of any absolute continuity assumptions. Concerning the forward direction, consider the points (1, 0) and (0, 1). These are points in the IPS and, since the IPS is a subset of the unit square, it follows that neither of these points lies in the interior of a line segment contained in the IPS. Therefore, by (the equivalence class version of) Theorem 2.6, the p-class corresponding to each of these points consists of a single s-class. Suppose now that neither measure is absolutely continuous with respect to the other. Then, by Lemma 3.29, neither (1, 0) nor (0, 1) is Pareto maximal or Pareto minimal and, hence, neither is strongly Pareto maximal or strongly Pareto minimal. Therefore, in this situation, the forward direction of Theorem 14.13 fails. It is not hard to see that this is the only possible violation of this result. In particular, if one of the measures is absolutely continuous with respect to the other, then Theorem 14.13 holds since, in this case, one of (1, 0) and (0, 1) will be strongly Pareto maximal and the other will be strongly Pareto minimal.

Next, we consider Theorem 14.14, which concerns the possible numbers of strongly Pareto maximal points versus Pareto maximal points that are not strongly Pareto maximal when there are more than two players. Certainly, the examples constructed in the proof of Theorem 14.14 show that each of the four given situations is possible in our present context. However, precisely as in the two-player context, Theorem 5.40 tells us that when the measures concentrate on disjoint sets then there is exactly one strongly Pareto maximal point and no Pareto maximal points that are not strongly Pareto maximal. Thus, our adjusted version of Theorem 14.14 is the following.

Theorem 14.18 *Fix $n \geq 2$. For each of the following conditions, there exists a cake C and measures $m_1, m_2, \ldots, m_n$ on C such that the given condition is satisfied:*

	Number of Strongly Pareto Maximal Points	*Number of Pareto Maximal Points That Are Not Strongly Pareto Maximal*
a.	1	0
b.	*Arbitrarily large finite*	*Uncountably infinite*
c.	*Countably infinite*	*uncountably infinite*
d.	*Uncountably infinite*	0
e.	*Uncountably infinite*	*Uncountably infinite*

Also, this list is complete in the sense that any other combination of numbers for the two given types of points is impossible, with one exception. We do not know whether, in general, for a given (finite) number k, there exists a cake and corresponding measures such that there are exactly k strongly Pareto maximal points (and thus an uncountably infinite number of Pareto maximal points that are not strongly Pareto maximal).

It is possible to be slightly more specific about the possibilities for finitely many strongly Pareto maximal points. We shall not establish a general result, but shall instead present an example to illustrate the idea. Assume that there are three players, that Player 1 and Player 2 have identical measures, and that these two measures on the one hand, and Player 3's measure on the other hand, concentrate on disjoint sets. The corresponding RNS and IPS are shown in Figures 14.5a and 14.5b, respectively. In this case, the Pareto maximal points are the points on the closed line segment in Figure 14.5b between the points p

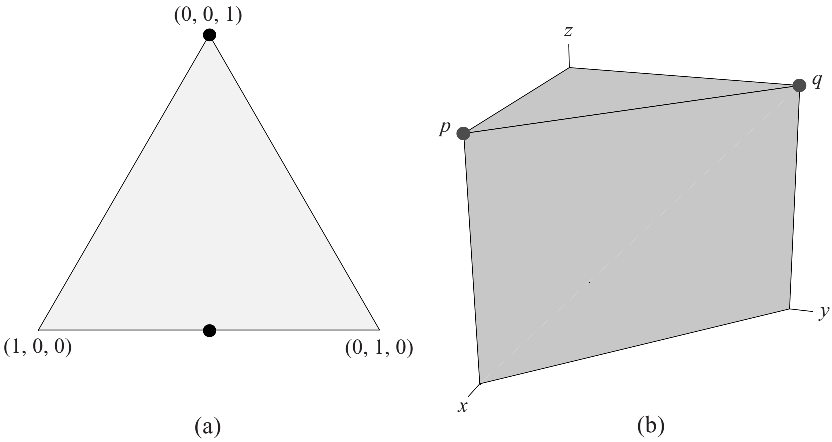


Figure 14.5

and q. Then the equivalence of parts a and f of Theorem 14.4 implies that p and q are the only strongly Pareto maximal points. This is in contrast with the situation when the measures are absolutely continuous with respect to each other. In that case, we saw (see the paragraph preceding the statement of Theorem 14.14) that when there are n players there are always at least n strongly Pareto maximal partitions. For more than three players, it is possible to generalize this idea to obtain IPSs that have various numbers of Pareto maximal points that are strictly between one and n.

We conclude this section by stating the chores version of Theorem 14.18. The proof is analogous and we omit it. Whereas situation a of Theorem 14.18 corresponds to the measures concentrating on disjoint sets, situation a of Theorem 14.19 corresponds to the measures concentrating on the complements of disjoint sets.

Theorem 14.19 *Fix $n \geq 2$. For each of the following conditions, there exists a cake C and measures $m_1, m_2, \ldots, m_n$ on C such that the given condition is satisfied:*

	Number of Strongly Pareto Minimal Points	*Number of Pareto Minimal Points That Are Not Strongly Pareto Minimal*
a.	1	0
b.	*Arbitrarily large finite*	*Uncountably infinite*
c.	*Countably infinite*	*Uncountably infinite*
d.	*Uncountably infinite*	0
e.	*Uncountably infinite*	*Uncountably infinite*

Also, this list is complete in the sense that any other combination of numbers for the two given types of points is impossible, with one exception. We do not know whether, in general, for a given (finite) number k, there exists a cake and corresponding measures such that there are exactly k strongly Pareto minimal points (and thus an uncountably infinite number of Pareto minimal points that are not strongly Pareto minimal).

14F. Fairness and Efficiency Together: Part 3

In this section, we revisit Section 12E. We make no assumptions about absolute continuity.

We recall the main theorem of that section, Theorem 12.32: There exists a partition that is envy-free and Pareto maximal. After we proved this result, we considered a strengthening. We showed (see Lemma 12.35) that if P is the envy-free and Pareto maximal partition from the proof of Theorem 12.32 and $m(P)$ is not on a line segment on the outer Pareto boundary of the IPS, then P is strongly envy-free. This result, combined with Theorem 12.14, told us (see Theorem 12.36) that if no two players are in relative agreement on any set that has positive measure to each of these two players (or, equivalently, if the RNS is not concentrated), then there exists a partition that is strongly envy-free and Pareto maximal. We are now in a position to show that, with this same assumption, there exists a partition that is strongly envy-free and strongly Pareto maximal.

Theorem 14.20 *If no two players are in relative agreement on any set that has positive measure to each of these two players (or, equivalently, if the* RNS *is not concentrated), then there exists a partition that is strongly envy-free and strongly Pareto maximal.*

Proof: Let P be the partition obtained as in the proof of Theorem 12.32. Then P is Pareto maximal and envy-free. Assume that no two players are in relative agreement on any set that has positive measure to each of these two players and, hence, that the RNS is not concentrated. We claim that P is strongly Pareto maximal and strongly envy-free.

Since the RNS is not concentrated, Theorem 12.14 tells us that there are no line segments on the outer Pareto boundary of the IPS. Then certainly $m(P)$, which is a point on the outer Pareto boundary, does not lie on a line segment on the outer Pareto boundary. It follows from Lemma 12.35 that P is strongly envy-free. (This part of the proof is a repeat of the proof of Theorem 12.36.)

To show that P is strongly Pareto maximal, we observe that since there are no line segments on the outer Pareto boundary of the IPS, and $m(P)$ is a point on the outer Pareto boundary, $m(P)$ obviously does not lie in the interior of a line segment contained in the IPS. It follows from the equivalence of parts a and f of Theorem 14.4 that P is strongly Pareto maximal. □

The chores version of Theorem 14.20 is the following. The proof is similar and we omit it.

Theorem 14.21 *If no two players are in relative agreement on any set that has positive measure to each of these two players (or, equivalently, if the* RNS *is not concentrated), then there exists a partition that is strongly c-envy-free and strongly Pareto minimal.*

15

Characterizing Pareto Optimality Using Hyperreal Numbers

In this chapter, we show how the use of hyperreal numbers can simplify our previous characterizations of Pareto optimality. In particular, we shall see that this approach will allow us to avoid the iterative procedures using partition sequence pairs, which involved a-maximization and b-maximization of convex combinations of measures in Chapter 7, and w-association in Chapter 10. Our new approach will also allow us to avoid the assumption (which we needed at times in Chapters 7 and 10) that each player receives a piece of cake that he or she believes to be of positive measure.

In Section 15A, we give the necessary background on hyperreals. In Section 15B, we illustrate the use of hyperreals by considering a two-player example. In Section 15C, we do the same for a three-player example. In Section 15D, we state and prove our new characterization. We make no general assumptions about absolute continuity in this chapter (although our examples in Sections 15B and 15C involve the failure of absolute continuity).

15A. Introduction

Although the foundations of calculus evolved over many years, Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716) are generally considered to be the inventors of modern calculus. Their work involved two different but related notions, limits and infinitesimals. Limits were eventually formalized using what has become known as the ε–δ method and is now the standard approach to calculus and mathematical analysis. Infinitesimals are simpler than limits, but defied efforts at formalization and so fell out of favor.

Infinitesimals were finally formalized in the early 1960s by A. Robinson [37] under the name of "non-standard analysis." His work led to applications of infinitesimals in a wide variety of areas, such as physics and economics (see [2, 12]). There are also modern calculus books that use infinitesimals rather than the limit approach (see [27, 29]).

An infinitesimal is a number that is different from zero, but is closer to zero than is any real number. Thus, an infinitesimal is not a real number. The hyperreal number system is the number system obtained by extending the usual real number system to include an infinitesimal and any other numbers that need to be added so that any "formula" that is true of the real numbers will be true of the hyperreal numbers. (We comment more on this idea later.) We shall not go into complete detail. For our purposes, we shall need only the following facts:

a. The *hyperreal number system*, which we denote by $\mathbf{R}_H$, is an extension of the real number system $\mathbf{R}$. Every real number is a hyperreal number.
b. $\mathbf{R}_H$ contains an *infinitesimal*. If ε is a positive infinitesimal, then $0 < \varepsilon < r$ for every positive real number r. If ε is a negative infinitesimal, then $0 > \varepsilon > r$ for every negative real number r.
c. All of the usual formulas that hold for the real number system hold for the hyperreal number system.

There are two standard methods for creating a hyperreal number system. One uses the notion of ultraproducts and the other uses the compactness theorem of mathematical logic (see, for example, [12, 37]). There are many hyperreal number systems. For our purposes, any hyperreal number system will suffice, and we shall simply refer to "the" hyperreal number system.

What do we mean by "formula" in statement c? As examples, we mean statements such as the following:

- For any two distinct numbers, there is a number between these two numbers.
- There is no biggest number.
- For any two numbers, there is a number that is equal to the product of these two numbers.

Formalizing this idea requires the methods of mathematical logic. An appropriate language is defined, which includes relevant constants (such as names for numbers), functions (such as addition and multiplication), relations (such as the "less than" relation), and the standard logical symbols (such as connectives, quantifiers, parentheses, and variable symbols). In this language, we can express the basic facts about the real numbers, including facts about arithmetic and the standard ordering of the real numbers, including the three preceding statements.

By statement *b*, the hyperreal number system $\mathbf{R}_H$ includes an infinitesimal ε. Notice if $\varepsilon > 0$ then $-\varepsilon < 0$, and if $\varepsilon < 0$ then $-\varepsilon > 0$. This tells us that $\mathbf{R}_H$ contains both positive and negative infinitesimals. Let ε be a positive infinitesimal. It is not hard to see that, for any non-zero integer k, $k\varepsilon$ is an infinitesimal. Hence, there are infinitely many infinitesimals.

Consider the statement "given any numbers κ and λ with $\lambda \neq 0$, there is a number that is equal to $\frac{\kappa}{\lambda}$." This is a statement that can be formalized in our language and is true of the real numbers. Therefore, it is true of the hyperreal numbers and so, for any positive infinitesimal ε, there is a hyperreal number that is equal to $\frac{1}{\varepsilon}$. Since $\varepsilon < r$ for every positive real number r, it follows that $\frac{1}{\varepsilon} < \frac{1}{r}$ for every positive real number r. This tells us that $\frac{1}{\varepsilon}$ is bigger than every real number. We shall refer to a hyperreal that is either bigger than every real number or smaller than every real number as an *infinite hyperreal*.

Notice that if r is a real number and ε is an infinitesimal, then $r + \varepsilon$ is neither real nor infinitesimal. It is a hyperreal number whose difference from a real number is infinitesimal. We say that $r + \varepsilon$ is *infinitesimally close* to the real number r. We write "$[\kappa] = r$" to mean that the hyperreal number κ is either equal to or is infinitesimally close to the real number r.

We will allow the use of hyperreal numbers that are not real numbers in certain settings but not in others. We continue to assume that all measures take on only real number values and, thus, the IPS is defined exactly as before. Also, our use of the Radon–Nikodym theorem in Chapter 9 to define the density functions f_i and the function f is as before, and so our definition of the RNS, which is the range of the function f, is unchanged.

We have also used real numbers for the coordinates of points in the simplex that are to be used

- to provide the coefficients for families of parallel hyperplanes,
- to provide the coefficients for the maximization of convex combinations of measures, or
- in the context of w-association.

(We recall that the first two of these uses are the same, and that these two are connected to the third by the RD function.) In this chapter, we allow the coordinates of these points to have hyperreal values that may or may not be real.

Let S_H denote the *hyperreal simplex* and let S_H^+ denote its interior. Thus, $S_H = \{(\kappa_1, \kappa_2, \ldots, \kappa_n) : \kappa_1, \kappa_2, \ldots, \kappa_n \in R_H; \kappa_1, \kappa_2, \ldots, \kappa_n \geq 0;$ and $\kappa_1 + \kappa_2 + \cdots + \kappa_n = 1\}$ and ${S_H}^+ = \{(\kappa_1, \kappa_2, \ldots, \kappa_n) : \kappa_1, \kappa_2, \ldots, \kappa_n \in R_H; \kappa_1, \kappa_2, \ldots, \kappa_n > 0;$ and $\kappa_1 + \kappa_2 + \cdots + \kappa_n = 1\}$. When choosing $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ to provide coefficients either for families of parallel hyperplanes or for convex combinations of measures, and when choosing $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ as in the context of w-association, we shall allow $\alpha \in S_H^+$ and $\omega \in S_H^+$ rather than insisting that $\alpha \in S^+$ and $\omega \in S^+$.

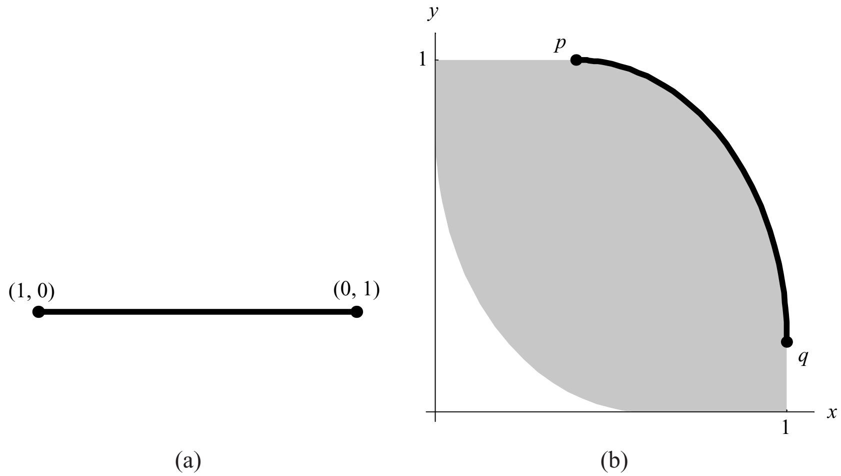


Figure 15.1

15B. A Two-Player Example

In this section, we introduce the use of hyperreals in the two-player context by examining a specific example.

Example 15.1 Consider Figure 15.1. This figure shows an RNS and the corresponding IPS for some cake C and measures m_1 and m_2 on C. In Figure 15.1a, our intention is that

- the points (1,0) and (0,1) each correspond to a piece of cake of positive measure and hence are in the RNS, and
- the remainder of the RNS is spread out over the open interval between (1,0) and (0,1). (We introduced the notion of a "spread-out" RNS in Chapter 12; see Example 12.6.)

These two facts, together with the ideas that we developed in Chapter 12, tell us that

- neither measure is absolutely continuous with respect to the other, and hence, the IPS in Figure 15.1b contains a vertical line segment going up from the point $(1, 0)$ and a horizontal line segment going to the right from the point $(0, 1)$, and
- the outer Pareto boundary of the IPS contains no line segments and meets the vertical and horizontal line segments smoothly (i.e., the outer boundary of the IPS has a unique tangent line at the point p and at the point q).

We have darkened the outer Pareto boundary of the IPS. The existence of C, m_1, and m_2 that yield the given IPS follows from Theorem 11.1.

Consider the RNS in Figure 15.1a. Recall that, for any $\omega \in S^+$, if P is a partition obtained by giving each player all of the cake that is associated with points of the RNS that are on that player's side of ω (and dividing up any cake associated with ω arbitrarily), then P is said to be w-associated with ω, and any such partition is Pareto maximal. However, not every Pareto maximal partition is obtained in this way. Let P be the partition that gives all of the cake associated with the point $(1, 0)$ of the RNS to Player 1 and gives the rest of the cake to Player 2. (Notice that this is the minimum amount of cake that Player 1 must receive in any Pareto maximal partition. This is the property of non-wastefulness, given by Definition 6.5.) If P were w-associated with some $\omega \in S^+$, then ω would have to be to the right of the point $(1, 0)$ but to the left of any point in the interior of the simplex. Clearly this is impossible in our usual setting, and this difficulty led us to our iterative approach, using the notion of w-association with a partition sequence pair (see Definitions 7.11 and 10.26, and Theorem 10.28). However, now that we have hyperreal numbers available, we will see that choosing an appropriate ω is quite easy and, thus, we have no need for this iterative approach.

Fix any infinitesimal $\varepsilon > 0$ and let $\omega = (1 - \varepsilon, \varepsilon)$. Then $\omega \in S_H^+$. The point ω is to the right of $(1, 0)$ and, since every point in the RNS that is associated with a piece of cake that goes to Player 2 is of the form (r_1, r_2), where r_1 and r_2 are real numbers with $r_1 < 1$ and $r_2 > 0$, it follows that every such point is to the right of ω. Hence, P is w-associated with ω. The same approach certainly works for the partition that gives all cake associated with $(0, 1)$ to Player 2 and gives all of the rest of the cake to Player 1. Such a partition is w-associated with the point $\omega = (\varepsilon, 1 - \varepsilon)$, where $\varepsilon > 0$ is infinitesimal. It follows from our work in Chapter 10 that any partition besides these two is Pareto maximal if and only if it is w-associated with some $\omega \in S^+$. Hence we see that, at least for the situation depicted in Figure 15.1a,

a partition P is Pareto maximal
if and only if
for some $\omega \in S_H^+$, P is w-associated with ω.

We have not presented a complete and general proof that this characterization works for all cakes and any two measures. This will follow from our characterization for the general context of n players, given in Section 15D.

Next, we consider the IPS in Figure 15.1b. Recall that, if a partition P is Pareto maximal, then, for some $\alpha \in S$, $m(P)$ is a point of first contact with the

IPS of the family of parallel lines with coefficients given by α. Let us focus our attention on the Pareto maximal point p. This is the point in the IPS that corresponds to the partition P discussed early. (P is the partition that gives all of the cake asssociated with the point (1, 0) of the RNS to Player 1 and gives the rest of the cake to Player 2.) Then $m(P) = p$. The only family of parallel lines with coefficients from S that makes first contact with the IPS at p is the family of horizontal lines, i.e., the family of the form $0x + 1y = c$, which is the family corresponding to the point $(0, 1) \in S$. The problem is that this family also makes first contact with the IPS at every point on the line segment connecting the points (0, 1) and p, and p is the only point on this line segment that is Pareto maximal. The best we were able to do with this sort of direct approach in Chapter 7 was two different "if-then" statements, as given by Theorem 7.10, rather than a genuine characterization. (Theorem 7.10 used the notion of maximization of convex combinations of measures, rather than the notion of points of first contact of families of parallel hyperplanes. These two ideas are equivalent. Also, we recall that in the two-player context a hyperplane is simply a line.)

In Section 7C, we solved this problem and obtained two closely related characterizations, a-maximization and b-maximization of a partition sequence pair, by developing an iterated approach (see Definitions 7.11, 7.12, and 7.17, and Theorems 7.13 and 7.18). For the IPS in Figure 15.1b, we discussed this iterative idea in Section 7C and illustrated it in Figure 7.5. We shall now see that hyperreals allow us to avoid this iteration.

Consider Figure 15.2. We would like a family of parallel lines that makes first contact with the IPS at p, but not at any other point on the line segment between the points (0, 1) and p, since no other point along this line segment is Pareto maximal. As we have seen, the horizontal family of parallel lines makes first contact with the IPS at all points along this line segment. The line of first contact is the line $y = 1$, and we have shown this line in the figure. It is easy to see that any other family of parallel lines with coefficients from S that is obtained by rotating slightly clockwise from horizontal will make first contact with the IPS not at p, but at some point on the outer boundary that is to the lower right of p. In the figure, the family of parallel lines of the form $\alpha_1 x + \alpha_2 y = c$, where $(\alpha_1, \alpha_2) \in S$, makes first contact with the IPS at point r. We have shown the line in this family that makes first contact with the IPS and have labeled it "$\alpha_1 x + \alpha_2 y = k_1$."

We need to tilt the horizontal family of parallel lines some positive amount clockwise (so that no point on the line segment between the points (0, 1) and p is a point of first contact, except for p), but not so much that p is no longer a point of

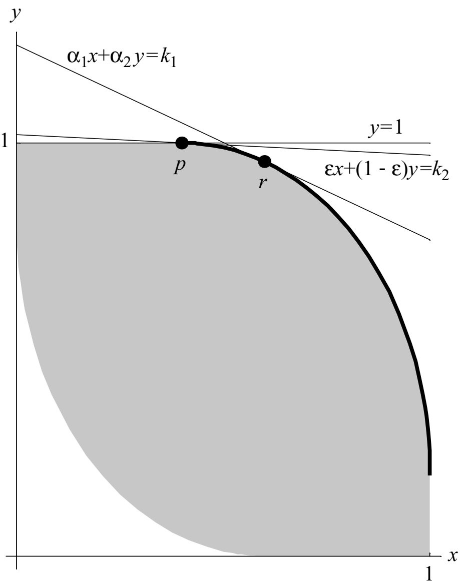


Figure 15.2

first contact. We do this by tilting this family an infinitesimal amount. Suppose that $\varepsilon > 0$ is an infinitesimal. Then $(\varepsilon, 1 - \varepsilon) \in S_H^+$, and we may consider the family of parallel lines of the form $\varepsilon x + (1 - \varepsilon)y = c$.

Claim The family of parallel lines of the form $\varepsilon x + (1 - \varepsilon)y = c$ makes first contact with the IPS in Figure 15.2 at the point p and at no other points.

Proof of Claim: Suppose $p = (p_1, 1)$ and let k_2 be such that the line in the given family that makes first contact with the IPS is the line $\varepsilon x + (1 - \varepsilon)y = k_2$, as indicated in the figure. Fix any other point $s = (s_1, s_2)$ on the outer boundary of the IPS. It suffices to show that the given family of parallel lines makes contact with p before it makes contact with s. Suppose, by way of contradiction, that this is not the case. Then $\varepsilon s_1 + (1 - \varepsilon)s_2 \geq \varepsilon p_1 + (1 - \varepsilon)(1) = \varepsilon p_1 + (1 - \varepsilon)$. We consider two cases.

Case 1: s is on the line segment between p and $(0, 1)$. Then $s_2 = 1$ and it follows that $\varepsilon s_1 + (1 - \varepsilon) \geq \varepsilon p_1 + (1 - \varepsilon)$ and, hence, $s_1 \geq p_1$. But since s is on the line segment between $(0, 1)$ and p, and $s \neq p$, we know that $s_1 < p_1$. This is a contradiction.

Case 2: s is not on the line segment between $(0, 1)$ and p. Then $s_2 < 1$. Since $\varepsilon s_1 + (1 - \varepsilon)s_2 \geq \varepsilon p_1 + (1 - \varepsilon)$, it follows that $\frac{\varepsilon}{1-\varepsilon} \geq \frac{1-s_2}{s_1-p_1}$. But $\frac{\varepsilon}{1-\varepsilon}$ is an infinitesimal divided by a positive hyperreal that is not infinitesimal and, hence, this quantity is infinitesimal. On the other hand, since s_2 is a real number less than one, and s_1 and p_1 are real numbers with $p_1 < s_1$, it follows that $1 - s_2$ and $s_1 - p_1$ are each positive real numbers. Hence, their quotient, $\frac{1-s_2}{s_1-p_1}$, is a positive real number. This contradicts the fact that $\frac{\varepsilon}{1-\varepsilon} \geq \frac{1-s_2}{s_1-p_1}$, since an infinitesimal cannot be greater than or equal to a positive real number, and completes the proof of the claim.

In a similar manner, it is not hard to see that the point q in Figure 15.1b is the point of first contact of the family of parallel lines that arise from the family of vertical lines by an infinitesimal counter-clockwise tilt. Also, as we already knew before the present chapter, any point on the outer Pareto boundary that is strictly between p and q is the point of first contact with the IPS of some family of parallel lines with coefficients given by some $\alpha \in S^+$. Finally, it is not hard to see that no point that is not on the outer Pareto boundary is a point of first contact with the IPS of family of parallel lines with coefficients given by some $\alpha \in S_H^+$. Hence, we see that, at least for the situation depicted in Figure 15.1b,

a partition P is Pareto maximal
if and only if
for some $(\alpha_1, \alpha_2) \in S_H^+$, the family of parallel lines of the form $\alpha_1 x + \alpha_2 y = c$ makes first contact with the IPS at $m(P)$.

Or, equivalently,

a partition P is Pareto maximal
if and only if
for some $(\alpha_1, \alpha_2) \in S_H^+$, P maximizes the convex combination of measures $\alpha_1 m_1 + \alpha_2 m_2$.

As before for the RNS and w-association, we note that we have not presented a complete and general proof that our characterization works for all cakes and any two measures. This will follow from our characterization for the general context of n players, given in Section 15D.

Recall Theorem 10.6: For any partition P, and any $\omega \in S^+$ and $\alpha \in S^+$ with $\alpha = \text{RD}(\omega)$, P is w-associated with ω if and only if the family of parallel hyperplanes with coefficients given by α makes first contact with the IPS at $m(P)$. (Theorem 10.6 refers to convex combinations of measures rather than the equivalent notion of points of first contact with the IPS of families of parallel hyperplanes. For the definition of the RD function, see Definition 10.5.) The RD function, which we originally defined to be a function from S^+ to S^+, extends naturally to a function from S_H^+ to S_H^+. Although we shall not do so, it is straightforward to show that with this extended definition Theorem 10.6

is true for any partition P, and any $\omega \in S_H^+$ and $\alpha \in S_H^+$ with $\alpha = \mathrm{RD}(\omega)$. We close this section by illustrating this for the ω and α that we considered in this section.

Let C, m_1, and m_2 be the cake and measures that yield the RNS and IPS shown in Figure 15.1, and let P be the partition that gives all of the cake associated with the point $(1, 0)$ of the RNS to Player 1 and all of the rest of the cake to Player 2. We found that, for any infinitesimal $\varepsilon > 0$, P is w-associated with $(1 - \varepsilon, \varepsilon)$. We compute $\mathrm{RD}(1 - \varepsilon, \varepsilon)$ as follows:

$$\mathrm{RD}(1-\varepsilon,\varepsilon) = \left(\frac{1}{\frac{1}{1-\varepsilon}+\frac{1}{\varepsilon}}\right)\left(\frac{1}{1-\varepsilon},\frac{1}{\varepsilon}\right) = \left(\frac{(\varepsilon)(1-\varepsilon)}{\varepsilon+1-\varepsilon}\right)\left(\frac{1}{1-\varepsilon},\frac{1}{\varepsilon}\right)$$
$$= (\varepsilon(1-\varepsilon))\left(\frac{1}{1-\varepsilon},\frac{1}{\varepsilon}\right) = (\varepsilon, 1-\varepsilon)$$

Thus, the extension of Theorem 10.6 to allow ω and α to be chosen from S_H^+ rather than S^+ implies that the family of parallel lines with coefficients given by $(\varepsilon, 1 - \varepsilon)$ makes first contact with the IPS at $m(P)$. This is precisely what we showed earlier in this section.

15C. Three-Player Examples

In this section, we shall examine two examples to illustrate the use of hyperreals in characterizing Pareto maximality when there are three players. As we saw in previous chapters, and shall review in this section, there are certain difficulties that arise in the three-player context that do no arise in the two-player context. Hyperreals provide a simple means of dealing with these difficulties.

Example 15.2 Consider Figure 15.3. Figure 15.3a shows an RNS for some cake C and measures m_1, m_2, and m_3, and Figure 15.3b shows the corresponding IPS. In the RNS picture, our intention is that each of the vertices of the simplex corresponds to a piece of cake of positive measure, and that the RNS corresponding to the rest of the cake is spread out throughout the interior of the simplex. Thus, except for the three vertices, no zero or one-dimensional subset of the simplex is associated with a piece of cake of positive measure. We also note that the RNS is neither concentrated nor separable. (For the definitions of concentrated and separable, see Definitions 12.9 and 12.15, respectively.) It may seem that the RNS is concentrated, since there are three points that are each associated with a piece of cake of positive measure. However, a careful reading of Definition 12.9 shows that it is not concentrated, since these points are vertices of the simplex. Another perspective on this issue is that the RNS

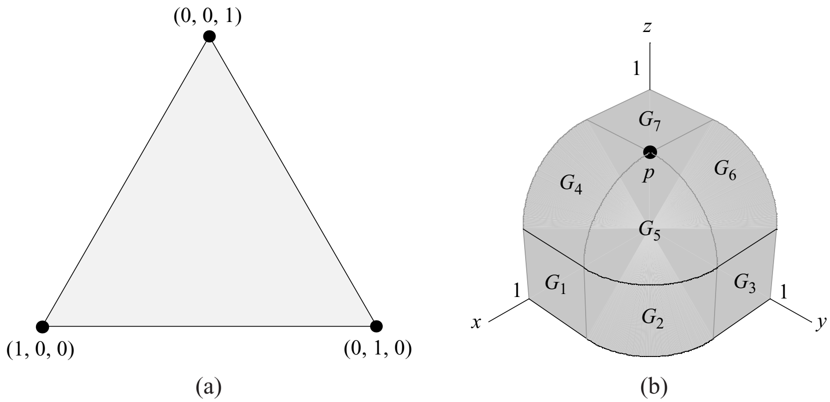


Figure 15.3

is not concentrated since no two players are in relative agreement on any set that has positive measure to each of these two players. (See Definition 12.11, the discussion following this definition, and the discussion of this notion in the absence of any assumptions about absolute continuity in Section 12D.)

The regions $G_1, G_2, \ldots, G_7$ that we have labeled in Figure 15.3b are meant to be closed regions on the outer boundary of the IPS. They therefore intersect on any common boundaries.

Our work in Chapter 12 enables us to see the correspondence between the RNS and the IPS. Let us begin by considering partitions of the form $P = \langle P_1, P_2, P_3 \rangle$ where $m_3(P_3) = 0$, i.e., partitions in which Player 3 thinks that he or she gets no cake. The corresponding RNS is a subset of the one-simplex consisting of the line segment between (1, 0, 0) and (0, 1, 0), and the corresponding IPS is the intersection of the full IPS with the xy plane. We can think of this new RNS as being obtained by projecting the full RNS along lines from (0, 0, 1) to this line segment. (We discussed this idea in Example 10.10 and illustrated it in Figure 10.4. Note that it is not clear to which point (0, 0, 1) should be projected. Since the cake associated with this point has measure zero to Player 1 and to Player 2, this issue can be ignored.) In this case, this new RNS is as in Figure 15.1a, which we discussed in the previous section. The corresponding IPS is similar to the IPS in Figure 15.1b. (The IPSs need not be the same, even though the RNSs are identical, since the precise amounts of cake associated with different parts of the RNS are not reflected in the RNS, but do affect the IPS.) Thus, we see that the intersection of the outer boundary of the full IPS with the xy plane consists of a line segment perpendicular to the x axis, a line segment perpendicular to the y axis, and a (non-straight line) curve between these line segments that meets each of them smoothly. This is the bottom curve of the two darker curves in Figure 15.3b.

Consider the cake associated with the point (0, 0, 1) of the RNS, which is Player 3's vertex. In the partitions described in the preceding paragraph, none of this cake was given to Player 3. If we now take this piece of cake away from Player 1 and Player 2 and give it to Player 3, we do not lower Player 1's or Player 2's evaluation of their own pieces, but Player 3 now believes that he or she has a piece of cake of positive measure. The set of points in the IPS corresponding to all partitions obtained in this way is the top curve of the two darker curves in Figure 15.3b. This curve is directly above the curve discussed in the previous paragraph. (It is a translation of the other curve by some fixed amount in the z direction. This fixed amount is Player 3's measure of the cake associated with the point (0, 0, 1) of the RNS.) Hence, the regions G_1, G_2, and G_3 on the outer boundary of the IPS are each perpendicular to the xy plane. This tells us that no point in any of these three regions of the outer boundary is on the outer Pareto boundary, with the possible exception of points on the top boundary of these regions, which is the top curve of the two darkened curves in the figure.

Notice that the RNS is symmetric with respect to x, y, and z. Thus, we may apply the preceding analysis with the roles of the players permuted. We find that the intersection of the outer boundary of the full IPS with the xz plane, and also with the yz plane, consists of two line segments, one perpendicular to each of the appropriate axes, and a (non-straight line) curve between these lines segments that meets each smoothly. Regions G_1, G_4, and G_7 on the outer boundary are each perpendicular to the xz plane, and regions G_3, G_6, and G_7 on the outer boundary are each perpendicular to the yz plane. This, combined with our discussion in the previous paragraph, tells us that no point of G_1, G_2, G_3, G_4, G_6, or G_7 is on the outer Pareto boundary, with the possible exception of points on the boundary of G_5.

It is not hard to see that the outer Pareto boundary consists precisely of (the closed) region G_5. Since the RNS is neither concentrated nor separable, we know from Theorems 12.14 and 12.18, respectively, that G_5 contains no line segments and no edge points. Also, since the part of the RNS that is in the interior of the simplex is spread out throughout the interior of the simplex, it follows that G_5 meets each of the other regions smoothly.

Recall one of the issues discussed in the previous section and illustrated in Figure 15.1. We considered a partition that gave all of the cake associated with (1, 0) to Player 1 and gave the rest of the cake to Player 2. Whereas our previous characterizations using w-association and using points of first contact of families of parallel lines with the IPS did not allow us to describe this partition, we found that we were able to do so if we allow the point ω, in the context of w-association, and the point α, which provided the coefficients for a family of parallel hyperplanes, to be chosen from S_H^+ rather than S^+. We consider two analogous situations for our present example. First, we consider the partition that gives each of two players only the cake associated with each player's vertex in the RNS and gives the rest of the cake to the third player. Next, we consider partitions that give one player only the cake associated with that player's vertex, and divides the rest of the cake between the other two players. We shall see that the first of these two situations is analogous to the situation discussed in the previous section for two players, whereas the second situation is not.

Returning to Figure 15.3, let P be the partition that gives all cake associated with the point $(1, 0, 0)$ of the RNS to Player 1, gives all cake associated with $(0, 1, 0)$ to Player 2, and gives the rest of the cake to Player 3. It is not hard to see that this partition is Pareto maximal and $m(P) = p$, where p is as in Figure 15.3b. (This is the minimum amount of cake that Player 1 and Player 2 must each receive in any Pareto maximal partition.) Our simple characterizations from previous chapters will not allow us to describe this situation (using either w-association or points of first contact of families of parallel planes with the IPS) and this is why we developed our iterated procedures in Sections 7C and 10C. As in the previous section, we can now use our simple approach with the aid of hyperreals.

First, consider the RNS. For any ω that is in the interior of the simplex, any partition that is w-associated with ω will give the cake associated with each of the three vertices of the simplex to the player corresponding to that vertex. We wish to choose ω so that any such partition that is w-associated with ω will give all cake associated with points in the interior of the simplex to Player 3. The idea is to choose ω sufficiently close to the line segment between $(1, 0, 0)$ and $(0, 1, 0)$. This is clearly impossible if $\omega \in S^+$. However, this is easy if we allow $\omega \in S_H^+$. Let $\varepsilon > 0$ be infinitesimal, fix positive real numbers ω_1 and ω_2 such that $\omega_1 + \omega_2 = 1$, and let $\omega = (\omega_1 - \varepsilon, \omega_2 - \varepsilon, 2\varepsilon)$. Then $\omega \in S_H^+$ and P is w-associated with ω.

Next, consider the IPS. As noted earlier, $m(P) = p$ and, thus, the goal is to find a family of parallel planes with positive coefficients that makes first contact with the IPS at p. Since all of the regions of the outer boundary of the IPS that meet at p meet in a smooth manner, it is not hard to see that any family of parallel hyperplanes with coefficients given by some $(\alpha_1, \alpha_2, \alpha_3) \in S^+$ will make first contact with the IPS at some point other than p. However, if we simply let α_1 and α_2 be any positive infinitesimals, and let $\alpha_3 = 1 - \alpha_1 - \alpha_2$, then we obtain the desired family.

We can also find the appropriate coefficients for a family of parallel planes that makes first contact with the IPS at p by using the RD function and Theorem 10.6. (As noted in the previous section, the RD function extends in a natural way to a function from S_H^+ to S_H^+, and Theorem 10.6 holds for this extended function.) As discussed earlier, for any infinitesimal $\varepsilon > 0$ and positive real numbers ω_1 and ω_2 with $\omega_1 + \omega_2 = 1$, P is w-associated with $\omega = (\omega_1 - \varepsilon, \omega_2 - \varepsilon, 2\varepsilon)$. We compute RD$(\omega)$ as follows:

$$\begin{aligned}
\mathrm{RD}(\omega) &= \mathrm{RD}(\omega_1 - \varepsilon, \omega_2 - \varepsilon, 2\varepsilon) \\
&= \left(\frac{1}{\frac{1}{\omega_1-\varepsilon} + \frac{1}{\omega_2-\varepsilon} + \frac{1}{2\varepsilon}}\right)\left(\frac{1}{\omega_1 - \varepsilon}, \frac{1}{\omega_2 - \varepsilon}, \frac{1}{2\varepsilon}\right) \\
&= \left(\frac{1}{1 + \frac{\omega_1-\varepsilon}{\omega_2-\varepsilon} + \frac{\omega_1-\varepsilon}{2\varepsilon}}, \frac{1}{\frac{\omega_2-\varepsilon}{\omega_1-\varepsilon} + 1 + \frac{\omega_2-\varepsilon}{2\varepsilon}}, \frac{1}{\frac{2\varepsilon}{\omega_1-\varepsilon} + \frac{2\varepsilon}{\omega_2-\varepsilon} + 1}\right)
\end{aligned}$$

We consider each of these three coordinates. Recalling that ε is a positive infinitesimal and ω_1 and ω_2 are positive real numbers, we see that $\frac{\omega_1-\varepsilon}{\omega_2-\varepsilon}$ is a positive hyperreal and $\frac{\omega_1-\varepsilon}{2\varepsilon}$ is an infinite positive hyperreal. It follows that $1 + \frac{\omega_1-\varepsilon}{\omega_2-\varepsilon} + \frac{\omega_1-\varepsilon}{2\varepsilon}$ is an infinite positive hyperreal, and, hence, $\frac{1}{1+\frac{\omega_1-\varepsilon}{\omega_2-\varepsilon}+\frac{\omega_1-\varepsilon}{2\varepsilon}}$ is a positive infinitesimal. Similarly, we find that $\frac{1}{\frac{\omega_2-\varepsilon}{\omega_1-\varepsilon}+1+\frac{\omega_2-\varepsilon}{2\varepsilon}}$ is a positive infinitesimal. Finally, we note that $\frac{1}{\frac{2\varepsilon}{\omega_1-\varepsilon}+\frac{2\varepsilon}{\omega_2-\varepsilon}+1} = 1 - \frac{1}{1+\frac{\omega_1-\varepsilon}{\omega_2-\varepsilon}+\frac{\omega_1-\varepsilon}{2\varepsilon}} - \frac{1}{\frac{\omega_2-\varepsilon}{\omega_1-\varepsilon}+1+\frac{\omega_2-\varepsilon}{2\varepsilon}}$ (which, of course, must be the case since RD$(\omega) \in S_H^+$). This is in agreement with our choice of $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Next we consider the second of the two situations mentioned earlier in this section. Let P be a partition that gives all cake associated with $(0, 0, 1)$ to Player 3 and divides the rest of the cake between Player 1 and Player 2. Clearly, there are many such partitions and not every one of these is Pareto maximal. In particular, if P is to be Pareto maximal, then the portion of cake that is to be divided between Player 1 and Player 2 must be divided in a Pareto maximal way. We want to use the notion of w-association to characterize which partitions of this form are Pareto maximal. If P is a partition that is w-associated with some $\omega \in S^+$, then P gives some cake to Player 3 that is not associated with the point $(0, 0, 1)$ and, hence, such a P is not of the type that we are presently considering. (This is analogous to the situation discussed in the previous section. In that situation, we saw that if ω is any point in the one-simplex of Figure 15.1a that is to the right of the point $(1, 0)$, then any partition that is w-associated with ω gives some cake to Player 1 that is associated with points in the interior of the simplex.) In Section 10C we considered the possibility of not insisting that ω be in S^+, but instead allowing $\omega = (0, 0, 1)$. This led us to Theorem 10.23, which did not give us a characterization of Pareto maximality but instead gave us two separate "if-then" statements. The problem with this approach, as we saw in Example 10.24 and we now see in the present example, is that this ω can make no distinctions between the points of C that go to Player 1 and those that go to Player 2. Thus, we see that we need a point that is closer to (0, 0, 1) than is any point in S^+ (and hence closer to (0, 0, 1) than any point in the RNS) but has a meaningful ratio (as opposed to $\frac{0}{0}$) between its first two coordinates. Of course, hyperreals provide precisely what we need.

Let us consider these types of partitions in terms of the IPS. As discussed earlier in this section, if P is such a partition, then $m(P)$ is on the top curve of the two darkened curves in Figure 15.3b. Since the outer Pareto boundary of the IPS consists precisely of region G_5, it follows that if P is a Pareto maximal partition that gives all of the cake associated with the point (0, 0, 1) of the RNS to Player 3 and divides the rest of the cake between Player 1 and Player 2, then $m(P)$ is on the bottom boundary of G_5. Since G_5 meets G_2 smoothly, we know that for no $\alpha \in S^+$ does the family of parallel planes with coefficients given by α make first contact with the IPS at a point along the lower boundary of G_5. What we need is a family of parallel planes that is not perpendicular to the xy plane but is infinitesimally tilted from such a family. In other words, we want a point $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ to provide the coefficients of a family of parallel hyperplanes where α_3 is positive but smaller than any positive real number. Again, we see that hyperreals provide us with exactly what we need.

We consider an example. Figure 15.4 shows the same RNS and the IPS as in Figure 15.3. In the RNS shown in Figure 15.4a, we have drawn a dashed

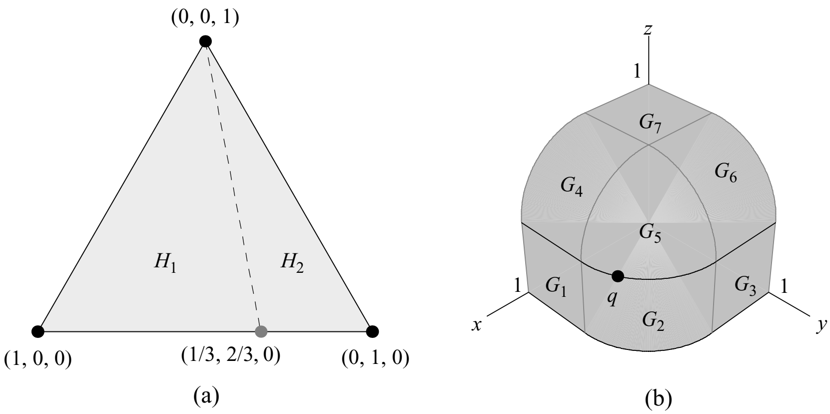


Figure 15.4

line from (0, 0, 1) to the point $(\frac{1}{3}, \frac{2}{3}, 0)$. (The point $(\frac{1}{3}, \frac{2}{3}, 0)$ is not in the RNS.) Our intention is that H_1 includes all points of the RNS that are to the left of the dashed line and H_2 includes all points of the RNS that are to the right of the dashed line. The points of the dashed line, except for the point (0, 0, 1), can be assigned arbitrarily to either H_1 or H_2, since this set of points is associated with a piece of cake of measure zero. Let Q be the partition that gives to Player 1 all cake associated with H_1, gives to Player 2 all cake associated with H_2, and gives to Player 3 all cake associated with (0, 0, 1). Then Q is the type of partition just described. We found that if Q is w-associated with some point ω, in the interior of the simplex, then ω must be closer to (0, 0, 1) than is any point in S^+ and must have the appropriate ratio between its first two coordinates. If we allow ω to be chosen from S^+_H, instead of only from S^+, then we can easily find such an ω so that Q is w-associated with ω. Fix an infinitesimal $\varepsilon > 0$ and let $\omega = (\varepsilon, 2\varepsilon, 1 - 3\varepsilon)$. Then $\omega \in S^+_H$ and Q is w-associated with ω.

Next, consider the IPS in Figure 15.4b. In this figure, we have labeled the point q, where $q = m(Q)$. To find a point whose coefficients yield a family of parallel planes that makes first contact with the IPS at q, we need a point $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that α_3 is positive but smaller than any positive real number. Thus, we may simply let α_3 be any positive infinitesimal. For any such α, the corresponding family of parallel planes will make first contact with the IPS somewhere along the top curve of the two darkened curves in Figure 15.4b. The proper choice for α_1 and α_2 will guarantee that q is the point of first contact. Notice that α_1 and α_2 are both positive hyperreals that are neither infinitesimal nor infinite.

We can be precise about the point α. Since the partition Q is w-associated $\omega = (\varepsilon, 2\varepsilon, 1 - 3\varepsilon)$, it follows that the family of parallel planes with coefficients given by RD(ω) makes first contact with the IPS at q. We compute RD(ω) as follows:

$$\begin{aligned}
\mathrm{RD}(\omega) &= \mathrm{RD}(\varepsilon, 2\varepsilon, 1 - 3\varepsilon) \\
&= \left(\frac{1}{\frac{1}{\varepsilon} + \frac{1}{2\varepsilon} + \frac{1}{1-3\varepsilon}}\right)\left(\frac{1}{\varepsilon}, \frac{1}{2\varepsilon}, \frac{1}{1-3\varepsilon}\right) \\
&= \left(\frac{1}{1 + \frac{1}{2} + \frac{\varepsilon}{1-3\varepsilon}}, \frac{1}{2 + 1 + \frac{2\varepsilon}{1-3\varepsilon}}, \frac{1}{\frac{1-3\varepsilon}{\varepsilon} + \frac{1-3\varepsilon}{2\varepsilon} + 1}\right) \\
&= \left(\frac{2}{3 + \frac{2\varepsilon}{1-3\varepsilon}}, \frac{1}{3 + \frac{2\varepsilon}{1-3\varepsilon}}, \frac{1}{\frac{1-3\varepsilon}{\varepsilon} + \frac{1-3\varepsilon}{2\varepsilon} + 1}\right)
\end{aligned}$$

The third coordinate is of the form $\frac{1}{(\text{infinite hyperreal}) + (\text{infinite hyperreal}) + 1}$ and is therefore infinitesimal. This is consistent with our preceding discussion. We also note that the ratio of the first coordinate to the second coordinate is two. This is what we would expect when applying the RD function to ω, since the ratio of the first coordinate to the second coordinate of ω is $\frac{1}{2}$.

The two situations examined in this section illustrate that, for the cake and measures corresponding to the RNS of Figures 15.3a and 15.4a,

- a partition that gives each of two players only the cake associated with that player's vertex in the RNS and gives the rest of the cake to the third player is Pareto maximal and is w-associated with some $\omega \in S_H^+$, and
- a partition that gives some player only the cake associated with that player's vertex in the RNS and divides the rest of the cake between the other two players is Pareto maximal if and only if it is w-associated with some $\omega \in S_H^+$.

Our work in Chapter 10 tells us that any partition that is not of one of these two types is Pareto maximal if and only if it is w-associated with some $\omega \in S^+$. Hence we see that, at least for the situation depicted in Figures 15.3a and 15.4a,

a partition P is Pareto maximal
if and only if
for some $\omega \in S_H^+$, P is w-associated with ω.

Similarly, based on our discussion of the points p and q in Figures 15.3b and 15.4b, respectively, we find that any point on the boundary of region G_5 is a point of first contact of a family of parallel planes with coefficients given by some $\alpha \in S_H^+$, and we know from our work in Chapter 7 that any interior point of G_5 is a point of first contact of a family of parallel planes with coefficients given by some $\alpha \in S^+$. Also, a point of the IPS that is not in G_5 is not a point of first contact with the IPS of any family of parallel planes with coefficients given by any $\alpha \in S_H^+$. Then, since G_5 is the outer Pareto boundary, it follows that, at least for the situation depicted in Figures 15.3b and 15.4b,

a partition P is Pareto maximal
if and only if
for some $\alpha \in S_H^+$, the family of parallel planes with coefficients given by α makes first contact with the IPS at $m(P)$.

Or, equivalently,

a partition P is Pareto maximal
if and only if
for some $\alpha \in S_H^+$, P maximizes the convex combination of the measures corresponding to α.

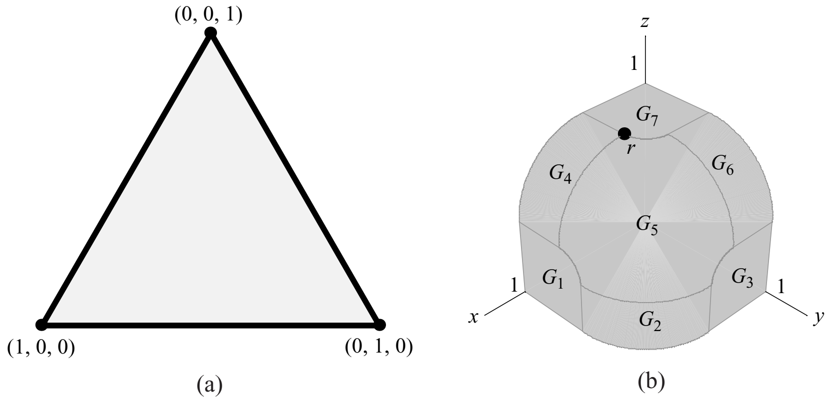


Figure 15.5

We have not presented a complete and general proof that our characterization works for all cakes and any three measures. This will follow from our general n-player characterization in the next section.

Next, we present an example that is similar to the one we have just considered, but is different in a significant way.

Example 15.3 The RNS and IPS in Figure 15.5 are obviously similar to the RNS and IPS in Figures 15.3 and 15.4. The difference in the RNSs is that in Figure 15.5a we have darkened the three open line segments on the boundary of the RNS. These line segments are not darkened in Figures 15.3a and 15.4a. This indicates that for the situation depicted in Figure 15.5a there is a piece of cake of positive measure that is associated with each of these line segments, and the associated points of the RNS are spread out along each of these line segments. We assume, as in Figures 15.3a and 15.4a, that there is a piece of cake of positive measure associated with each vertex of the simplex. Then each of the line segments making up the boundary of the RNS is as in the two-player example illustrated in Figure 15.1a. We also assume, again as in Figures 15.3a and 15.4a, that there is a piece of cake of positive measure whose associated subset of the RNS is spread out over the entire interior of the simplex.

The effect of this change in the IPS can be seen by comparing Figure 15.5b with Figure 15.3b or 15.4b. As an example, we focus on region G_7 in each figure. In both situations, this region corresponds to giving to Player 3 all bits of cake that have any value to him or her and dividing the rest of the cake between Player 1 and Player 2. In other words, any cake not associated with points of the RNS that are on the closed line segment between $(1, 0, 0)$ and

(0, 1, 0) is given to Player 3, and any cake that is associated with points on this line segment is divided between Player 1 and Player 2. We can see from Figures 15.3a and 15.4a that, in this case, some of the cake that is to be divided between Player 1 and Player 2 is associated with the point (1, 0, 0), some with the point (0, 1, 0), and none is associated with any point in the interior of the line segment between these two points. This tells us that on this piece of cake the measures of Player 1 and of Player 2 concentrate on disjoint sets. Hence, the associated part of the IPS in Figures 15.3b and 15.4b is a rectangular region, G_7. On the other hand, in the IPS of Figure 15.5a, there is a positive-measure piece of cake associated with the open line segment between (1, 0, 0) and (0, 1, 0), in addition to cake of positive measure that is associated with each of these points. As noted earlier, this is as in Figure 15.1a. Hence, the IPS corresponding to this division of cake between Player 1 and Player 2 is as in Figure 15.1b. This yields the curved part of G_7 in Figure 15.5b. Since the RNS is symmetric with respect to the three players, this analysis also explains the curved parts of regions G_1 and G_3. As was the case for the situation depicted in Figures 15.3b and 15.4b, we see that the outer Pareto boundary of the IPS is (the closed) region G_5. Also, since the part of the RNS that is in the interior of the simplex is spread out throughout the interior of the simplex, it follows that G_5 meets each of the other regions smoothly.

Using the RNS of Figure 15.5a, let $R = \langle R_1, R_2, R_3 \rangle$ be the partition where

- R_1 is the set of all bits of cake associated with the point (1, 0, 0), or with points on the open line segment between (1, 0, 0) and (0, 1, 0);
- R_2 is the set of all bits of cake associated with the point (0, 1, 0); and
- R_3 is the set of all bits of cake associated with the point (0, 0, 1), with points on the open line segment between (0, 0, 1) and (1, 0, 0), with points on the open line segment between (0, 0, 1) and (0, 1, 0), or with any interior point of the simplex.

We claim that R is Pareto maximal. We establish this using partition ratios. We compute the partition ratios as follows. (For the relevant definition and notation, see Definition 8.20 and Notation 8.21.)

$$\mathrm{pr}_{12} = \sup\left\{\frac{m_2(A)}{m_1(A)} : A \subseteq R_1 \text{ and either } m_1(A) > 0 \text{ or } m_2(A) > 0\right\} = \infty^*$$

$$\mathrm{pr}_{21} = \sup\left\{\frac{m_1(A)}{m_2(A)} : A \subseteq R_2 \text{ and either } m_1(A) > 0 \text{ or } m_2(A) > 0\right\} = 0$$

$$\mathrm{pr}_{13} = \sup\left\{\frac{m_3(A)}{m_1(A)} : A \subseteq R_1 \text{ and either } m_1(A) > 0 \text{ or } m_3(A) > 0\right\} = 0$$

$$\mathrm{pr}_{31} = \sup\left\{\frac{m_1(A)}{m_3(A)} : A \subseteq R_3 \text{ and either } m_1(A) > 0 \text{ or } m_3(A) > 0\right\} = \infty^*$$

$$\mathrm{pr}_{23} = \sup\left\{\frac{m_3(A)}{m_2(A)} : A \subseteq R_2 \text{ and either } m_2(A) > 0 \text{ or } m_3(A) > 0\right\} = 0$$

$$\mathrm{pr}_{32} = \sup\left\{\frac{m_2(A)}{m_3(A)} : A \subseteq R_3 \text{ and either } m_2(A) > 0 \text{ or } m_3(A) > 0\right\} = \infty^*$$

Next, we compute all cyclic products. (For the relevant arithmetic rules, see Definition 8.22.)

$$\mathrm{pr}_{12}\mathrm{pr}_{21} = \mathrm{pr}_{13}\mathrm{pr}_{31} = \mathrm{pr}_{23}\mathrm{pr}_{32} = (\infty^*)(0) = 0$$
$$\mathrm{pr}_{12}\mathrm{pr}_{23}\mathrm{pr}_{31} = (\infty^*)(0)(\infty^*) = 0$$
$$\mathrm{pr}_{32}\mathrm{pr}_{21}\mathrm{pr}_{13} = (\infty^*)(0)(0) = 0$$

Since each cyclic product is less than one, it follows from Theorem 8.24 that R is Pareto maximal. (We could also have also established the Pareto maximality of R using Theorem 6.4 with $\gamma = \langle\{3\}, \{1\}, \{2\}\rangle$.)

The point $m(R) = r$ of the IPS is as shown in Figure 15.5b. It is the point of intersection of regions G_4, G_5, and G_7. If we wish to characterize the Pareto maximal points as in our example in the section, then we must find $\omega \in S_H^+$ and $\alpha \in S_H^+$ such that R is w-associated with ω, and the family of parallel planes with coefficients given by α makes first contact with the IPS at r. (Of course, this is one problem, not two, since $\mathrm{RD}(\omega) = \alpha$.) We shall do so in the next section. For now, we simply observe that finding such an ω and α is harder than for the previous example in this section. We consider the IPS and the problem of finding α. The issues concerning the RNS and ω are similar.

Recall our discussion concerning the point p in Figure 15.3b and the point q in Figure 15.4b. For the point p in Figure 15.3b, we found that if we let α_1 and α_2 be any positive infinitesimals and let $\alpha_3 = 1 - \alpha_1 - \alpha_2$, then the family of parallel planes with coefficients given by $(\alpha_1, \alpha_2, \alpha_3)$ makes first contact with this IPS at p. And, for the point q in Figure 15.4b, we found that if we let α_3 be any positive infinitesimal, then there are positive hyperreals α_1 and α_2 that are neither infinitesimal nor infinite and are such that the family of parallel planes with coefficients given by $(\alpha_1, \alpha_2, \alpha_3)$ makes first contact with this IPS at q. The present situation is different. Suppose that $(\alpha_1, \alpha_2, \alpha_3) \in S_H^+$ is such that the family of parallel planes with coefficients given by $(\alpha_1, \alpha_2, \alpha_3)$ makes first contact with this IPS at r. In order that this family not make contact with an interior point of region G_4, G_5, or G_6 before it makes contact with r, it must be that $\alpha_3 = 1 - \varepsilon$ for some positive infinitesimal ε. Then $\alpha_1 + \alpha_2 = \varepsilon$ and, hence,

α_1 and α_2 are each infinitesimal. But if this family is to make first contact with r and not with some other point along the boundary between G_5 and G_7, then, in some sense, α_2 must be infinitesimal compared to α_1. We shall make sense of this using the notion of "higher-order hyperreals" in the next section.

15D. The Characterization

We illustrated our characterization theorem in the case of two and three players in Examples 15.1 and 15.2, respectively. In this section, we consider the general case of n players. We assume that $m_1, m_2, \ldots, m_n$ are measures on some cake C that may or may not be absolutely continuous with respect to each other. As in previous chapters, the term "almost every" refers to the measure $\mu = m_1 + m_2 + \cdots + m_n$ unless otherwise stated. We will have occasion to use the term "almost every" in one other sense. If $\delta \subseteq \{1, 2, \ldots, n\}$ then, as in Chapter 10, we let $\mu^\delta = \sum_{i \in \delta} m_i$. When we mean "almost every" with respect to this measure, we shall explicitly say so. If both uses of "almost every" occur in the same context, then we shall always specify to which measure this refers.

Our characterization involves the notion of a partition P being w-associated with some $\omega \in S_H^+$. After presenting this result, we shall obtain, as an easy corollary, a characterization involving maximization of convex combinations of measures (which, as we have seen, is equivalent to a characterization involving points of first contact of families of parallel hyperplanes with the IPS).

We need not give the definition of w-associated, since it is precisely the same as in Definition 10.4. We shall need to evaluate the truth of inequalities of the form $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$, as in part a of the definition. In Chapter 10, we adopted various conventions regarding this inequality. (For example, we declared that the expression "$\frac{0}{0} \geq 0$" is always false.) We continue to follow these same conventions in this section. However, we do need to extend these conventions because of the presence of hyperreal numbers that are not real. We recall that, for each $i = 1, 2, \ldots, n$ and $a \in C$, $f_i(a)$ is a non-negative real number. Thus, as was the case in Chapter 10, each $\frac{f_i(a)}{f_j(a)}$ is a non-negative real number divided by a non-negative real number and, hence, is equal to zero, a positive real number, ∞, or else is of the form $\frac{0}{0}$. However, for each $i = 1, 2, \ldots, n$, ω_i is a non-negative hyperreal and, hence, each $\frac{\omega_i}{\omega_j}$ is equal to zero, a positive infinitesimal, a positive hyperreal that is neither infinitesimal nor infinite, an infinite hyperreal, ∞, or else is of the form $\frac{0}{0}$. (We consider the quotient of any positive hyperreal, infinitesimal or not, with zero to be ∞.)

The truth or falsity of most of the new inequalities involving hyperreals follows from the basic facts on hyperreals given in Section 15A. The following are natural extensions of our previous conventions in Chapter 10. For any positive hyperreal κ (infinitesimal, infinite, or neither),

- $\infty \geq \kappa$ is always true and
- $\frac{0}{0} \geq \kappa$ is always false.

Our characterization theorem is the following.

Theorem 15.4 *A partition P is Pareto maximal if and only if it is w-associated with ω for some $\omega \in S_H^+$.*

This result takes the place of Theorem 10.28, which characterized Pareto maximality using the notion of w-association with a partition sequence pair. The present result is obviously simpler than that result. In particular, it is not an iterated approach, as was Theorem 10.28. Theorem 15.4 will easily yield a corollary that will take the place of Theorems 7.13 and 7.18, which characterized Pareto maximality using the notions of a-maximization of a partition sequence pair and b-maximization of a partition sequence pair, respectively. Our choice of which previous result to replace by proving a simpler theorem, and which to replace by an easy corollary to that theorem, was arbitrary.

Before proving Theorem 15.4, we state and discuss a definition, give an informal description of the idea behind the theorem, and then return to Example 15.3, which we introduced at the end of the previous section.

Definition 15.5 For hyperreals $\kappa, \lambda > 0$, κ is a *higher-order hyperreal* than λ if and only if $\frac{\kappa}{\lambda}$ is infinitesimal. If neither κ nor λ is higher order than the other, then we shall say that κ and λ *have the same order*.

(The notion of higher-order hyperreal certainly makes sense for negative hyperreals. We chose to insist that κ and λ be positive in Definition 15.5 since we shall only need to apply this notion to positive hyperreals and because it will simplify our presentation slightly.)

Notice that, if κ and λ are positive hyperreals and κ is a higher-order hyperreal than λ, then $\kappa < \lambda$. Hence, higher-order hyperreals are closer to zero. We also observe that if κ has the same order as λ, then $\frac{\kappa}{\lambda}$ is neither infinitesimal nor infinite and, hence, is either equal to or is infinitesimally close to a positive real number.

"Having the same order" is an equivalence relation on the set of all positive hyperreals. For any two of the induced equivalence classes, every hyperreal in one of the classes is less than every hyperreal in the other. Hence, the usual ordering of the hyperreals induces an ordering of the associated equivalence classes.

It is easy to see that there are infinitely many equivalence classes since, for any infinitesimal $\varepsilon > 0$, the hyperreals $1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots,$ are all positive and, for any non-negative integers m and n with $m < n$, $\frac{\varepsilon^n}{\varepsilon^m} = \varepsilon^{n-m}$ is infinitesimal. Hence, ε^n is of higher order than ε^m.

Next, we give an informal perspective on Theorem 15.4. Suppose that partition P is Pareto maximal. By Theorem 10.28, P is w-associated with some partition sequence pair (ω, γ). Set $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$. (For the definitions of partition sequence pair and of w-association with a partition sequence pair, see Definitions 7.11 and 10.26. In particular, we assume that P and (ω, γ) satisfy condition bi of Definition 10.26.) We recall that in Chapter 10 we developed an iterative perspective on this notion. We imagine that at the first stage cake is given out to players named by γ_1, at the second stage cake is given out to the players named by γ_2, etc. By condition bi of Definition 10.26, we know that each player receiving cake at any given stage of this process believes that the cake given out at previous stages has measure zero. Using our "social hierarchy" terminology (see the discussion following the proof of Theorem 7.13), we may say that players named by later γ_ks have higher social status than players named by earlier γ_ks, since each player believes that the only bits of cake given out at earlier stages of the process are bits of cake that he or she does not care about. The different stages in the iteration correspond to different levels in the social hierarchy. We shall now see that by using hyperreals we no longer need to use an iterative approach. We can distinguish between different levels in this hierarchy by using different orders of hyperreals. This is not reflected in the statement of Theorem 15.4, but will be central to the proof. Suppose that the partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ is w-associated with $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S_H^+$. For any $i, j = 1, 2, \ldots, n$, if ω_i is a higher-order hyperreal than ω_j, then $\frac{\omega_i}{\omega_j}$ is infinitesimal. Therefore, for any $a \in C$, if $f_i(a) > 0$, then $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$, regardless of the value of $f_j(a)$. This tells us that no player receives a piece of cake that has positive value to some other player whose associated component in ω is of higher order. In other words, players associated with higher-order hyperreals have higher social status than players associated with lower-order hyperreals.

Finally, before proving Theorem 15.4, we return to Example 15.3, which we considered at the end of the previous section. We discussed the Pareto maximal point r in the IPS of Figure 15.5b. We found that if some family of parallel hyperplanes with coefficients given by $(\alpha_1, \alpha_2, \alpha_3)$ is to make first contact with the IPS at r, then we must have $\alpha_3 = 1 - \varepsilon$ for some positive infinitesimal ε. This implies that $\alpha_1 + \alpha_2 = 1$ and, hence, that α_1 and α_2 are each infinitesimal. We also saw that "α_2 must be infinitesimal compared to α_1." In the last section, this was meant to be an informal notion. Now we can be precise. We simply insist that α_2 be a higher-order hyperreal than α_1. Fix any positive infinitesimal ε and set $\alpha_3 = 1 - \varepsilon$. Since $\alpha_1 + \alpha_2 = \varepsilon$, we can simply let $\alpha_2 = \varepsilon^2$ and $\alpha_1 = \varepsilon - \varepsilon^2$. (Since ε is a positive infinitesimal, $\varepsilon^2 < \varepsilon$ and, therefore, $\alpha_1 = \varepsilon - \varepsilon^2 > 0$.) Then $\alpha_1 + \alpha_2 = \varepsilon$ and $\frac{\alpha_2}{\alpha_1} = \frac{\varepsilon^2}{\varepsilon - \varepsilon^2} = \frac{\varepsilon}{1-\varepsilon}$. Since $1 - \varepsilon$ is a positive hyperreal that is not infinitesimal, it follows that $\frac{\varepsilon}{1-\varepsilon}$ is infinitesimal and, hence, α_2 is a higher-order hyperreal than α_1. The family of parallel planes with coefficients given by $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (\varepsilon - \varepsilon^2, \varepsilon^2, 1 - \varepsilon)$ makes first contact with the IPS of Figure 15.5b at the point r.

How about the corresponding RNS in Figure 15.5a? Let R be a partition such that $m(R) = r$, and suppose that R is w-associated with $\omega = (\omega_1, \omega_2, \omega_3) \in S_H^+$. Since all bits of cake associated with interior points of the line segment connecting the points $(1, 0, 0)$ and $(0, 1, 0)$ are given to Player 1, $\frac{\omega_1}{\omega_2}$ must be infinitesimal and, hence, ω_1 is a higher-order hyperreal than ω_2. And, since all bits of cake associated with interior points of the simplex or with points on the open line segment between the points $(0, 0, 1)$ and $(1, 0, 0)$ are given to Player 3, $\frac{\omega_3}{\omega_1}$ must be infinitesimal; hence, ω_3 is a higher-order hyperreal than ω_1. We now apply the RD function and Theorem 10.6 to the α from the previous paragraph to obtain an ω with which R is w-associated, and then we compare the result of this computation with what we have just discovered about ω.

By the extension of the RD function and of Theorem 10.6 to the hyperreal setting, which we discussed previously, we know that R is w-associated with the point RD(α). Set $\omega = (\omega_1, \omega_2, \omega_3) = RD(\alpha)$. We compute the coordinates of ω as follows:

$$
\begin{aligned}
\omega = (\omega_1, \omega_2, \omega_3) &= \mathrm{RD}(\alpha) = \mathrm{RD}\left(\varepsilon - \varepsilon^2, \varepsilon^2, 1 - \varepsilon\right) \\
&= \left(\frac{1}{\frac{1}{\varepsilon-\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{1}{1-\varepsilon}}\right)\left(\frac{1}{\varepsilon - \varepsilon^2}, \frac{1}{\varepsilon^2}, \frac{1}{1-\varepsilon}\right) \\
&= \left(\frac{1}{1 + \frac{\varepsilon-\varepsilon^2}{\varepsilon^2} + \frac{\varepsilon-\varepsilon^2}{1-\varepsilon}}, \frac{1}{\frac{\varepsilon^2}{\varepsilon-\varepsilon^2} + 1 + \frac{\varepsilon^2}{1-\varepsilon}}, \frac{1}{\frac{1-\varepsilon}{\varepsilon-\varepsilon^2} + \frac{1-\varepsilon}{\varepsilon^2} + 1}\right) \\
&= \left(\frac{1}{1 + \frac{1-\varepsilon}{\varepsilon} + \varepsilon}, \frac{1}{\frac{\varepsilon}{1-\varepsilon} + 1 + \frac{\varepsilon^2}{1-\varepsilon}}, \frac{1}{\frac{1}{\varepsilon} + \frac{1-\varepsilon}{\varepsilon^2} + 1}\right)
\end{aligned}
$$

We consider each of the three coordinates. We first observe that $\frac{1-\varepsilon}{\varepsilon}$ and $\frac{1-\varepsilon}{\varepsilon^2}$ are each of the form $\frac{\text{positive hyperreal that is not infinitesimal}}{\text{positive infinitesimal}}$ and, hence, are infinite positive hyperreals. Therefore $1 + \frac{1-\varepsilon}{\varepsilon} + \varepsilon$ and $\frac{1}{\varepsilon} + \frac{1-\varepsilon}{\varepsilon^2} + 1$ are each infinite positive hyperreals. It follows that $\omega_1 = \frac{1}{1+\frac{1-\varepsilon}{\varepsilon}+\varepsilon}$ and $\omega_3 = \frac{1}{\frac{1}{\varepsilon}+\frac{1-\varepsilon}{\varepsilon^2}+1}$ are each positive infinitesimals.

Next, we consider $\omega_2 = \frac{1}{\frac{\varepsilon}{1-\varepsilon}+1+\frac{\varepsilon^2}{1-\varepsilon}}$. Note that $\frac{\varepsilon}{1-\varepsilon}$ and $\frac{\varepsilon^2}{1-\varepsilon}$ are each of the form $\frac{\text{positive infinitesimal}}{\text{positive hyperreal that is not infinitesimal}}$ and, hence, are each positive infinitesimals. It follows that $\frac{\varepsilon}{1-\varepsilon} + 1 + \frac{\varepsilon^2}{1-\varepsilon}$ is a hyperreal that is infinitesimally larger than one. This implies that $\omega_2 = \frac{1}{\frac{\varepsilon}{1-\varepsilon}+1+\frac{\varepsilon^2}{1-\varepsilon}}$ is a hyperreal that is infinitesimally smaller than one.

Finally, we compute the ratio $\frac{\omega_3}{\omega_1}$:

$$\frac{\omega_3}{\omega_1} = \frac{\left(\frac{1}{\frac{1}{\varepsilon}+\frac{1-\varepsilon}{\varepsilon^2}+1}\right)}{\left(\frac{1}{1+\frac{1-\varepsilon}{\varepsilon}+\varepsilon}\right)} = \frac{1 + \frac{1-\varepsilon}{\varepsilon} + \varepsilon}{\frac{1}{\varepsilon} + \frac{1-\varepsilon}{\varepsilon^2} + 1} = \varepsilon$$

Since ε is infinitesimal, this implies that ω_3 is a higher-order hyperreal than ω_1. Since ω_1 is infinitesimal and ω_2 is a positive hyperreal that is not infinitesimal, it follows that ω_1 is a higher-order hyperreal than ω_2. Thus, $\omega = (\omega_1, \omega_2, \omega_3)$ is in agreement with our preceding discussion.

Proof of Theorem 15.4: Fix a partition P. For the forward direction, we assume that P is Pareto maximal. By Theorem 10.28, P is w-associated with some partition sequence pair (ω'', γ). Suppose that $\omega'' = (\omega''_1, \omega''_2, \ldots, \omega''_n)$, $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$, and let us assume that P and (ω'', γ) satisfy condition bi of Definition 10.26. (We may assume this, without loss of generality, since, if P and $(\omega'', \gamma\ )$ instead satisfy condition bii of the definition, then P and the partition sequence pair (ω'', γ') satisfy condition bi, where $\gamma' = \langle \gamma_t, \gamma_{t-1}, \ldots, \gamma_1 \rangle$.)

Recall that ω'' is a sequence of positive real numbers and γ is a partition of $\{1, 2, \ldots, n\}$. We shall use ω'' to define $\omega \in S_H^+$ so that P is w-associated with ω. We first define a sequence of positive hyperreals $(\omega'_1, \omega'_2, \ldots, \omega'_n)$. Fix any positive infinitesimal ε. For each $i = 1, 2, \ldots, n$, let $\omega'_i = \varepsilon^k \omega''_i$, where k is such that $i \in \gamma_k$.

As we shall see, the sequence $(\omega'_1, \omega'_2, \ldots, \omega'_n)$ has the desired ratios between terms, but we must divide by the sum of the terms to obtain a point in S_H^+. Let $\lambda = \omega'_1 + \omega'_2 + \cdots + \omega'_n$ and set $\omega = (\omega_1, \omega_2, \ldots, \omega_n) = (1/\lambda)(\omega'_1, \omega'_2, \ldots, \omega'_n)$. Then $\omega \in S_H^+$.

We claim that P is w-associated with ω. We must show that, for all distinct $i, j = 1, 2, \ldots, n$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every $a \in P_i$. Fix such an i and j and assume that $i \in \gamma_k$ and $j \in \gamma_{k'}$. We consider three cases.

Case 1: $k < k'$. By condition bi of Definition 10.26, $f_j(a) = 0$ for almost every $a \in P_i$. Hence, for almost every such a, $\frac{f_i(a)}{f_j(a)} = \frac{\text{positive number}}{0} = \infty$ and, hence, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$.

Case 2: $k = k'$. By condition a of Definition 10.26, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every (with respect to μ^{γ_k}) $a \in P_i$. Since $i \in \gamma_k$, this implies that $m_i(\{a \in P_i : \frac{f_i(a)}{f_j(a)} < \frac{\omega_i}{\omega_j}\}) = 0$. By the Pareto maximality of P, it follows that P is non-wasteful and, hence, $\mu(\{a \in P_i : \frac{f_i(a)}{f_j(a)} < \frac{\omega_i}{\omega_j}\}) = 0$. This tells us that $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$ for almost every (with respect to μ) $a \in P_i$.

Case 3: $k > k'$. Then $\frac{\omega_i}{\omega_j} = \frac{(1/\lambda)\omega_i{}'}{(1/\lambda)\omega_j{}'} = \frac{\omega_i{}'}{\omega_j{}'} = \frac{\varepsilon^k \omega_i{}''}{\varepsilon^{k'} \omega_j{}''} = (\varepsilon^{k-k'})\frac{\omega_i{}''}{\omega_j{}''}$. Since $k > k'$, it follows that $\varepsilon^{k-k'}$ is infinitesimal, and since ω_i'' and ω_j'' are both positive real numbers, we know that $\frac{\omega_i''}{\omega_j''}$ is a positive real number. Hence, $\frac{\omega_i}{\omega_j} = (\varepsilon^{k-k'})\frac{\omega_i''}{\omega_j''}$ is infinitesimal. We claim that $f_i(a) > 0$ for almost every $a \in P_i$. If this were not the case, then there would exist some $A \subseteq P_i$ such that $m_i(A) = 0$ but A has positive measure to some other player. This violates the fact that P is Pareto maximal and, hence, is non-wasteful. Therefore, for almost every $a \in P_i$, $f_i(a) > 0$, and thus $\frac{f_i(a)}{f_j(a)}$ is either equal to a positive real number or else is equal to infinity. Since $\frac{\omega_i}{\omega_j}$ is infinitesimal, it follows in either case that $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$.

This establishes that P is w-associated with ω and, hence, completes the proof of the forward direction of the theorem.

For the reverse direction, we assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S_H^+$ and that P is w-associated with ω. For $i, j = 1, 2, \ldots, n$, we shall write $\omega_1 \sim \omega_2$ to denote the fact that ω_1 and ω_2 have the same order. The relation "$\sim$" is an equivalence relation on the set $\{\omega_1, \omega_2, \ldots, \omega_n\}$, and the usual ordering of the hyperreals induces an ordering of the associated equivalence classes.

We shall again use Theorem 10.28. We wish to define a partition sequence pair (ω'', γ) with which P is w-associated. Let $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_t \rangle$ be the partition of $\{1, 2, \ldots, n\}$ satisfying that

a. for each $k = 1, 2, \ldots, t$, $\{\omega_i : i \in \gamma_k\}$ is one of the equivalence classes induced by the relation "$\sim$" and
b. for each $k, k' = 1, 2, \ldots, t$ with $k < k'$, every element of the equivalence class associated with $\gamma_{k'}$ is a higher-order hyperreal and, hence, is smaller than every element of the equivalence class associated with γ_k.

For each $k = 1, 2, \ldots, t$, if $\omega_i, \omega_j \in \gamma_k$ then, as discussed earlier in this section, $\frac{\omega_i}{\omega_j}$ is either equal to or is infinitesimally close to a positive real number. This implies that $\frac{\omega_i}{\lambda_k}$ is infinitesimally close to a positive real number, where $\lambda_k = \sum_{i \in \gamma_k} \omega_i$. We recall that $[\frac{\omega_i}{\lambda_k}]$ denotes the real number to which this hyperreal is equal or is infinitesimally close. Thus, $[\frac{\omega_i}{\lambda_k}]$ is a positive real number. For each

$i = 1, 2, \ldots, n$, define ω_i'' as follows:

$$\omega_i'' = \left[\frac{\omega_i}{\lambda_k}\right], \text{ where } i \in \gamma_k$$

Set $\omega'' = (\omega_1'', \omega_2'', \ldots, \omega_n'')$. Then each ω_i'' is a positive real number and it is straightforward to verify that, for each $k = 1, 2, \ldots, t$, $\sum_{i \in \gamma_k} \omega_i{}'' = 1$. Hence, (ω'', γ) is a partition sequence pair. We claim that P is w-associated with (ω'', γ). We must show that P and $(\omega'', \gamma\ )$ satisfy the conditions of Definition 10.26.

We first consider condition a of Definition 10.26. Fix some $k = 1, 2, \ldots, t$ and distinct $i, j \in \gamma_k$. We must show that $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i''}{\omega_j{}''}$ for almost every (with respect to μ^{γ_k}) $a \in P_i$. Since P is w-associated with ω, we know that, for almost every (with respect to μ) $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$. This implies that, for almost every (with respect to μ^{γ_k}) $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$. For each such a, since $\frac{f_i(a)}{f_j(a)}$ is a positive real number and $\frac{\omega_i}{\omega_j}$ is equal to or is infinitesimally close to a positive real number, it follows that $\frac{f_i(a)}{f_j(a)} \geq [\frac{\omega_i}{\omega_j}]$. But $[\frac{\omega_i}{\omega_j}] = [\frac{(\omega_i/\lambda_k)}{(\omega_j/\lambda_k)}] = \frac{[(\omega_i/\lambda_k)]}{[(\omega_j/\lambda_k)]} = \frac{\omega_i''}{\omega_j''}$. Hence, for almost every (with respect to μ^{γ_k}) $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i''}{\omega_j{}''}$. This establishes condition a of Definition 10.26.

Finally, we show that P and (ω'', γ) satisfy condition bi of Definition 10.26. Fix some $k = 1, 2, \ldots, n$. We must show that

(*) for almost every $a \in C$, $a \in \bigcup_{i \in \gamma_k} P_i$ if and only if $a \notin \bigcup_{k'<k} \bigcup_{i \in \gamma_{k'}} P_i$, $f_j(a) > 0$ for some $j \in \gamma_k$, and $f_{j'}(a) = 0$ for all $j' \in \gamma_{k'}$ with $k' = k+1, k+2, \ldots, t$.

For the forward direction of (*), let

$A = \{a \in C : a \in \bigcup_{i \in \gamma_k} P_i$ and either $a \in \bigcup_{k'<k} \bigcup_{i \in \gamma_{k'}} P_i$ or $f_j(a) = 0$ for all $j \in \gamma_k$ or $f_{j'}(a) > 0$ for some $j' \in \gamma_{k'}$ with $k' = k+1, k+2, \ldots, t\}$

and assume, by way of contradiction, that A has positive measure. Clearly, if $a \in \bigcup_{i \in \gamma_k} P_i$, then $a \notin \bigcup_{k'<k} \bigcup_{i \in \gamma_{k'}} P_i$. It follows that either $B_1 = \{a \in \bigcup_{i \in \gamma_k} P_i : f_j(a) = 0$ for all $j \in \gamma_k\}$ or $B_2 = \{a \in \bigcup_{i \in \gamma_k} P_i : f_{j'}(a) > 0$ for some $j' \in \gamma_{k'}$ with $k' = k+1, k+2, \ldots, t\}$ has positive measure.

Suppose first that B_1 has positive measure. Then, for some $i \in \gamma_k$, $D = \{a \in P_i : f_j(a) = 0$ for all $j \in \gamma_k\}$ has positive measure. Then, for some $i' \notin \gamma_k$ and some positive-measure $E \subseteq D$, $f_i(a) = 0$ and $f_{i'}(a) > 0$ for every $a \in E$. For each such a, $\frac{f_i(a)}{f_{i'}(a)} = 0$. But $\frac{\omega_i}{\omega_{i'}}$ is the ratio of positive hyperreals and, hence, is a positive hyperreal. It follows that for every $a \in E$, $\frac{f_i(a)}{f_{i'}(a)} < \frac{\omega_i}{\omega_{i'}}$. Therefore, since $E \subseteq P_i$ and E has positive measure, it is not true that for almost every $a \in P_i \frac{f_i(a)}{f_{i'}(a)} \geq \frac{\omega_i}{\omega_{i'}}$. This contradicts the fact that P is w-associated with ω.

Next, we suppose that B_2 has positive measure. Then, for some $k' = k+1$, $k+2, \ldots t$ and some $i \in \gamma_k$ and $j' \in \gamma_{k'}$, $D = \{a \in P_i : f_{j'}(a) > 0\}$ has positive measure. For every $a \in D$, $f_{j'}(a) > 0$, and, hence, $\frac{f_i(a)}{f_{j'}(a)}$ is either zero or a positive real number. Consider the term $\frac{\omega_i}{\omega_{j'}}$. This is the ratio of positive hyperreals. Since $k < k'$, we know that $\omega_{j'}$ is a higher-order infinitesimal than ω_i. This implies that $\frac{\omega'_j}{\omega_i}$ is infinitesimal and, hence, that $\frac{\omega_i}{\omega_{j'}}$ is an infinite hyperreal. It follows that, for every $a \in D$, $\frac{f_i(a)}{f_{j'}(a)} < \frac{\omega_i}{\omega_{j'}}$. Therefore, since $D \subseteq P_i$ and D has positive measure, it is not true that for almost every $a \in P_i$, $\frac{f_i(a)}{f_{j'}(a)} \geq \frac{\omega_i}{\omega_{j'}}$. This contradicts the fact that P is w-associated with ω.

Finally, we must establish the reverse direction of (*). Let

$$A = \{a \in C : a \notin \bigcup_{i\in\gamma_k} P_i, a \notin \bigcup_{k'<k} \bigcup_{i\in\gamma_{k'}} P_i, f_j(a) > 0 \text{ for some } j \in \gamma_k, \text{ and } f_{j'}(a) = 0 \text{ for all } j' \in \gamma_{k'} \text{ with } k' = k+1, k+2, \ldots, t\}$$

and assume, by way of contradiction, that A has positive measure. Since P is a partition of C, any $a \in C$ must be in some P_i. Hence, $A = \{a \in C : a \in \bigcup_{k'>k} \bigcup_{i\in\gamma_{k'}} P_i, f_j(a) > 0 \text{ for some } j \in \gamma_k, \text{and } f_{j'}(a) = 0 \text{ for all } j' \in \gamma_{k'} \text{ with } k' = k+1, k+2, \ldots, t\}$. This implies that, for some $k' = k+1, k+2, \ldots t$ and some $i \in \gamma_{k'}$ and $j \in \gamma_k$, $B = \{a \in P_i : f_j(a) > 0 \text{ and } f_i(a) = 0\}$ has positive measure. For any $a \in B$, $\frac{f_i(a)}{f_j(a)} = 0$. We know that $\frac{\omega_i}{\omega_j}$ is the ratio of positive hyperreals and, hence, is a positive hyperreal. It follows that, for every $a \in B$, $\frac{f_i(a)}{f_j(a)} < \frac{\omega_i}{\omega_j}$. Thus, since B has positive measure, it is not true that, for almost every $a \in P_i$, $\frac{f_i(a)}{f_j(a)} \geq \frac{\omega_i}{\omega_j}$. This contradicts the fact that P is w-associated with ω and, hence, establishes that P is w-associated with the partition sequence pair (ω'', γ). By Theorem 10.28, it follows that P is Pareto maximal. This completes the proof of the theorem. □

Corollary 15.6 *A partition P is Pareto maximal if and only if it maximizes the convex combination of measures corresponding to some $\alpha \in S_H^+$.*

Proof: The corollary follows immediately from Theorem 10.6 (which, as we have already discussed, extends to our present setting). □

The corollary takes the place of Theorems 7.13 and 7.18, which characterized Pareto maximality using the notions of a-maximization of a partition sequence pair and b-maximization of a partition sequence pair,respectively.

By the correspondence between maximization of convex combinations of measures and points of first contact with the IPS of families of parallel hyperplanes, the corollary implies the following:

A partition P is Pareto maximal
if and only if
for some $\alpha \in S_H^+$, $m(P)$ is a point of first contact with the IPS of the family of parallel hyperplanes with coefficients given by α.

The chores versions of Theorem 15.4 and Corollary 15.6 are the following. The proofs are similar and we omit them.

Theorem 15.7 *A partition P is Pareto minimal if and only if it is chores w-associated with* ω *for some* $\omega \in S_H^+$.

Corollary 15.8 *A partition P is Pareto minimal if and only if it minimizes the convex combination of measures corresponding to some* $\alpha \in S_H^+$.

16

Geometric Object # 1d

The Multicake Individual Pieces Set (MIPS) Symmetry Restored

By Lemma 2.3, the IPS is always symmetric about the point $(\frac{1}{2}, \frac{1}{2})$ when there are two players. In particular, given any point in the IPS, we obtain the reflection of that point about $(\frac{1}{2}, \frac{1}{2})$ by simply having the two players trade pieces. This provides a one-to-one correspondence between the set of Pareto maximal points and the set of Pareto minimal points. However, we have seen that there is no analogous symmetry when there are more than two players. (See the discussion following Corollary 4.6, the concluding comments in Chapter 7, Theorem 11.5, and the discussion before and after Theorem 11.5. Corollary 4.9 revealed a type of symmetry, but not a precise symmetry about a particular point.) In this chapter, we show that the IPS can be viewed as part of a larger and more general structure, the Multicake Individual Pieces Set, or MIPS. The MIPS has nice symmetry properties that are not generally present in the IPS when there are more than two players.

In Section 16A, we consider the MIPS for three players. (At the end of that section, we comment on why the two-player situation is trivial and uninteresting.) In Section 16B, we consider the general case of n players. We make no general assumptions about absolute continuity in this chapter.

16A. The MIPS for Three Players

We assume throughout this section that there are three players, Player 1, Player 2, and Player 3, with corresponding measures m_1, m_2, and m_3, respectively. We recall that in contrast with the two-player situation the IPS need not be symmetric about any point. One way to see this is to consider two extremes. If the measures are identical, then the IPS is equal to the two-simplex and, hence, is symmetric about the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and no other point. If the measures concentrate on disjoint sets, then the IPS is the unit cube and, hence, is symmetric about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and no other point. It follows that there is no one point about which the IPS is always symmetric.

We now define a more general structure of which the IPS is a part. By way of motivation, notice that if we choose to specify the size of the piece of cake that a player does not get, then we have conveyed the same information as if we instead specify the size of the piece of cake that a player does get. We recall that Part denotes the set of all partitions of C.

Definition 16.1

a. *IPS(3,1)* $= \{(m_1(P_1), m_2(P_2), m_3(P_3)) : \langle P_1, P_2, P_3\rangle \in \text{Part}\}$.

b. *IPS(3,2)* $= \{(m_1(P_2 \cup P_3), m_2(P_1 \cup P_3), m_3(P_1 \cup P_2)) : \langle P_1, P_2, P_3\rangle \in \text{Part}\}$.

c. The *Multicake Individual Pieces Set*, or *MIPS*, is the union of IPS(3, 1) and IPS(3, 2).

IPS(3, 1) is another name for the three-player IPS. Notice that IPS(3, 2) $= \{(m_1(C\backslash P_1), m_2(C\backslash P_2), m_3(C\backslash P_3)) : \langle P_1, P_2, P_3\rangle \in \text{Part}\}$. Thus, we may view each point in IPS(3, 2) as corresponding to a particular partition, where each player's coordinate corresponds to that player's evaluation of the size of the piece of cake that this player does not receive. We will discuss the reason for the names "IPS(3, 1)," "IPS(3, 2)," and "MIPS," in the next section.

Claim IPS(3, 1) and IPS(3, 2) are reflections of each other about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Proof of Claim: For any point (p_1, p_2, p_3),

$(p_1, p_2, p_3) \in$ IPS(3, 1)
if and only if
for some partition $\langle P_1, P_2, P_3\rangle$, $m_1(P_1) = p_1$, $m_2(P_2) = p_2$, and $m_3(P_3) = p_3$
if and only if
for some partition $\langle P_1, P_2, P_3\rangle$, $m_1(C\backslash P_1) = 1 - p_1$, $m_2(C\backslash P_2) = 1 - p_2$, and $m_3(C\backslash P_3) = 1 - p_3$
if and only if
$(1 - p_1, 1 - p_2, 1 - p_3) \in$ IPS(3, 2).

Since $(1 - p_1, 1 - p_2, 1 - p_3)$ is the reflection of (p_1, p_2, p_3) about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, this establishes the claim.

The claim implies that the MIPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Let us reconsider the two extreme examples that we considered earlier. If the measures are identical, then IPS(3, 1), our usual IPS, is equal to the simplex,

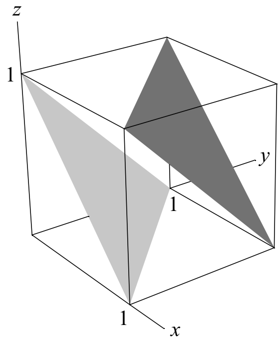


Figure 16.1

which is the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1), together with its interior. In this case, IPS(3, 2) is equal to the triangle with vertices (0, 1, 1), (1, 0, 1), and (1, 1, 0), together with its interior. Notice that, for example, the points (1, 0, 0) and (0, 1, 1) from IPS(3, 1) and IPS(3, 2), respectively, correspond to the same partition, namely $\langle C, \emptyset, \emptyset \rangle$, the partition that gives all of the cake to Player 1. This is so since $(m_1(C), m_2(\emptyset), m_3(\emptyset)) = (1, 0, 0)$ and $(m_1(C \backslash C), m_2(C \backslash \emptyset), m_3(C \backslash \emptyset)) = (m_1(\emptyset), m_2(C), m_3(C)) = (0, 1, 1)$. The MIPS in this situation, which is the union of these two triangles, is shown in Figure 16.1. In the figure, we have drawn the unit cube and have drawn IPS(3, 1) lighter and IPS(3, 2) darker. Clearly, this MIPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

The other extreme case that we considered earlier is when the measures concentrate on disjoint sets. In this case, both IPS(3, 1) and IPS(3, 2) consist of the entire unit cube and, hence, so does the MIPS. Therefore, this MIPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Let us now refer to the outer Pareto boundary and the inner Pareto boundary as the *(3,1)-outer Pareto boundary* and the *(3,1)-inner Pareto boundary*, respectively. Of course, these are each part of the boundary of IPS(3, 1), the usual IPS. We shall denote the corresponding notions for IPS(3, 2) as the *(3,2)-outer Pareto boundary* and the *(3,2)-inner Pareto boundary*, respectively. It is not hard to see that a point is on the (3, 1)-outer Pareto boundary if and only if its reflection about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is on the (3, 2)-inner Pareto boundary, and a point is on the (3, 1)-inner Pareto boundary if and only if its reflection about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is on the (3, 2)-outer Pareto boundary. There is a natural perspective on this correspondence that involves partitions of C. In the standard setting, each player wants to get as much cake as possible. This is the same as saying that each player wants the cake received by the other players to be as small as possible. Thus, asking for a "big" point in IPS(3, 1) is equivalent to asking for a "small" point in IPS(3, 2). The same holds for the chores setting, where we see that asking for a "small" point in IPS(3, 1) is equivalent to asking for a "big" point in IPS(3, 2).

In this section, we have found that by broadening our perspective to see the IPS as a part of a larger structure, the MIPS, the symmetry that was lost in going from the two-player setting to the three-player setting is restored. We shall see in the next section that although this is true when there are more than three players, there is additional structure that arises in a natural way.

In concluding this section, we comment on why we chose to begin by considering three players instead of two. For two players, we have IPS $= \{(m_1(P_1), m_2(P_2) : \langle P_1, P_2\rangle \in \text{Part}\}$. If we define a "new" set in the style of Definition 16.1, it would be the set $\{(m_1(P_2), m_2(P_1)) : \langle P_1, P_2\rangle \in \text{Part}\}$, and it is easy to see that this is the same as the IPS. Thus, this perspective yields nothing new when there are only two players.

16B. The MIPS for the General *n*-Player Context

We begin by considering the four-player context. We assume that there are four players, Player 1, Player 2, Player 3, and Player 4, with corresponding measures m_1, m_2, m_3, and m_4, respectively. The definitions of the sets IPS(3, 1) and IPS(3, 2), given in Definition 16.1, generalize in a natural way to this setting (although the reasons for our choice of names for these sets is not yet clear). We let

$$\text{IPS}(4, 1) = \{(m_1(P_1), m_2(P_2), m_3(P_3), m_4(P_4)) : \langle P_1, P_2, P_3, P_4\rangle \in \text{Part}\}$$

and

$$\begin{aligned}\text{IPS}(4, 3) = \{(&m_1(P_2 \cup P_3 \cup P_4), m_2(P_1 \cup P_3 \cup P_4), m_3(P_1 \cup P_2 \cup P_4),\\ &m_4(P_1 \cup P_2 \cup P_3)) : \langle P_1, P_2, P_3, P_4\rangle \in \text{Part}\}.\end{aligned}$$

The relationship between these two sets is as in the three-player context. In particular,

- each of these sets is the reflection of the other about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and,
- a point is on the (4, 1)-outer Pareto boundary if and only if its reflection about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is on the (4, 3)-inner Pareto boundary, and a point is on the (4, 1)-inner Pareto boundary if and only if its reflection about the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is on the (4, 3)-outer Pareto boundary (with the obvious definitions of (4, 1)-outer Pareto boundary, (4, 3)-inner Pareto boundary, etc.).

However, we are not going to define the MIPS for the four-player context to be the union of these two sets. Let us return to the three-player setting and see that there is another natural way to define this MIPS that agrees with Definition 16.1 for three players, but gives additional structure beyond the union of the sets IPS(4, 1) and IPS(4, 3) when there are four players. (Of course, our notation suggests that there is something that we will call "IPS(4, 2).") Then, we shall present a general definition of the MIPS for any number of players.

Recall that

$$\text{IPS}(3, 1) = \{(m_1(P_1), m_2(P_2), m_3(P_3)) : \langle P_1, P_2, P_3\rangle \in \text{Part}\}$$

and

$$\begin{aligned}\text{IPS}(3, 2) &= \{(m_1(P_2 \cup P_3), m_2(P_1 \cup P_3), m_3(P_1 \cup P_2)) : \langle P_1, P_2, P_3\rangle \in \text{Part}\}\\ &= \{(m_1(C\backslash P_1), m_2(C\backslash P_2), m_3(C\backslash P_3)) : \langle P_1, P_2, P_3\rangle \in \text{Part}\}.\end{aligned}$$

We observe that $\langle P_1, P_2, P_3\rangle$ is a partition of C if and only if $P_1, P_2, P_3 \subseteq C$ and every $a \in C$ is in exactly one of the P_i. Hence, we may write $\text{IPS}(3, 1) = \{(m_1(P_1), m_2(P_2), m_3(P_3)) : P_1, P_2, P_3 \subseteq C$ and every $a \in C$ is in exactly one of these sets$\}$. For any partition $\langle P_1, P_2, P_3\rangle$ of C, every $a \in C$ is in exactly two of the sets $P_1 \cup P_2, P_1 \cup P_3, P_2 \cup P_3$. On the other hand, given any $Q_1, Q_2, Q_3 \subseteq C$, if every $a \in C$ is in exactly two of these sets, then $\langle C\backslash Q_1, C\backslash Q_2, C\backslash Q_3\rangle$ is a partition of C. This tells us that $\text{IPS}(3, 2) = \{(m_1(Q_1), m_2(Q_2), m_3(Q_3)) : Q_1, Q_2, Q_3 \subseteq C$ and every $a \in C$ is in exactly two of these sets$\}$.

Next, we apply this alternative approach to the four-player context. We may write

- $\text{IPS}(4, 1) = \{(m_1(R_1), m_2(R_2), m_3(R_3), m_4(R_4)) : R_1, R_2, R_3, R_4 \subseteq C$ and every $a \in C$ is in exactly one of these sets$\}$ and
- $\text{IPS}(4, 3) = \{(m_1(R_1), m_2(R_2), m_3(R_3), m_4(R_4)) : R_1, R_2, R_3, R_4 \subseteq C$ and every $a \in C$ is in exactly three of these sets$\}$.

It is easy to see that these definitions are equivalent to the previous definitions of IPS(4, 1) and IPS(4, 3). Our reasons for this notation (i.e., IPS(3, 1), IPS(3, 2), IPS(4, 1), and IPS(4, 3)) should now be clear. Also, it is clear that there is another set whose definition follows in a natural way from this approach. This is the set $\text{IPS}(4, 2) = \{(m_1(R_1), m_2(R_2), m_3(R_3), m_4(R_4)) : R_1, R_2, R_3, R_4 \subseteq C$ and every $a \in C$ is in exactly two of these sets$\}$. We define the MIPS for four players to be the union of the three sets IPS(4, 1), IPS(4, 2), and IPS(4, 3). We formalize this for the general n-player setting as follows.

Definition 16.2

a. For each $k = 1, 2, \ldots, n-1$, $\langle R_1, R_2, \ldots, R_n \rangle$ is a *k-partition* of C if and only if $R_1, R_2, \ldots, R_n \subseteq C$ and for every $a \in C$, $|\{i \leq n : a \in R_i\}| = k$.

b. For each $k = 1, 2, \ldots, n-1$, $IPS(n, k) = \{(m_1(R_1), m_2(R_2), \ldots, m_n(R_n)) : \langle R_1, R_2, \ldots, R_n \rangle$ is a k-partition of $C\}$.

c. $MIPS = \bigcup_{k=1}^{n-1} \text{IPS}(n, k)$.

Notice that $\langle R_1, R_2, \ldots, R_n \rangle$ is a partition of C if and only if it is a 1-partition of C. It is easy to see that our previous definitions of IPS(3, 1), IPS(3, 2), IPS(4, 1), and IPS(4, 3) are consistent with Definition 16.2.

The following claim generalizes the claim from the previous section.

Claim For each $k = 1, 2, \ldots, n-1$, IPS(n, k) and IPS$(n, n-k)$ are reflections of each other about the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in \mathbf{R}^n$.

Proof of Claim: Fix any $k = 1, 2, \ldots, n-1$. For any point $(r_1, r_2, \ldots, r_n) \in \mathbf{R}^n$,

$(r_1, r_2, \ldots, r_n) \in \text{IPS}(n, k)$

if and only if

for some $R_1, R_2, \ldots, R_n \subseteq C$, $\langle R_1, R_2, \ldots, R_n \rangle$ is a k-partition of C and $m_1(R_1) = r_1, m_2(R_2) = r_2, \ldots, m_n(R_n) = r_n$

if and only if

for some $R_1, R_2, \ldots, R_n \subseteq C$, $\langle C \backslash R_1, C \backslash R_2, \ldots, C \backslash R_n \rangle$ is an $(n-k)$-partition of C and $m_1(C \backslash R_1) = 1 - r_1, m_2(C \backslash R_2) = 1 - r_2, \ldots, m_n (C \backslash R_n) = 1 - r_n$

if and only if

for some $S_1, S_2, \ldots, S_n \subseteq C$, $\langle S_1, S_2, \ldots, S_n \rangle$ is an $(n-k)$-partition of C and $m_1(S_1) = 1 - r_1, m_2(S_2) = 1 - r_2, \ldots, m_n(S_n) = 1 - r_n$

if and only if

$(1 - r_1, 1 - r_2, \ldots, 1 - r_n) \in \text{IPS}(n, n-k)$.

Since $(1 - r_1, 1 - r_2, \ldots, 1 - r_n)$ is the reflection of $(r_1, r_2, \ldots, r_n)$ about the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, this establishes the claim.

It follows from the claim that the MIPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Notice that, if n is odd, then $n-1$ is even and the $n-1$ sets IPS$(n, 1)$, IPS$(n, 2)$, $\ldots$, IPS$(n, n-1)$ come in pairs that are reflections of each other about the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. On the other hand, if n is even, then there are an odd number of such sets. All of these sets except for IPS$(n, \frac{n}{2})$ come in pairs that are reflections of each other about the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. The claim tells us that IPS$(n, \frac{n}{2})$ is its own reflection about the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. In other words, IPS$(n, \frac{n}{2})$ is symmetric about this point. (When $n = 2$, this is simply Lemma 2.3, which tells us that in this case the IPS is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$.) Thus, we see that the MIPS is a natural structure of which our usual IPS is a part, and which has nice symmetry properties.

We conclude by giving a different perspective on IPS(n, k). This perspective, which we shall not formalize but will present informally, explains the use of the term "multicake." Fix some $k = 1, 2, \ldots, n-1$. We imagine k copies of the cake C, and we let D_k be the disjoint union of these copies. (We can make these copies disjoint by, for example, changing each element of C to an ordered pair, with first coordinate the original element of C, and second coordinate 1 for the first copy of C, 2 for the second copy, etc.) We can view the measures as being measures on D_k in the obvious way. Then, for each $i = 1, 2, \ldots, n$, $m_i(D_k) = k$. Call a partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of D_k a *proper partition* if and only if, for each $a \in C$, each P_i contains at most one copy of a. As usual, we let $m(P) = (m_1(P_1), m_2(P_2), \ldots, m_n(P_n))$. Then it is not hard to see that IPS$(n, k) = \{m(P) : P$ is a proper partition of $D_k\}$. Hence, we can view IPS(n, k) as arising from a collection of copies of the original cake C. This explains the use of the term multicake in the "Multicake Individual Pieces Set," which is the union of the IPS(n, k).

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